5 Light-cone quantization – basic setup

5.1 Light-cone gauge

- The flat gauge we used so far leaves some "residual gauge freedom" unfixed (related to the null states we have encountered).
- We will now make this explicit and fix the gauge further.
- Recall general diffeomorphisms:

$$\xi^a \to \xi'^a = \xi'^a(\xi) \; ; \quad h_{ab} \to h'_{ab}(\xi') = h_{cd}(\xi) \frac{\partial \xi^c}{\partial \xi'^a} \frac{\partial \xi^d}{\partial \xi'^b}$$

- Infinitesimal version:

$$\xi'^a = \xi^a + \epsilon^a(\xi) \; ; \qquad h'_{ab}(\xi) = h_{ab}(\xi) - \epsilon^c \partial_c h_{ab} - \left(\partial_a \epsilon^c h_{cb} + \partial_b \epsilon^c h_{ac}\right)$$

$$= h_{ab} - \left(D_a \epsilon_b + D_b \epsilon_a\right) \qquad \text{(Check!)}$$

- We only need the flat-space version:

$$\eta_{ab} \to \eta_{ab} - \left(\partial_a \epsilon_b + \partial_b \epsilon_a\right)$$

- In addition, consider infinitesimal Weyl rescalings:

$$\eta_{ab} \to \eta_{ab}(1 + 2\omega)$$

$\Rightarrow$ "Weyl compensates Diff" if $\boxed{2\omega \eta_{ab} = \partial_a \epsilon_b + \partial_b \epsilon_a}$

This specific combination of Weyl & Diff is our residual gauge freedom.

- The above condition represents 3 indep. eqs., which in light-cone coordinates are:

(1) $2\omega \eta_{+-} = \partial_+ \epsilon_- + \partial_- \epsilon_+$

(2) $\partial_+ \epsilon_+ = 0 \; ; \quad \partial_- \epsilon_- = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\partial_+ \epsilon^- = 0 \; ; \quad \partial_- \epsilon^+ = 0$$

- Thus, (2) simply means that $\epsilon^- = \epsilon^-(\sigma^-)$, $\epsilon^+ = \epsilon^+(\sigma^+)$.
Moreover (1) can always be implemented by choosing an appropriate fct. ω.

- We see that the group of residual gauge-trfs. is $\left(\mathrm{Diff}(S^1)\right)^2$.
We may recall that $\mathrm{Diff}(S^1)$ is generated by the Witt-algebra (classically) and the Virasoro algebra (in the quantum theory).

- Not surprisingly, these algebras are precisely the algebras of our constraints. The latter correspond to the statement $T_{ab} = 0$ in flat gauge. Thus, our constraints simply said that phys. states must be invariant under residual gauge freedom.

- With this understanding, we clearly see a logical alternative to OCQ: Fix the residual gauge freedom classically. Then we will need fewer constraints when we quantize.

- The gauge condition is roughly $\tau \sim X^+ +$ const.
In other words, we measure time on the WS using the pullback the coordinate-fct. X^+. By definition, there are then no X^+-oscillators.

<u>Explicitly:</u> Define $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^1)$.

$$\underbrace{\partial_+\partial_- X^+(\sigma^+,\sigma^-) = 0}_{\uparrow\ \text{EOM}} \quad \Rightarrow \quad \partial_+\partial_-\Big(X^+ - \underbrace{\frac{2\pi\alpha'}{\ell}p^+\tau}\Big) = 0 \qquad \tau = \tfrac{1}{2}(\sigma^+ + \sigma^-)$$

(We do this gauge fixing on solutions. It can also be done off-shell, but this is not essential for us)

We deliberately choose this to to correspond to the centre-of-mass motion of our previous solutions.

$\Rightarrow \exists$ fcts. $f(\sigma^+)$ & $g(\sigma^-)$ such that

$$X^+ - \left(x^+ + \frac{2\pi\alpha'}{\ell} p^+ \tau\right) = f(\sigma^+) + g(\sigma^-)$$

Use residual gauge freedom to reparametrize σ^+ & σ^- s.t.

$$\frac{2\pi\alpha'}{\ell} p^+ \cdot \frac{\sigma^+}{2} - f(\sigma^+) = \frac{2\pi\alpha'}{\ell} p^+ \cdot \frac{\sigma^{+\prime}}{2}$$

$$\& \quad \frac{2\pi\alpha'}{\ell} p^+ \cdot \frac{\sigma^-}{2} - g(\sigma^-) = \frac{2\pi\alpha'}{\ell} p^+ \cdot \frac{\sigma^{-\prime}}{2} .$$

Dropping the prime, this gives: $\boxed{X^+ = \frac{2\pi\alpha'}{\ell} p^+ \tau}$

light-cone gauge

A straightforward calculation shows: (check this!)

$$T_{ab} = 0 \quad \Longleftrightarrow \quad (\dot{X} \pm X')^2 = 0$$

(here $\cdot / '$ denote τ/σ derivatives

and $(\ldots)^2 \equiv (\ldots)^\mu (\ldots)_\mu$.)

