Inflationary correlation functions without IR divergences

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Outline

- IR divergences in δN formalism
- Including fluctuations of the Hubble scale
- Geometry of the reheating surface
- IR-safe 2-point correlator
- Tensor modes / Higher correlators / Explicit calculation

IR divergences in δN formalism

Starobinsky '85, Sasaki/Stewart '95
Wands/Malik/Lyth/Liddle '00
Lyth/Malik/Sasaki '04

- Consider some late, constant-energy-density surface (reheating surface):

$$ds^2 = e^{2\zeta} dx^i \, (e^{\gamma})_{ij} \, dx^j \, .$$

- Ignoring γ_{ij} for the moment, one has

$$\begin{aligned} \zeta(x) &= N(\varphi + \delta\varphi(x)) - N(\varphi) \\ &= N_\varphi \delta\varphi(x) + \frac{1}{2} N_{\varphi\varphi} \delta\varphi(x)^2 + \cdots \end{aligned}$$

Lyth/Rodriguez '05

- Consider the curvature correlator:

$$\langle \zeta_k \zeta_p \rangle = N_\varphi^2 \langle \delta\varphi_k \delta\varphi_p \rangle + \frac{1}{4} N_{\varphi\varphi}^2 \langle (\delta\varphi^2)_k (\delta\varphi^2)_p \rangle + \cdots$$

- Focus on the second term:

$$\sim N_{\varphi\varphi}^2 \int_{q,l} \langle \delta\varphi_q \delta\varphi_{k-q} \delta\varphi_l \delta\varphi_{p-l} \rangle \,.$$

- Use

$$\delta\varphi_q \sim \frac{H}{q^{3/2}} a_q$$

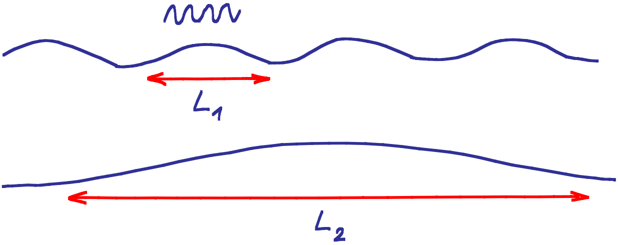
to find the leading-log contribution from $q, l \ll k, p$:

$$N_{\varphi\varphi}^2 H^4(k) \int \frac{d^3q}{q^3} \quad \sim \quad N_{\varphi\varphi}^2 H^4(k) \ln(kL) \,.$$

Intuitive physical picture:

- Long-wavelength modes affect measured short-wavelength fluctuations (e.g. L_1).
- Modes outside the 'box size' can be absorbed in constant ζ-background and are irrelevant (e.g. L_2).

Lyth '07



Fluctuations of the Hubble scale

- Even if only for conceptual reasons, we do care about very large L, relevant for the late observer.
- Obviously, the technical origin of the effect is the dependence of $N_\varphi(\varphi)$ on $\delta\varphi_q$ with $q \ll k$.
- Hence, the Hubble scale H should be modified analogously:

$$\delta\varphi(x) \sim \int_k \frac{e^{-ikx}}{k^{3/2}} \; H(\varphi(t_k) + \delta\bar{\varphi}(x)) \; a_k \,,$$

where

$$\delta\bar{\varphi}(x) \sim \int_{q \ll k} \frac{e^{-iqx}}{q^{3/2}} \; a_q \,.$$

- Using this modified $\delta\varphi$ in $\zeta = N(\varphi + \delta\varphi) - N(\varphi)$ and expanding in both $\delta\varphi$ and $\delta\bar{\varphi}$, one finds

$$\langle \zeta_k \zeta_p \rangle \sim \frac{\delta^3(k+p)}{k^3} \left[N_\varphi^2 H^2 + \frac{1}{2}(H^2 \ln kL) \frac{d^2}{d\varphi^2}(N_\varphi^2 H^2) \right] .$$

- With $H^2 \ln kL \sim \langle \delta\bar{\varphi}^2 \rangle_{1/k}$ this gives

$$\mathcal{P}_\zeta(k) \sim N_\varphi^2 H^2 + \frac{1}{2} \langle \delta\bar{\varphi}^2 \rangle_{1/k} \frac{d^2}{d\varphi^2}(N_\varphi^2 H^2) .$$

