4 Old covariant quantization – explicit construction of states

We have: Fock space $\mathcal{H}$:
- direct sum of all $|0,p\rangle$
- all creation operators $\overset{(\sim)}{\alpha}{}^{\mu}_{m}$, $m<0$ can act

Phys. Subspace: all $|\psi\rangle$ in $\mathcal{H}$ with

$$\boxed{(\overset{(\sim)}{L}_m + a\delta_m)|\psi\rangle = 0\,, \quad m \geq 0}$$

4.1 Open string

Vacuum: $|0,p\rangle$, $\hat{p}|0,p\rangle = p|0,p\rangle$, let $m>0$

$$0 \overset{!}{=} L_m|0,p\rangle = \frac{1}{2}\sum_n \alpha_{m-n}\alpha_n|0,p\rangle = 0 \;\forall p \text{ since either}$$

$n>0$ or $(m-n)>0$.

$\Rightarrow$ Only the L_0 - constraint is relevant:

$$\underbrace{0 \overset{!}{=} (L_0+a)|0,p\rangle}_{\text{"Mass-shell condition"}} = \Big(\alpha'\hat{p}^2 + \underbrace{\underbrace{\sum_{n>0}\alpha_{-n}\cdot\alpha_n}_{\text{"}\hat{N}\text{" ("level number operator")}}}_{\text{gives zero in our case}} + a\Big)|0,p\rangle$$

$$\Rightarrow p^2 = -\frac{a}{\alpha'}\,, \quad \text{i.e. } m^2 = -p^2 = \frac{a}{\alpha'}$$

(We found one scalar particle with mass$^2 = a/\alpha'$.)

First excited level: general state is $\zeta_\mu \alpha^\mu_{-1}|0,p\rangle$

(Characterized by polarization vector ζ.)

Now the mass shell condition reads:

$$0 = (L_0 + a)\, \zeta_\mu \alpha^\mu_{-1} |0,p\rangle = (\alpha' p^2 + \underbrace{\alpha^\mu_{-1} \alpha_{1\mu}} + a)\, \zeta_\mu \alpha^\mu_{-1} |0,p\rangle$$

this term is now relevant, but not the higher terms.

$$\Downarrow$$

$$0 = (\alpha' p^2 + 1 + a)\, \zeta \cdot \alpha_{-1} |0,p\rangle$$

we will use this to abbreviate index contraction

$$\Rightarrow m^2 = \frac{1+a}{\alpha'} .$$

- In addition to the mass-shell condition, now the L_1-condition also becomes non-trivial:

$$0 = L_1\, \zeta\cdot\alpha_{-1} |0,p\rangle = \left(\frac{1}{2} \sum_n \alpha_{1-n} \cdot \alpha_n\right) \zeta \cdot \alpha_{-1} |0,p\rangle$$

- For $n \leq -1$, α_{1-n} always annihilates the state.
  For $n \geq 2$, α_n — " — .

$$\Rightarrow \text{sum over } n = -1, 0 \quad \Rightarrow \quad 0 = (\alpha_1 \cdot \alpha_0 + \alpha_0 \cdot \alpha_1)\, \zeta \cdot \alpha_{-1} |0,p\rangle$$

$$\Rightarrow 0 = \zeta \cdot \alpha_0 |0,p\rangle \quad \Rightarrow \quad \zeta \cdot p |0,p\rangle = 0 \quad \Rightarrow \quad \underline{\underline{\zeta \cdot p = 0}}$$

"The polarization vector must be transversal."

- Let us also calculate the norm of the state:

$$\langle 0,p| (\zeta\cdot\alpha_{-1})^\dagger (\zeta \cdot \alpha_{-1}) |0,p\rangle = \langle 0,p|0,p\rangle\, \zeta^2$$

$\equiv \zeta_\mu \zeta^\mu$

<u>Can distinguish three cases for level-1 states:</u>

a) <u>$a < -1$</u> $\Rightarrow$ $m^2 < 0$ $\Rightarrow$ p space-like

$\Rightarrow \exists$ time-like ζ with $\zeta \cdot p = 0$ $\Rightarrow$ $\exists$ negative-norm states, case excluded!

b) $\underline{a=-1}$ $\Rightarrow$ $m^2=0$ $\Rightarrow$ p light-like

$\Rightarrow$ $(D-1)$ vectors ξ with $\xi \cdot p = 0$:

- 1 longitudinal, $\xi \parallel p$ $\Rightarrow$ zero-norm state
- $(D-2)$ transverse, $\xi^2 > 0$ $\Rightarrow$ positive-norm states

