3 Old covariant quantization – basic setup

- Before quantizing, we need to go from the lagrangian to the hamiltonian formulation.
- For systems with gauge invariance (≡ systems with constraints), this is highly non-trivial.
  (cf. Dirac's "Lectures on QM"; Henneaux/Teitelboim; Matschul; Wipf.)
- We will not be able to present the very interesting & important story of the "quantization of systems with constraints" but only use some basic elements (to get our "stringy" results of interest).
- Nevertheless, it is worthwhile to try and understand the basic issues using our simple toy model, the relativistic particle

3.1 Towards the hamiltonian treatment of the relativistic particle

- Recall: $S = -m \int d\tau \sqrt{-\dot{X}^2}$; $X^\mu = X^\mu(\tau)$

  $\dot{X}^2 = \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}$

$$\Rightarrow p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m \frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}$$

- Define $\phi \equiv p^2 + m^2$
- Work it out: $p^2 + m^2 = m^2 \frac{\dot{X}^2}{\sqrt{-\dot{X}^2}^{\,2}} + m^2 = 0$

identity; true even before imposing EOMs

[This is called a <u>primary</u> constraint.]

$\Rightarrow$ impossible to express $\dot{x}^\mu$ in terms of p^μ. Must restrict ourselves to submanifold of phase space defined by constraint.

- Work out H: $\quad \dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} - L = -m\sqrt{-\dot{x}^2} - (-m\sqrt{-\dot{x}^2}) \equiv 0.$

[This is due to time-diff.-invariance and is the origin of the constraint found above.]

- Use more general H: $\quad H \longrightarrow H + \underbrace{\sum_k c_k \phi_k}$

lin. comb. of all constraints

(It's ok to add them since they vanish on constrained phase space.)

- Our case: $\quad H = \frac{N}{2m}(p^2 + m^2)$
- Poiss. eq.: $\quad \dot{x}^\mu = \{x^\mu, H\} = \frac{N}{m} p^\mu \quad \checkmark$

($\Rightarrow N^2 = -\dot{x}^2$, i.e. N is an arbitrary parameter fixing the freedom of choosing a time variable. $N=1 \Leftrightarrow$ eigentime.)

Alternatively: <u>Polyakov treatment</u>

$$S = -\frac{m}{2}\int dt \sqrt{-h}\left(h^{-1}\dot{x}^2 + 1\right) \quad ; \quad \Rightarrow p^\mu = \frac{m}{\sqrt{-h}}\dot{x}^\mu$$

<u>h-EOM</u>: $\quad -\frac{1}{2\sqrt{-h}}\left(h^{-1}\dot{x}^2 + 1\right) - \sqrt{-h}\left(h^{-2}\dot{x}^2\right) = 0 \quad \Rightarrow \underbrace{\frac{\dot{x}^2}{(-h)} + 1 = 0}$

This is our old constraint $p^2 + m^2 = 0$.
(It now follows only after using EOMs and is hence called "secondary".)

<u>X-EOM:</u> $\partial_\tau\left(\dot{X}^\mu/\sqrt{-h}\right) = 0 \underset{\text{using } h\text{-EOM}}{\Longrightarrow} \partial_\tau\left(\dot{X}^\mu/\sqrt{-\dot{X}^2}\right) = 0$

Now, we <u>can</u> express the $\dot{X}^\mu$ through p^μ as usual.
But quantizing the system by declaring all pairs (X^μ, p^μ) independent would be wrong. All straight lines, i.e. all fcts. with $\ddot{X}^\mu = 0$, would be solutions. We <u>must</u> introduce $p^2 + m^2 = 0$ as a constraint into hamiltonian and later quantum formalism, i.e.

$$\left\{ \frac{\delta S}{\delta h} \sim \underset{\text{"EM-tensor"}}{\underset{\uparrow}{\text{"}T_{ab}\text{"}}} = 0 \iff p^2 + m^2 = 0 \right\}$$ must be enforced in canon. quantization

As before, this is a result of the gauge-inv. of our system under time-diffeomorphisms.
(see BLT & Zwiebach for more details)

