9 Modern covariant quantization — BRST

9.1 BRST symmetry (Becchi, Rouet, Stora, Tyutin)

- For notational simplicity, we will work in euclidean signature, s.t. $S = \int d^dx\,(T+V)$ and the exponential factor is $\exp(-S)$.

- Generically, we have $Z = \int D\phi \exp(-S_\phi[\phi])$

  S_ϕ: index to distinguish this from extra terms below

  ϕ: in our case $X^\mu(\xi)$ & $h_{ab}(\xi)$

- Write the gauge conditions as $F^A[\phi] = 0$. (In our case $h_{ab} - \delta_{ab} = 0$.)

  A: index labelling the different independent conditions.

- Our last section showed that Z may be expressed as

$$Z = \int D\phi\, DB_A\, Db_A\, Dc^\alpha \exp\Big(-S_\phi[\phi] - S_{gf}[B,\phi] - S_{FP}[b,c,\phi]\Big)$$

  with $S_{gf} = -iB_A F^A[\phi]$ (s.t. the B_A-integration implies $F^A = 0$)

  and $S_{FP} = b_A c^\alpha \underbrace{\delta_\alpha F^A[\phi]}_{\text{Variation of } F^A \text{ w.r.t. the gauge parameter } \alpha.}$ (s.t. the c/b-integrations calculate the FP-determinant)

- Thus, α labels the gauge group generators and $[\delta_\alpha, \delta_\beta] = f_{\alpha\beta}{}^{\gamma}\delta_\gamma$ is the algebra of infinitesimal gauge transformations.

(Note: In our case c^α is the Grassmann version of our ϵ^a, generating diffeomorphism. In principle, one would expect a 3rd generator corresponding to Weyl trfs. and "ω". But we have been able to explicitly perform the integration over ω, so our FP-determinant is only 2×2, not 3×3.)

Key claim: $S \equiv S_\phi + S_{gf} + S_{FP}$ is invariant under the BRST-transformation

$$\left.\begin{aligned} \delta_{BRST}\,\phi &= -i\epsilon c^\alpha \delta_\alpha \phi \\ \delta_{BRST}\,B_A &= 0 \\ \delta_{BRST}\,b_A &= \epsilon B_A \\ \delta_{BRST}\,c^\alpha &= \frac{i}{2}\epsilon c^\beta c^\gamma f_{\beta\gamma}{}^\delta \end{aligned}\right\}$$

This is a generalized version of a gauge trf., where c^α takes the place of the gauge parameter and ϵ is a Grassmann-type infinitesimal parameter.

(Check the invariance of S!)

- We can apply Noether's theorem and derive the conserved charge, called "BRST charge Q" or "BRST operator". As usual, $\delta_{BRST} = \epsilon Q$.

9.2 BRST-quantization: general structure

- If we quantize our theory with fields ϕ, B, b, c, we will get a much too large Fock space. We can restrict it by demanding gauge invariance of amplitudes. To do so, consider an infinitesimal change of the gauge condition: $F^A \to F'^A = F^A + \delta F^A$.
- This implies a change of $S_{gf} + S_{FP}$ by $\delta(S_{gf} + S_{FP})$.
- A physical amplitude should not change, i.e.

$$\delta\left(\int_{\text{initial}}^{\text{final}} D\phi \cdots Dc\, \exp\left(-S_\phi - S_{gf} - S_{FP}\right)\right) = 0.$$

$$\Rightarrow \int_{\text{initial}}^{\text{final}} D\phi \cdots Dc \, \exp(-S) \, \delta(S_{gf} + S_{FP}) = 0$$

$$\Updownarrow$$

$$\langle \text{final} | \, \delta(S_{gf} + S_{FP}) | \text{initial} \rangle = 0$$

- Now we observe: $\delta_{BRST}(b_A F^A) = i\epsilon(S_{gf} + S_{FP})$ (easy to check!)

- Hence, we need to demand that $\delta(\delta_{BRST}(b_A F^A))$ vanishes under the path integral or between physical states. Since Q is a fermionic operator, we have:

$$\delta(\delta_{BRST}(b_A F^A)) = \delta(\{Q, b_A F^A\}) = \underset{\substack{\uparrow \\ \text{vanishes by gauge} \\ \text{condition}}}{\{\delta Q, b_A F^A\}} + \underset{\substack{\uparrow \\ \text{need to} \\ \text{demand that} \\ \text{this is zero!}}}{\{Q, \delta(b_A F^A)\}}$$

- Thus, assuming also that Q is hermitian, all we need to demand is $\boxed{Q|\text{phys}\rangle = 0}$

- Key fact: $\boxed{Q^2 = 0}$ (Q is nilpotent)

  (To check: $\delta^{\epsilon'}_{BRST} \delta^{\epsilon}_{BRST}(\cdot\cdot) = 0$, e.g.

  $\delta^{\epsilon'}_{BRST} \delta^{\epsilon}_{BRST} b_A = \delta^{\epsilon'}_{BRST}(\epsilon B_A) = 0$, etc.)

