7 Light-cone quantization - Fock space & symmetries

7.1 Fock space

Recall: (closed) $[\alpha^i_m, \alpha^j_n] = m\,\delta^{ij}\delta_{m+n}$; $[x^i, p^j] = i\delta^{ij}$; $[x^-, p^+] = -i$

- no independent p^-, instead: $m^2 = \frac{4}{\alpha'}(N_\perp + a)$, $N_\perp = \tilde{N}_\perp$

  with $m^2 = 2p^+p^- - p_\perp^2$ & $a = -(D-2)/24$

- Only transverse oscillators $\Rightarrow$ Fock space = Hilbert space

Open: No $\tilde{\alpha}$; Due to different relation of p^μ & α_0^μ we have:

$$m^2 = \frac{1}{\alpha'}(N_\perp + a) \quad ; \quad \text{all else is the same.}$$

Start with open string:

Level 0: $|0,p\rangle$, $m^2 = a/\alpha' = -1/\alpha'$; tachyon

Level 1: $\alpha^i_{-1}|0,p\rangle$, $m^2 = (1+a)/\alpha' = 0$; vector

Note: $a=-1$ is required by Lorentz symm. For $a > -1$ we would have a massive vector with D-2 d.o.f. : ↯

Level 2: $\alpha^i_{-1}\alpha^j_{-1}|0,p\rangle$ & $\alpha^i_{-2}|0,p\rangle$, $m^2 = (2+a)/\alpha' = 1/\alpha'$

Before discussing this in detail, we make a small excursion:

7.2 Young Tableaux

- Consider SU(N) tensor representations: $t^{i_1\cdots i_n} \rightarrow U^{i_1}{}_{j_1}\cdots U^{i_n}{}_{j_n}\, t^{j_1\cdots j_n}$

- Their irreducible components can be found by systematic symmetrization / antisymmetrization of their indices:

$t^i \longrightarrow \boxed{i}$ (Indices are assigned to boxes.)

$t^{ij} \longrightarrow \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \;\hat{=}\; t^{ij} + t^{ji}$ (symm.)

$\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array} \;\hat{=}\; t^{ij} - t^{ji}$ (antisymm.)

$t^{ijk} \longrightarrow \begin{array}{|c|c|c|}\hline i & j & k \\ \hline\end{array} \;\hat{=}\; t^{ijk} + t^{jki} + t^{kij} + t^{jik} + t^{ikj} + t^{kji}$

$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array} \;\hat{=}\; t^{ijk} + t^{jki} + t^{kij} - t^{jik} - t^{ikj} - t^{kji}$

$$\begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array} \;\hat{=}\; \underbrace{\underbrace{t^{ijk} + t^{jik}}_{\text{symmetrize rows}} - t^{kji} - t^{jki}}_{\text{then antisymmetrize columns}}$$

- The tensors defined on the r.h. side correspond to irreducible sub-representations of our original (general) tensor repres.

- This scheme generalizes to higher tensors, e.g.

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1}\end{array} \text{ is a subrepres. of } t^{i_1 \cdots i_8} \;\hat{=}\; \underbrace{\square \otimes \square \otimes \cdots \otimes \square}_{8\text{-fold}}$$

- <u>Rows</u>: symmetrize
  <u>Columns</u>: antisymmetrize

- Key rule: Rows never become longer going from top to bottom.
  Columns never become longer going from left to right.

- Math. Background: Young tableaux actually classify representations of the "symmetric groups" S_n. Since S_n commutes with the $SU(N)$ action on tensors, this can be used to decompose $SU(N)$ tensor representations.

- This carries over to $SO(N)$ with two important caveats:
  1) For $SO(N)$, in addition to (anti-)symmetrization, one can use traces (implemented by δ_{ij}) to build projectors.
  2) One should in general also include spinors to build the original tensor product representation.

- We will use Young tableaux below in a naive, purely notational way. You should consult textbooks, e.g. Georgi or Hamermesh, for more details.

- In addition to boxes, we will use "•" for the trivial, 1-dimensional representation.