3.2 Quantizing the closed string

- Use the Polyakov action and flat gauge: $S = -\frac{T}{2}\int d^2\xi\, \partial^a X_\mu \partial_a X^\mu$.
- This is just a 2d QFT with d scalars X^μ.
- Canonical momenta:

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T\dot{X}^\mu$$

$$H = \int_0^\ell d\sigma\, \underbrace{\left(\dot{X}\cdot\Pi - \mathcal{L}\right)}$$

$$= \frac{T}{2}\left(\dot{X}^2 + X'^2\right) = T\left((\partial_+ X)^2 + (\partial_- X)^2\right)$$

- We find at the Poiss.-Bracket level and then impose at the commutator level:

(without the "i")

$$[\Pi_\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\delta(\sigma-\sigma')\,\delta_\mu{}^\nu \quad ; \quad [X,X] = [\Pi,\Pi] = 0$$

- We recall the oscillator expansion given earlier, where now

$$(\alpha_m^\mu)^+ = \alpha_{-m}^\mu \; , \; (\tilde{\alpha}_m{}^\mu)^+ = \tilde{\alpha}_{-m}^\mu$$ by hermiticity of the X^μ.

- It is straightforward to work out:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu} \; ; \; [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu} \; ; \; [p^\mu, x^\nu] = -i\eta^{\mu\nu}$$

$$(\,\delta_k \equiv \delta_{k,0}\,)$$

- We also note that the integral of the "momentum density" Π^μ belonging to the field X^μ reads:

$$p^\mu = \int_0^\ell d\sigma\, \Pi^\mu = T\int_0^\ell d\sigma\, \dot{X}^\mu = \frac{1}{2\pi\alpha'}\int_0^\ell d\sigma \left(\frac{\pi\alpha'}{\ell}\cdot p^\mu \cdot 2\tau\right)^{\!\cdot} = p^\mu .$$

This proves that the coeff p^μ in our expansion is really the physical 4-momentum of our string.

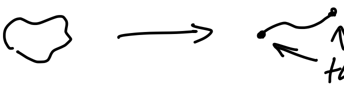
<u>Comments</u>:
- Please check the commut. relations using

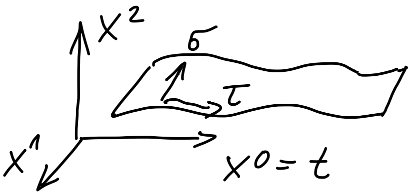
$$\int_0^\ell d\sigma \int_0^\ell d\sigma'\, e^{2\pi i m\sigma/\ell}\, e^{2\pi i n\sigma'/\ell}\, \delta(\sigma-\sigma') = \ell\,\delta_{m+n}$$

etc.

- Note that the terms with x^μ, p^ν do not appear in conventional QFT since there $\ell = \infty$ and then x^μ, i.e. the average field value, is non-dynamical.

3.3 Quantizing the open string

Everything, from classical analysis to quantization, can be repeated for the open string, i.e.  two boundaries.



$\sigma \in (0, \ell)$

$$S = -\frac{T}{2}\int d^2\xi\,(\partial^a X^\mu)(\partial_a X_\mu)\,;\quad \delta S = -T\int d^2\xi\,(\partial^a X^\mu)(\partial_a \delta X_\mu)$$

$$= -T\int d^2\xi\,\underbrace{(\partial^2 X^\mu)}_{\text{EOM}}\delta X_\mu \underbrace{- T\int_{-\infty}^{\infty} d\tau\,(\partial^\sigma X^\mu)\,\delta X_\mu\Big|_{\sigma=0}^{\sigma=\ell}}_{\text{Boundary term}}$$

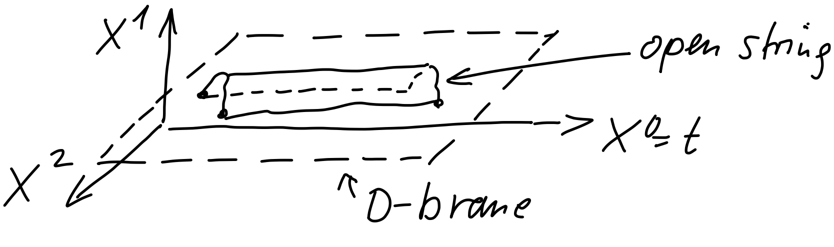
- Bd. term vanishes for
  1) Neumann b.c.s : $\partial_\sigma X^\mu = 0$
  2) Dirichlet b.c.s : $\delta X^\mu = 0$