$\Rightarrow$ States of the form $Q|\chi\rangle$ (with arbitrary $|\chi\rangle$) are always physical. They are also orthogonal to any phys. state:

$$\langle \text{phys} | Q | \chi \rangle = \langle \chi | Q | \text{phys} \rangle^* = 0 .$$

Hence, these states are "null" according to our earlier definition.

$\Rightarrow$ The true Hilbert space is $\boxed{\mathcal{H}_{BRST} \equiv \operatorname{Ker} Q / \operatorname{Im} Q}$

- In other words: $\boxed{\mathcal{H}_{BRST} \text{ is the cohomology of } Q}$

(This is mathematically a very natural definition.
Cf. e.g. "H^p is the cohomology of d acting on Ω^p.")

9.3 BRST quantization of the bosonic string

Recall from last section: $L_m = L_m^X + L_m^g + a\delta_m$

with $L_m^X = :\frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\alpha_n: +$

$L_m^g = :\sum_{n=-\infty}^{\infty}(m-n)\, b_{m+n}c_{-n}:$

(analogous eqs. for $\tilde{L}$ are suppressed)

[Specifically, $L_0^X = \frac{\alpha_0{}^2}{2} + \sum_1^\infty \alpha_{-n}\alpha_n$

$L_0^g = \sum_1^\infty n\left(b_{-n}c_n + c_{-n}b_n\right)$

$\Downarrow$

Rewriting original expression from T_{ab} in form of L_m^X contributes $-D/24$ to a. Ghosts contribute $\underset{\substack{\uparrow \\ \text{b \& c part}}}{+2/24}$. $\Rightarrow a = \frac{-26+2}{24} = -1$.]

- Now we want explicitly obtain Q in terms of the above operators.
- By Noether's theorem, we associate a current j_{BRST} to the symmetry δ_{BRST}. The latter is essentially "Diff", with ϵ^a replaced by c^a and declared to be part of the symm. trf. Thus, we get the same current as for "Diff", but with an extra factor c^a:

$$j^a_{BRST} \sim T^{ab}c_b .$$

- Explicitly, one finds (we report the open string case for brevity):

$$Q = \sum_{-\infty}^{\infty} :\left(L^X_{-m} + \frac{1}{2} L^g_{-m} + a\delta_m\right) c_m :$$

$$= \sum_{-\infty}^{\infty} \left(L^X_{-m} + a\delta_m\right) c_m + : \sum_{m,n=-\infty}^{\infty} \frac{m-n}{2} b_{m+n} c_{-n} c_{-m} :$$

- One can now straightforwardly check the nilpotency relation:

$$Q^2 = \frac{1}{2}\{Q,Q\} = \frac{1}{2} \sum_{m,n=-\infty}^{\infty} \left(\underbrace{[L_m, L_n] - (m-n) L_{m+n}}_{\sim \text{Virasoro anomaly}} \right) c_{-m} c_{-n}$$

(This is not surprising, since BRST is built on the presence of a proper group symm with a Lie algebra.)

$\Rightarrow$ BRST needs $Q^2 = 0$ and hence, again, $D = 26$ & $a = -1$ is enforced.

- We now proceed with the Fock space construction.
- The so-called anti-ghost B_A is eliminated by quantizing around the vacuum $h_{ab} = \delta_{ab}$, where our "F^A" vanishes.

- The novel part is in the fermionic b_m, c_m oscillators.
- We start with b_0, c_0 subalgebra: $c_0^2 = b_0^2 = 0$; $\{c_0, b_0\} = 1$.
Define $|\downarrow\rangle$ by $b_0|\downarrow\rangle = 0$. Define $|\uparrow\rangle \equiv c_0|\downarrow\rangle$.
Obviously, $c_0|\uparrow\rangle = 0$. Moreover, $b_0|\uparrow\rangle = b_0 c_0|\downarrow\rangle = (1 - c_0 b_0)|\downarrow\rangle = |\downarrow\rangle$. Thus, we have two states with, by convention, c_0 being the raising and b_0 the lowering operator.

- Completely analogous 2-state representations exist for each of the pairs c_1, b_{-1} ; b_1, c_{-1} ; c_2, b_{-2} ; .... . The Fock space is the tensor product.

- As opposed to the bosonic case, the algebra does not distinguish between creator & annihilator (and hence we can't tell who is "vacuum" and who "excited state"). This distinction is made by the ghost-part of the Hamiltonian: L_0^g.

- However, no term with c_0, b_0 is present in L_0^g, so it is up to us to define $|\uparrow\rangle$ or $|\downarrow\rangle$ as the vacuum.

- The correct choice is $|\downarrow\rangle$. In other words, physical states $|\chi\rangle$ are built by applying α_{-m}'s to $\underline{\underline{|\downarrow\rangle}}$.