7.3 Explicit spectra

- We reproduce two key tables from BLT:

Open string

level	$m^2 \cdot \alpha'$	$SO(24)$-rep.	Little group	Little group rep.
0	-1	$\lvert 0\rangle$; • ; (1)	$SO(25)$	• ; (1)
1	0	$\alpha^i_{-1}\lvert 0\rangle$; $\square$; (24)	$SO(24)$	$\square$; (24)
2	1	$\alpha^i_{-2}\lvert 0\rangle$, $\alpha^i_{-1}\alpha^j_{-1}\lvert 0\rangle$ $\square$, $\square\square$, • ; $(24)+(299)+(1)$	$SO(25)$	$\square\square$; (324)
3	2	$\alpha^i_{-3}\lvert 0\rangle$, $\alpha^i_{-2}\alpha^j_{-1}\lvert 0\rangle$, $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\lvert 0\rangle$ $\square$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}+\square\square+$ • $\square\square\square+\square$ (24) $(276)+(299)+(1)$ $(2576)+(24)$	$SO(25)$	$\square\square\square+\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (2900) (300)

(BLT also give level 4)

- The remarkable point about this table is that "light-cone" $SO(24)$ representations can be combined into $SO(25)$ representations, as required by the full Lorentz-symm. and for massive states.

Closed string

level	$m^2 \cdot \alpha'$	$SO(24)$-rep.	Little group	Little group rep.
0	-4	$\|0\rangle$; $\bullet$; (1)	$SO(25)$	$\bullet$; (1)
1	0	$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\|0\rangle$; $\square\!\square + \boxminus + \bullet$ $(299) + (276) + (1)$	$SO(24)$	$\square\!\square + \boxminus + \bullet$ $(299) + (276) + (1)$
2	4	$\alpha^i_{-2}\alpha^j_{-2}\|0\rangle$, $\alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-1}\tilde{\alpha}^l_{-1}\|0\rangle$, $\alpha^i_{-2}\tilde{\alpha}^j_{-1}\tilde{\alpha}^k_{-1}\|0\rangle$, $\alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-2}\|0\rangle$ .... $\longrightarrow$	$SO(25)$	The miraculous matching with $SO(25)$-reps. continues.

7.4 Lorentz symmetry

- By choosing l.c. gauge, we have given up manifest target-space Lorentz-invariance. We have quantized and now need to check whether Lorentz-invariance is still a feature of the quantum theory. (For first positive signs see table above.)

- It is straightforward to use Noether's procedure to derive the conserved charge associated with $SO(1,D-1)$-symm. (cf. BLT or Zwiebach for rather explicit derivations):

$$\Rightarrow J^{\mu\nu} = T\int_0^\ell d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right)$$

$$= \underbrace{x^\mu p^\nu - x^\nu p^\mu}_{\ell^{\mu\nu}} \underbrace{- i\sum_1^\infty \frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right)}_{+ \; E^{\mu\nu}} + \underbrace{(\cdot\cdot\, \tilde{\alpha} \,\cdot\cdot)}_{+\; \tilde{E}^{\mu\nu}}$$

(from c.o.m. motion) (internal angular momentum from oscillators) Drop for open string

- We expect that the class. Lie algebra is realized by operator commutation relations:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho}J^{\nu\sigma} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + \begin{pmatrix}\mu \leftrightarrow \nu \\ \& \rho \leftrightarrow \sigma\end{pmatrix}.$$

- One can straightforwardly check that this is mostly the case.
- A key exception arises for J^{i-}:
- The general commutator above implies $[J^{i-}, J^{j-}] = 0$.
- A tedious (cf. GSW) calculation using the α, p, x commutators shows (for the open string):

$$[J^{i-}, J^{j-}] = -\frac{1}{\alpha'(p^+)^2}\sum_{m=1}^\infty \left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right)\Delta_m \qquad \text{with}$$

$$\Delta_m \equiv m\,\frac{26-D}{24} + \frac{1}{m}\left(\frac{D-2}{24} + a\right)$$

("a" appears because it is part of the expression for α_0^- in terms of α_0^+ and the transverse oscillators)

<u>Comment:</u> The result is also quoted in Zwiebach, but a calculation appears only in GSW.

|| <u>We see:</u> <u>The anomaly only vanishes for $D=26$ and $a=-1$</u> ||

<u>Comment:</u> The manifestly pos.def. l.c.-Hilbert space can be mapped to the OCQ-Hilbert space $\Rightarrow$ "No ghost theorem" (cf. GSW & P).