<u>Physics interpretation:</u>

1) Shift symm. in X^μ preserved $\Rightarrow$ momentum conservation respected
   $\Rightarrow$ mom. can not flow off string at endpoints
   $\Rightarrow$ <u>endpoints move freely in target space</u>

2) $X^\mu(\tau, 0)$ & $X^\mu(\tau, \ell)$ fixed $\Rightarrow$ <u>endpoints attached to "D-brane"</u>

- <u>In general</u>, some of the X^μ obey Neumann b.c.s (i.e. move freely), the others obey Dirichlet b.c.s (are fixed). This corresponds to the string ending on a D-brane living in a hyperplane determined by the values of the "Dirichlet"-X^μ's. <u>Example:</u>

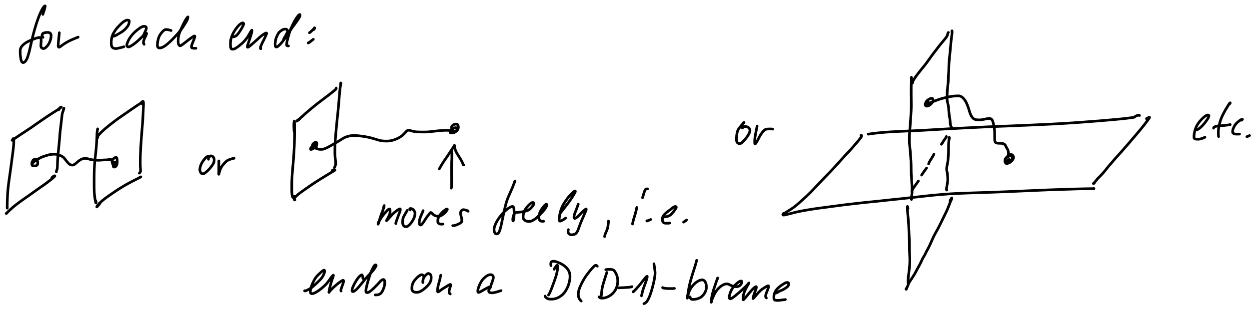


X^0, X^2 – Neumann

X^1 – Dirichlet

- Naming convention: A D-brane with p spatial dim.s. is called "Dp-brane". The string ending on it then has $(p+1)$ Neumann and $(D-p-1)$ Dirichlet b.c.s. Our example shows a D1-brane in $D=3$ target space dims.

- Strings can end on different branes and hence have different b.c.s for each end:



- One can derive mode expansions for NN, DD, ND & DN b.c.s. We only give the NN case:

$$X^\mu = x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\pi n\tau/\ell} \cos\left(\frac{\pi n \sigma}{\ell}\right).$$

(Note that $2\pi\alpha'$ was $\pi\alpha'$ for X_L & X_R of the closed string.)

For DD, ND, DN the p^μ-term is absent. The x^μ-term is fixed by the D-brane position(s). See BLT for the explicit eqs.

- A very useful way to think about the open string is to start with closed string on a doubled interval, $\sigma \in (-\ell, \ell)$, with (symbolically)

$$X^\mu = \dots + \sim \sum \frac{i}{n}\left(\tilde{\alpha}_n^\mu e^{-2\pi i n \sigma^+/\ell} + \alpha_n^\mu e^{-2\pi i n \sigma^-/\ell}\right)$$

and demand symmetry under $\sigma \to -\sigma$ ("mod out a $\mathbb{Z}_2$"). This enforces Neumann b.c.s at the "fixed points" $\sigma = 0$ & $\sigma = \ell$ and $\tilde{\alpha}_n = \alpha_n$. Thus:

$$X^\mu = \dots + \sim \sum \frac{i}{n} \alpha_n^\mu e^{-2\pi i n\tau/\ell} \cos\left(\frac{2\pi n \sigma}{\ell}\right).$$

We see the left & right-moving excitations are not independent, as you expect from reflecting b.c.s.