Justification: • $\underset{\substack{\uparrow \\ \text{w/o. ghost excitations}}}{Q|\chi\rangle} \overset{!}{=} 0 \Rightarrow \underbrace{\left[(L_0^X + a)c_0 + \sum_{m>0} c_{-m} L_m^X\right]|\chi\rangle \overset{!}{=} 0}_{\text{This leads precisely to the familiar constraints.}}$

- By contrast, if we had started from $|\uparrow\rangle$ and built $|\chi\rangle$ by applying α_{-m}^{μ}'s to this "vacuum", then $c_0|\chi\rangle = 0$ would be automatic and no L_0^X-constraint would result.

Thus, in more detail we have:

> The Hilbert space is the Q-cohomology of the Fock space built on $|0,p,\downarrow\rangle$, $|0,p,\uparrow\rangle$. In addition, the part built on $|0,p,\uparrow\rangle$ is is exluded from the physical spectrum.

(Equivalently: We impose the extra constraint $b_0|\psi\rangle = 0$.)

Explicit construction

First, on easily checks that $\{Q, b_0\} = L_0$. Then $b_0|\psi\rangle = 0$ $\Rightarrow L_0|\psi\rangle = 0$ for all physical states. Now recall that $L_0 = \alpha' p^2 + N + a$, with $N = N_X + N_g$ the "total" level.

$$\Rightarrow m^2 = \frac{1}{\alpha'}(N+a)$$

, i.e. mass shell condition guaranteed at all levels.

⓪ Level zero: $|0,p,\downarrow\rangle$, set $a=-1$

$$0 \overset{!}{=} Q|0,p,\downarrow\rangle = \Big[\sum_m (L^X_{-m} - \delta_m)c_m + \sum_{m,n}\frac{m-n}{2} :b_{m+n}c_{-m}c_{-n}:\Big]|0,p,\downarrow\rangle$$

If we drop all terms containing at least one annihilator, all that's left of the square bracket is $(\alpha' p^2 - 1)c_0 \Rightarrow m^2 = -\frac{1}{\alpha'}$ (tachyon)

Note: Since Q does not change p^2 or the level, this state can not be Q-exact, i.e. expressible as $Q|\psi\rangle$. If it were, it would have to be $Q|0,p,\downarrow\rangle$, but this is zero. Thus, there are no null states at level zero.

① Level one: $|\psi\rangle = (\varepsilon\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0,p,\downarrow\rangle$

- Consider how the $(L^X_0 - 1)$-part & ghost part of Q act on these three terms:

A) $(L^X_0 - 1)c_0(\varepsilon\cdot\alpha_{-1})|0,p,\downarrow\rangle = c_0(\alpha'p^2+1-1)(\varepsilon\cdot\alpha_{-1})|0,p,\downarrow\rangle = 0 \quad (p^2=0)$

B) $(L^X_0 - 1)c_0(\beta b_{-1})|0,p,\downarrow\rangle = c_0(\alpha'p^2-1)\beta b_{-1}|0,p,\downarrow\rangle$

C) $(L^X_0 - 1)c_0(\gamma c_{-1})|0,p,\downarrow\rangle = c_0(\alpha'p^2-1)\gamma c_{-1}|0,p,\downarrow\rangle$

Cancellation

D) $\sum\frac{m-n}{2}:b_{m+n}c_{-m}c_{-n}:(\varepsilon\cdot\alpha_{-1})|0,p,\downarrow\rangle = 0$

E) – " – $(\beta b_{-1})|0,p,\downarrow\rangle = c_0\beta b_{-1}|0,p,\downarrow\rangle$

F) – " – $(\gamma c_{-1})|0,p,\downarrow\rangle = c_0\gamma c_{-1}|0,p,\downarrow\rangle$

$\Rightarrow$ We got zero. So we only need to enforce that $\sum_{m\neq 0} L^X_m c_{-m}$ annihilates our state. In fact, only the terms with $m = \pm 1$ contribute. We find:

$$0 \overset{!}{=} \left(p\cdot\alpha_1 c_{-1} + \alpha_{-1}\cdot p\, c_1 \right)\left(\varepsilon\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1} \right)|0, p, \downarrow\rangle$$

$$= \Big((\varepsilon\cdot p)\, c_{-1} + (\alpha_{-1}\cdot p)\,\beta \Big)|0, p, \downarrow\rangle \qquad (*)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

transversality of ε $\qquad$ $\beta = 0$

Thus, we learned that at level one:

- b-excitations are forbidden (since $\beta = 0$)
- c-excitations are gauge freedom
- X-excitations are (residual) gauge freedom

$\}$ We see this by applying Q to a level-one state with

[see eq. $(*)$] $\Leftarrow$ $\beta \neq 0$ & $\varepsilon\cdot p \neq 0$

<u>Higher levels:</u> One can show systematically that this pattern continues.

<u>Comment:</u> At the end of Sect. 4.2, Polchinski gives a general argument for the need to eliminate $|\uparrow\rangle$ states.