It follows that

$$-2\underbrace{(\dot{X} \pm X')^+}_{= \frac{2\pi\alpha'}{\ell} p^+ \text{ by our choice of light-cone gauge}}(\dot{X} \pm X')^- + (\dot{X} \pm X)^i (\dot{X} \pm X)^i = 0$$

$(i = 2, \cdots, D-1)$

$$\Rightarrow (\dot{X} \pm X')^- = (\dot{X} \pm X')^2_\perp \Big/ \left(\frac{4\pi\alpha'}{\ell}\right) p^+ \qquad (*)$$

We see that X^- has seized to be an independent field (up to a normalization constant). We can use our mode decomposition for the X^μ and find, by Fourier-decomposing $(*)$:

(cf. Zwiebach, Ch. 13)

$$\tilde{\alpha}_n^- = \frac{\sqrt{2/\alpha'}}{p^+} \tilde{L}_n^\perp \quad ; \quad \alpha_n^- = \frac{\sqrt{2/\alpha'}}{p^+} L_n^\perp \qquad (**)$$

$$\text{with } \overset{(\sim)}{L}{}_n^\perp \equiv \frac{1}{2} \sum_{m \in \mathbb{Z}} \overset{(\sim)}{\alpha}{}_m^i \overset{(\sim)}{\alpha}{}_{n-m}^i \quad , \quad \text{where } i \in \{2, \cdots, D-1\}$$

Crucially, $p^- = \sqrt{\frac{2}{\alpha'}}\, \alpha_0^- = \sqrt{\frac{2}{\alpha'}}\, \tilde{\alpha}_0^-$ is fixed in terms of the other variables.

5.2 Light-cone quantization

- It is straightforward to work out the action ($\to$ BLT):

$$S = \frac{1}{4\pi\alpha'} \underbrace{\int d\tau d\sigma \left((\dot{X}^i)^2 - (X'^i)^2 \right)}_{\text{as usual}} - \underbrace{\int d\tau\, p^+ \dot{x}^-}_{\text{This is all that's left of the } X^+X^-\text{-terms in l.c. gauge.}}$$

Here: $x^- \equiv \frac{1}{\ell} \int d\sigma\, X^-$ — this is the only independent dynamical part of X^-.

The canonical momenta are

$$p_- \equiv -p^+ = \frac{\partial L}{\partial \dot{x}^-} \quad , \quad \Pi^i = \frac{1}{2\pi\alpha'} \dot{X}^i = \frac{\partial L}{\partial \dot{X}^i}$$

(for $i = 2 \cdots D-1$ nothing changed)

The mode decomposition of the X^i is as before. The canonical commut. relations for p^+, x^- & $\Pi^i(\sigma), X^i(\sigma)$ then give:

$$[x^-, p^+] = -i \quad ; \quad [x^i, p^j] = i\delta^{ij} \quad ; \quad [\alpha_n^i, \alpha_m^j] = n\,\delta^{ij}\,\delta_{m+n}$$

(& same for $\tilde{\alpha}$.)

- The key new point upon quantization is (as also in OCQ) the normal ordering ambiguity affecting $\overset{(\sim)}{L}_0$ in (**).

- Formally following the derivation, we find an expression $\sim \sum_{n\neq 0} \alpha_{-n}\alpha_n$. It is convenient to split this in L_0 (normal ordered by definition) and normal ordering constant, as before:

$$p^+p^- = \underbrace{\frac{1}{2}(p_\perp)^2}_{\text{from } \alpha_0^i\alpha_0^i} + \frac{2}{\alpha'}\cdot\frac{1}{2}\sum_{n\neq 0} \overset{(\sim)}{\alpha}{}^\perp_{-n} \cdot \overset{(\sim)}{\alpha}{}^\perp_n$$

$$p^+p^- = \frac{1}{2}(p_\perp^2) + \frac{2}{\alpha'}\left(\overset{(\sim)}{N}_\perp + a\right)$$

$\overset{(\sim)}{N}_\perp = \sum_{n>0} \overset{(\sim)}{\alpha}{}^\perp_{-n} \cdot \overset{(\sim)}{\alpha}{}^\perp_n$; normal ordered by definition

Since, in our conventions, $m^2 = 2p^+p^- - (p_\perp)^2$, we have

$$m^2 = \frac{4}{\alpha'}\left(\overset{(\sim)}{N}_\perp + a\right).$$

We can now be very explicit in that

$$a = \frac{1}{2}\sum_{n\neq 0}\alpha^\perp_{-n}\cdot\alpha^\perp_n - \sum_{n>0}\alpha^\perp_{-n}\cdot\alpha^\perp_n = (D-2)\cdot\frac{1}{2}\sum_{n<0}\underbrace{(\alpha_n\alpha_{-n} - \alpha_{-n}\alpha_n)}_{=n}$$

"single α"

$$a = (D-2)\cdot\frac{1}{2}\left(\sum_{n=1}^{\infty} n\right)$$

from commutators of higher and higher modes.

- The divergence comes from the UV (of our 2d QFT) and should be physically understood, regularized, and removed. We will do all that shortly. For now, just a shortcut giving the correct result:

$$\sum_{n=1}^{\infty} 1 \to \lim_{s\to -1}\left(\sum_{n=1}^{\infty} n^{-s}\right) \equiv \lim_{s\to -1} \zeta(s) = \zeta(-1) = \underline{\underline{-\frac{1}{12}}}.$$

"Riemann zeta function"

$\Rightarrow a = -\frac{D-2}{24} \Rightarrow$ Need $D=26$ for the critical case (i.e. for massless photon / graviton), as argued more vaguely before.

(Comment: We could have made this argument for a at the level of OCQ, but the result would have been wrong since the prefactor would have been D, not $(D-2)$. This reduction is very similar the photon not having the naive D but only $D-2$ phys. d.o.f., cf. "Gupta-Bleuler-Quantization.)

Summary: $m^2 = \frac{4}{\alpha'}(N_\perp + a) = \frac{4}{\alpha'}(\tilde{N}_\perp + a)$; $a = -\frac{D-2}{26}$

followed from our expression for p^-, which is not independent

This is still a proper constraint, arising because we can express p^- independently through the left & right moving oscillators.

(We still have level-matching: $N_\perp = \tilde{N}_\perp$)

$\Rightarrow$ Everything very similar to OCQ, but no $\pm$-oscillators, just the physical transverse modes. Fock space construction will have no negative-norm issue. See soon...