6 Normal ordering constant and Casimir energy

6.1 Cutoff regularization

Recall that, in l.c.-gauge, we had

$$S = \int d\tau\, L = \frac{1}{4\pi\alpha'}\int d\tau\, d\sigma \left(\dot{X}_\perp^2 - X_\perp'^2\right) - \int d\tau\, p^+ \dot{x}^-$$

sum only over $\mu = i = 2 \cdots D-1$

$$\Downarrow$$

$$H_{lc} = -p^+\dot{x}^- + \int_0^l d\sigma\, \Pi_i \dot{X}^i - L = \underbrace{\frac{1}{4\pi\alpha'}\int_0^l d\sigma \left(\dot{X}_\perp^2 + X_\perp'^2\right)}$$

This is now like in a standard QFT.
No trace of the neg.-sign X^0-field!

- Our normal-ordering constant is, by definition, determined by the vacuum expectation value of this Hamiltonian:

$$E(l) = \langle 0,p|\, H_{lc}\, |0,p\rangle = \langle 0,p|\left(\frac{\pi}{l}\sum_{n\neq 0}\left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\right) + \frac{\pi\alpha'}{l}p_\perp^2\right)|0,p\rangle$$

- This is divergent and the divergence calls for adding a cosmological-constant counterterm to the action:

$$S \to S - \int d^2\xi \sqrt{-h}\,\lambda \quad ; \quad H_{lc} \to H_{lc} + l\lambda .$$

$$\Rightarrow E(l) = \frac{\pi\alpha'}{l}p_\perp^2 + \frac{2\pi}{l}\underbrace{(D-2)\sum_{n>0} n} + l\lambda$$

regularize by factor $\exp(-n/l\Lambda)$

cutoff.

allow to diverge appropriately, $\lambda = \lambda(\Lambda)$, to make E finite.

- By our previous definition of a, $a = \frac{D-2}{2}\sum_{n=1}^{\infty} n$, with with renormalization assumed, we have: $E(\ell) = \frac{\pi\alpha'}{\ell} p_\perp^2 + \frac{2\pi}{\ell}\cdot 2a$

- By comparing the last two expressions, we see that $4\pi a/\ell$ is a very physical quantity: It is the Casimir energy of a QFT on a cylinder. The explicit technical result is:

$$\underline{\frac{4\pi a}{\ell} = \frac{2\pi}{\ell}(D-2)\sum_{n>0} n e^{-n\ell/\Lambda} + \ell\lambda(\Lambda)}.$$

- Working it out: $(\alpha = 1/\ell\Lambda)$

$$\sum_{n>0} n e^{-n/\ell\Lambda} = \sum_{n>0} n e^{-\alpha n} = -\partial_\alpha \sum_{n>0} e^{-\alpha n} = -\partial_\alpha \frac{1}{1-e^{-\alpha}}$$

$$= \frac{e^{-\alpha}}{(1-e^{-\alpha})^2} = \frac{1-\alpha+\alpha^2/2}{(\alpha-\alpha^2/2+\alpha^3/6)^2} + O(\alpha) = \frac{1}{\alpha^2}\cdot\frac{1-\alpha+\alpha^2/2}{(1-\alpha/2+\alpha^2/6)^2} + O(\alpha)$$

$$= \frac{1}{\alpha^2}\cdot\frac{1-\alpha+\alpha^2/2}{1-\alpha+7\alpha^2/12} + O(\alpha) = \frac{1}{\alpha^2}\cdot\frac{1+\alpha^2/2}{1+7\alpha^2/12} + O(\alpha)$$

$$= \frac{1}{\alpha^2}\left(1-\frac{1}{12}\alpha^2\right) + O(\alpha) = (\Lambda\ell)^2 - \frac{1}{12} + O(1/\Lambda\ell)$$

$\uparrow$

Can be cancelled by $\lambda(\Lambda) \sim \Lambda^2$.

$\lambda(\Lambda)$ has no chance to influence the crucial $(-1/12)$-term because of the wrong ℓ-dependence.

This term is a UV-insensitive observable in 2d QFT, determining the Casimir energy.

$$a = \frac{1}{2}(D-2)\left(-\frac{1}{12}\right) = -\frac{D-2}{24} = \underline{\underline{-1}} \text{ for } D = 26.$$

6.2 Regularization independence

- We have argued on physical grounds (QFT on cylinder) that a is physical and hence regularization-independent

- More precisely, the Casimir energy is $\sim 1/\ell$ on dim.-grounds and a cosmol.-constant giving $\sim \ell \Lambda^2$ can not affect this.

- Any smooth cutoff should work: $e^{-n/\ell\Lambda} \to f(n/\ell\Lambda)$

  with $f \to 0$ as $\Lambda \to \infty$.

- But this is in general non-trivial – cf. the discussion of the 4d case in Itzykson/Zuber's book on QFT.

- Let us have a quick look at how dim.-reg. would work:

- The standard QFT expression for the vacuum energy is

$$E \sim V_{d-1} \int \frac{d^{d-1}\bar{p}}{(2\pi)^{d-1}} \cdot \frac{1}{2}\omega_{\bar{p}} \quad ; \quad \omega_{\bar{p}} = \sqrt{\bar{p}^2 + m^2}$$

- Let $m = 0$ and compactify on a circle with circumference $\ell = 2\pi R$. Then one component of $\bar{p}$ becomes discrete and we find:

$$E = \frac{1}{2} 2\pi R \, V_{d-2} \underbrace{\frac{1}{2\pi R} \sum_n}_{\text{"}dp\text{"}} \underbrace{\int \frac{d^{d-2}\bar{p}}{(2\pi)^{d-2}} \sqrt{\left(\frac{n}{R}\right)^2 + \bar{p}^2}}_{\sim \left(\frac{n}{R}\right)^{d-1}}$$

on dimensional grounds

- We see immediately that dim.reg. corresponds to replacing $\sum_n n \to \sum_n n^{d-1}$ and taking the limit $d \to 2$ for our case. Just like with $\zeta(s)$!