- In summary, we have: $[p^\mu, x^\nu] = -i\eta^{\mu\nu}$ (for NN)

  $[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu}$ (i.e. "half" of the closed string)

3.4 Constructing the Fock space

- Operator algebra: $\underbrace{\alpha_m^\mu\,;\ \tilde{\alpha}_m^\mu \text{ (if closed)}}_{\text{oscillators}}\,;\ \underbrace{\hat{p}^\mu\,;\ \hat{x}^\mu}_{\text{momentum/position}}$

  ($\alpha_m^{\mu\,+} = \alpha_{-m}^\mu$ etc.) (hermitian)

- Assume Hilbert space repres. exists.
- Can diagonalize $\hat{p}^\mu$ or $\hat{x}^\mu$. Choose p^μ.

$$\Rightarrow \mathcal{H} = \bigoplus_p \mathcal{H}(p)$$ where $\mathcal{H}(p)$ is the subspace on which $\hat{p} = (\hat{p}^0, \dots, \hat{p}^{D-1})$ is diagonal with eigenvalues $p = (p^0, \dots, p^{D-1})$.

- Focus on one of the eigenspaces $\mathcal{H}(p)$, i.e., $\hat{p}$ is replaced by the real vector p. Build a repres. of the α's using $[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu}$ i.e. $[\alpha_n^\mu, \alpha_m^{\nu+}] = |m|\,\delta_{m,n}\,\eta^{\mu\nu}$

- As usual, define vacuum $|0,p\rangle$:

$$\alpha_m^\mu |0,p\rangle = 0 \quad \forall\, m>0,\ \mu,\ p$$

(we suppress analogous expressions with $\tilde{\alpha}_m^\mu$.)

- Define $$\mathcal{H} = \text{span}\left\{ \alpha_m^\mu \alpha_n^\nu \cdots |0,p\rangle \right\}$$

  ↑ any number of α's ; all $\mu, \nu, \dots$; all $m,n<0$; all p.

- Immediate problem: While $[\alpha_m^i, \alpha_m^{i+}] = +m$, as it should be,
  $(i,j = 1 \cdots D-1)$ $\qquad [\alpha_m^0, \alpha_m^{0+}] = -m$, not suitable for the standard oscillator interpretation.

- One could think of "switching the roles" of α_m^0 & α_m^{0+}, but that would destroy the Lorentz-trf.-properties of our representation.

- In fact, the wrong-sign commutator is disastrous for the physics interpretation: $[a,a^+] = -1 \Rightarrow |a^+|0\rangle|^2 = \langle 0|a a^+|0\rangle = -\langle 0|0\rangle$
  $\Rightarrow a^+|0\rangle$ is a neg.-norm-state or "ghost"
  (not to be confused with Fadeev-Popov-ghost)

- One inspiration for how to proceed comes from QED:
  $A_\mu \longrightarrow a_\mu, a_\mu^+$, with a_0, a_0^+ having a wrong-sign commutator.
  We recall that the resolution was to impose the gauge-fixing-condition on all phys.-states: $\quad (\partial_\mu A^\mu)\Big|_{\text{annih.-part.}} |\psi\rangle = 0 .$

- Another inspiration comes from the relativistic particle.
  There, we saw that the constraint $p^2 + m^2 = 0$ <u>had</u> to be imposed in the hamiltonian frame work. This constraint came from $\delta S/\delta h = 0$. It has remained crucial even after h disappeared by gauge fixing.

- Here we can proceed in exact analogy:
  $$\frac{\delta S}{\delta h_{ab}} = 0 \quad \Rightarrow \quad T_{ab} = 0 ,$$
  need to implement this at hamiltonian level & in QM.

- It will be convenient to use light-cone coordinates, σ^+/σ^-.