This corresponds precisely to the phys. states found in Gupta-Bleuler quantization of QED. We hence expect a $U(1)$ gauge theory to emerge. This case is called <u>"Critical ST"</u>

c) $\underline{a > -1}$ $\Rightarrow$ $m^2 > 0$ $\Rightarrow$ p time-like $\Rightarrow$ $(D-1)$ space-like ξ

$\Rightarrow$ expect $(D-1)$ polarizations of massive vector field. At this level OK, but problematic at interacting and higher-loop level. <u>"Non-critical ST"</u>

(Consistency appears to require sacrificing target-space Poinc.-invariance. Applicability to real-world physics still unclear ....)

<u>Second excited level:</u>

General state is $|\psi\rangle = (\epsilon_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu} + \epsilon_{\mu}\alpha_{-2}^{\mu})|0,p\rangle$

Mass-shell constraint: $(L_0 + a)|\psi\rangle = 0$ $\Rightarrow$ $m^2 = \dfrac{2+a}{\alpha'}$

In addition we now need to consider the L_1 & L_2 constraints. One then has to work out allowed polarizations, masses & norms of corresponding states etc. We will not go into this in any detail since we will soon study approaches where the results we would find follow more easily. These are

- For now, we only present an argument for $D=26$ based on a simple example state (cf. GSW for more details):

  Consider $|\phi\rangle = \{ c_1\, \alpha_{-1}\cdot\alpha_{-1} + c_2\cdot p\cdot\alpha_{-2} + c_3 (p\cdot\alpha_{-1})^2 \} |0,p\rangle$

  Focus on $a=-1$ and impose

  $$(L_0 - 1)|\phi\rangle = 0 \; ; \; L_1|\phi\rangle = 0 \; ; \; L_2|\phi\rangle = 0$$

  Derive constraints on c_1, c_2, c_3. Express c_2, c_3 through c_1

  $$\Rightarrow \quad \langle\phi|\phi\rangle = \frac{2c_1^2}{25}(D-1)(D-26)$$

- $\Rightarrow$ We need $D \leq 26$ for consistency. $D=26$ is special because, like for level one, we obtain (a large number of) zero-norm states, signalling a large gauge symm. Thus, $D=26$ "belongs to" $a=-1$. The critical string mentioned before really has $a=-1$ <u>and</u> $D=26$.

- With more work, one can show that

  a) $a < -1$, excluded (ghosts)

  b) $a=-1$, $D=26$ is a very special case governed by a large gauge symm. This makes the theory solvable and consistent including all levels, interactions, loops. $\Rightarrow$ <u>"Critical string"</u>

  (We will see that the key point here is that the Weyl symm. straightforwardly survives quantization.)

  c) $a > -1$, $D < 26$ – may be consistent with caveats <u>"Non-critical string"</u>

4.2 Formal summary of OCQ

- Start with Fock space $\mathcal{H}$
- Define subspace $\mathcal{H}_{phys}$ of "phys. states" by $(L_m + a\delta_m)|phys\rangle = 0$ $(m \geq 0)$
- States $|\psi\rangle$ with $\langle\psi|phys\rangle = 0$ for all phys. states are called "spurious".
- States which are both spurious and physical are called "null". $(\mathcal{H} \supset \mathcal{H}_{phys} \supset \mathcal{H}_{null})$
- $\underset{\sim}{\mathcal{H}} \equiv \mathcal{H}_{phys} / \underset{\substack{\uparrow \\ \text{residual gauge symm., very large in crit. case}}}{\mathcal{H}_{null}}$ is the Hilbert space.
- Mass-shell condition in critical case (open string):

$$-p^2 = m^2 = \frac{1}{\alpha'}(N-1) \quad \text{with} \quad \underset{\substack{\uparrow \\ \text{"level"}}}{N} \equiv \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$$

- The phys. states at level: $N = 0$ – tachyon
  $N = 1$ – massless vector
  $N \geq 2$ – "excited" string states

4.3 Closed string

- We discuss only the critical case. (As noted above, a better justification of this will follow soon.) Thus, $D = 26$, $a = -1$.