- $T_a{}^a = 0 \iff T_{+-} = T_{-+} = 0$ (identically)

- We only have to impose $\underline{T_{++} = 0 \;\; \& \;\; T_{--} = 0}$

- $T_{ab} = -\frac{1}{\alpha'}\left(G_{ab} - \frac{1}{2}h_{ab}\left(h^{cd}G_{cd}\right)\right) \underset{\text{flat}}{\Longrightarrow} \begin{array}{l} T_{++} = -\frac{1}{\alpha'}(\partial_+ X)\cdot(\partial_+ X) \\ T_{--} = -\frac{1}{\alpha'}(\partial_- X)\cdot(\partial_- X) \end{array}$

- $\partial_+ X_R = 0 \;\&\; \partial_- X_L = 0 \Longrightarrow \begin{array}{l} T_{++} = -\frac{1}{\alpha'}(\partial_+ X_L)\cdot(\partial_+ X_L) \\ T_{--} = -\frac{1}{\alpha'}(\partial_- X_R)\cdot(\partial_- X_R) \end{array}$

- It will be convenient to work not with T_{++}/T_{--} themselves but with their Fourier modes:

$$L_m \equiv -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma\, e^{2\pi i n\sigma/\ell}\, T_{--} = \frac{1}{2}\sum_m \alpha_{m-n}\cdot\alpha_n$$

$$\tilde{L}_m \equiv -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma\, e^{2\pi i n\sigma/\ell}\, T_{++} = \frac{1}{2}\sum_m \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \,.$$

- Here we have introduced the convenient notation

$$p^\mu\sqrt{\frac{\alpha'}{2}} \equiv \alpha_0^\mu \equiv \tilde{\alpha}_0^\mu \,.$$

- For the open string, only the L_m are relevant. But now: $p^\mu\sqrt{2\alpha'} \equiv \alpha_0^\mu$.
- Straightforward to check:

$$L_m^+ = L_{-m} \;;\; \tilde{L}_m^+ = \tilde{L}_{-m} \;;\; L_0 + \tilde{L}_0 = \frac{\ell}{2\pi}H$$

- $\underline{\text{Crucial point:}}$ L_0 has an ordering ambiguity since α_m & α_{-m} stand next to each other.
We (re-)define: $\boxed{L_0 \equiv \frac{1}{2}\alpha_0^2 + \sum_{n=1}^\infty \alpha_{-n}\cdot\alpha_n}$

(normal ordered)

- With this definition (and some work → tutorials) one can show:

$$[L_m, L_n] = \underbrace{(m-n) L_{m+n}}_{\text{classical}} + \underbrace{A(m)\,\delta_{m+n}}_{\text{anomaly}} \qquad \| \text{ Virasoro algebra}$$

with $A(m) \equiv \frac{D}{12}(m^3 - m)$.

- At the hamiltonian level, before quantization, we would have found $\{L_m, L_n\} = (m-n) L_{m+n}$ $\|$ Witt algebra.

- This reflects simply classical reparametrization invariance of the S^1. The reason is that T_{ab} (being the current belonging to P_a) generates <u>local</u> translations, i.e. Diff's.

- To see the Witt algebra arise explicitly, consider the space of fcts. on a circle, $f(\theta) = f(\theta + 2\pi)$.

- Local translations, $f(\theta) \to f(\theta + a(\theta))$, are generated by operators

$$D[a] = a(\theta)\,\partial_\theta \;.$$

- Consider the basis $D_n = i e^{in\theta}\partial_\theta$ and look at its Lie algebra:

$$[D_m, D_n] = [ie^{im\theta}\partial_\theta, ie^{in\theta}\partial_\theta] = i(m-n)e^{i(m+n)\theta}\partial_\theta = (m-n) D_{m+n}$$

✓

- We want to impose $T_{ab} = 0$, i.e. $L_m = 0$, on phys. states. But this is too strong since

$$0 = \langle \text{phys} | [L_m, L_{-m}] | \text{phys} \rangle = A(m) \langle \text{phys} | \text{phys} \rangle \neq 0 \; ↯ .$$

- For our purposes sufficient: Demand that

$$\boxed{(L_m + a\delta_m)\,|phys\rangle = 0 \qquad \forall m \geq 0}$$

So far unknown constant accounting for the ordering ambiguity in L_0 which we have not yet tried to resolve.

- This is similar to / inspired by QED, where only $(\partial A)|_{\text{annih. part}}$ is used to constrain phys. states.

- It is indeed sufficient since our condition implies

$$\langle phys.|L_m|phys\rangle = 0 \quad \forall m:$$

Indeed, take $m>0$ & observe: $\langle phys|L_{-m}|phys\rangle = \overline{\langle phys|L_m|phys\rangle}$
$= 0$ ✓