- We now have twice as many oscillators & constraints: $L_m \to L_m, \tilde{L}_m$.
- The following re-organization of the L_0-constraints is useful:

$$(L_0 + a)|phys\rangle = 0\,;\; (\tilde{L}_0 + a)|phys\rangle = 0 \longrightarrow (L_0 - \tilde{L}_0)|phys\rangle = 0\,;\; (L_0 + \tilde{L}_0 + 2a)|phys\rangle = 0$$

- We have $L_0 = N + \frac{\alpha_0^2}{2} = N + \frac{\alpha'}{4} p^2$ (recall that the relation between α_0 & p changes for closed strings)

In this language we get the constraints

(I) $(N - \tilde{N}) |phys\rangle = 0$ "level matching"

(II) $(N + \tilde{N} + 2a + \frac{\alpha'}{2} p^2) |phys\rangle = 0$ "mass shell condition"

As in the open case, we can now analyse each level:

0) vacuum: $N = \tilde{N} = 0 \Rightarrow m^2 = -p^2 = \frac{4a}{\alpha'} = -\frac{4}{\alpha'}$; tachyon

1) first excited level: $N = \tilde{N} = 1 \Rightarrow m^2 = -p^2 = \frac{2}{\alpha'}(2 + 2a) = 0$

- Now the L_1 & $\tilde{L}_1$ constraints are also non-trivial. We write the general state at level 1 as
$$\xi_{\mu\nu} \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0,p\rangle$$
and find
$$\xi_{\mu\nu} p^{\mu} = 0 \quad \& \quad \xi_{\mu\nu} p^{\nu} = 0.$$

- Formally, these are 2D constraints on ξ. But we see that, given 1st D constraints, a certain linear combination of the 2nd set of constraints is already satisfied: $p^{\mu}(\xi_{\mu\nu} p^{\nu}) = 0$.
Thus, only 2D-1 constraints are independent and we are left with
$$\underline{D^2 - (2D-1) \text{ phys. states at level 1.}}$$

- It is easy to show that $\langle phys|phys\rangle \sim \xi_{\mu\nu}\xi^{\mu\nu}$.

$\Rightarrow$ We can immediately write down two sets of (D-1) null states:

$$\xi^{(1)}_{\mu\nu} = \alpha_\mu p_\nu \; ; \; \xi^{(2)}_{\mu\nu} = p_\mu \beta_\nu \quad \text{with } \alpha\cdot p = 0 \;\&\; \beta\cdot p = 0 .$$

- Lin. space of null states has dim. $2(D-1)-1$ since the state with $\xi_{\mu\nu} \sim p_\mu p_\nu$ belongs to both sets.

- The null states correspond to gauge freedom, i.e.

$$\xi_{\mu\nu} \to \xi_{\mu\nu} + \xi^{(1)}_{\mu\nu} + \xi^{(2)}_{\mu\nu} \quad \text{is a gauge trf.}$$

- The expected Hilbert space dim. is then

$$D^2 - (2D-1) - (2D-3) = (D-2)^2 .$$

- To see that this subspace really is phys. with positive norm, choose $p = (1,1,0,\dots,0)$. Then the physical ("transverse") ξ take the form

$$\xi = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & \xi_t \end{pmatrix} \leftarrow (D-2)\times(D-2) \text{ matrix.}$$

- ξ_t transforms under the "little group" $SO(D-2)$.

  The subgroup of $SO(1,D-1)$ leaving p invariant.

- Decompose the rank-2 tensor repres. $(D-2)^2$ of ξ_t under $SO(D-2)$ in its irreducible components:

$$(D-2)^2 = \underbrace{\binom{D-2}{2}}_{\text{antisymm.}} + \underbrace{1}_{\text{trace}} + \underbrace{\left\{ \binom{D-2}{2} + (D-2) - 1 \right\}}_{\text{symm., traceless}}$$

antisymm. $\downarrow$ antisymm. tensor field $B_{\mu\nu}$ — scalar $=$ dilaton ϕ — graviton $G_{\mu\nu}$

(analogue of A_μ with field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.
Here: $H_{\mu\nu\rho} = 3\,\partial_{[\mu}B_{\nu\rho]}$ and $\mathcal{L} \supset \sim H_{\mu\nu\rho}H^{\mu\nu\rho}$

2) level two: $m^2 = \frac{2}{\alpha'}(N + \tilde{N} + 2a) = \frac{4}{\alpha'}(N + a) = \frac{4}{\alpha'}$

$\Rightarrow$ massive states ; we will not discuss them for the moment.

Summary (critical) closed string: $N = \tilde{N}$ (level matching)

$$m^2 = \frac{4}{\alpha'}(N-1)$$

$\Rightarrow$ level 0: tachyon ; level 1: $G_{\mu\nu}, B_{\mu\nu}, \phi$; level ≥ 2: massive excitations