- We now replace the 'time variable' $\bar{\varphi}$ by $\ln k = -\bar{\zeta}$:

$$\frac{d}{d\varphi} = \left(\frac{d \ln k}{d\varphi} \right) \left(\frac{d}{d \ln k} \right) = N_\varphi \frac{d}{d \ln k} .$$

Geometry of the reheating surface

- We find

$$\mathcal{P}_\zeta(k) \;=\; \left(1 \;-\; \langle\bar{\zeta}\rangle \frac{d}{d\ln k} \;+\; \frac{1}{2}\langle\bar{\zeta}^2\rangle \frac{d^2}{d(\ln k)^2}\right)\, \mathcal{P}^0_\zeta(k)\,,$$

  where $\mathcal{P}^0_\zeta$ is the (almost scale-invariant) tree-level spectrum.

- This are obviously the first terms of the Taylor expansion of

$$\mathcal{P}_\zeta(k) \;=\; \langle\, \mathcal{P}^0_\zeta(ke^{-\bar{\zeta}})\,\rangle\,,$$

  where $\langle ..\rangle$ is the average in $\bar{\zeta}$ (defined in patches of size $1/k$) over a box of size L.

- Can we get this simple result more directly?

see also Giddings/Sloth '10

IR-safe correlation functions

- Define the almost scale-invariant spectrum as

$$\mathcal{P}_\zeta(k) \;\sim\; k^3 \int_y e^{iky} \,\langle\, \zeta(x)\zeta(x+y)\,\rangle\,.$$

- This is sensitive to the box-size L since the physical meaning of y depends on the strongly varying background $\bar{\zeta}$.
- However, we can avoid this by selecting pairs of points using the invariant distance $z = y\,e^{\bar{\zeta}}$. The z-dependence of the correlator.

$$\langle\, \zeta(x)\zeta(x + ze^{-\bar{\zeta}})\,\rangle$$

is then a background-independent and hence IR-safe object.

related to Urakawa/Tanaka '10 ?

- Its Fourier transform is our desired IR-safe power spectrum:

$$\mathcal{P}^0_\zeta(k) \sim k^3 \int_z e^{ikz} \langle \zeta(x)\zeta(x + ze^{-\bar{\zeta}}) \rangle .$$

- The original IR-sensitive power spectrum follows as

$$\begin{aligned}
\mathcal{P}_\zeta(k) &\sim k^3 \int_y e^{iky} \langle \zeta(x)\, \zeta(x+y) \rangle \\
&\sim k^3 \int_y e^{iky} \langle \zeta(x)\, \zeta(x + (ye^{\bar{\zeta}})e^{-\bar{\zeta}}) \rangle \\
&\sim \langle\, (ke^{-\bar{\zeta}})^3 \int_z \exp(ike^{-\bar{\zeta}}z)\; \zeta(x)\, \zeta(x + ze^{-\bar{\zeta}}) \rangle \\
&\sim \langle \mathcal{P}^0_\zeta(ke^{-\bar{\zeta}}) \rangle
\end{aligned}$$

in agreement with our previous result.

Tensor modes

- Our IR-safe power spectrum immediately generalizes to the case of background tensor modes:

$$\mathcal{P}^0_\zeta(k) \sim k^3 \int_z e^{ikz} \langle \zeta(x)\zeta(x + e^{-\bar{\zeta}}(e^{-\bar{\gamma}/2} z))\rangle \,.$$

- As before, the length of z is the invariant distance between the two points in the correlator.
- The calculation of the IR-sensitive spectrum produces an extra term since

$$\int d^3(e^{-\bar{\zeta}} e^{-\bar{\gamma}/2} z) = e^{-3\bar{\zeta}} \int d^3 z \,.$$

The factor k^3 is not automatically changed to $(e^{-\bar{\gamma}/2} k)^3$.

- We find

$$\mathcal{P}_\zeta(k) = \left\langle \, (e^{-\bar{\gamma}/2}\hat{k})^{-3} \, \mathcal{P}^0_\zeta(e^{-\bar{\zeta}-\bar{\gamma}/2}k) \, \right\rangle ,$$

  where $\hat{k}$ is a unit-vector in k-direction.

- Expanding in leading non-trivial order in the background (and assuming $\langle\bar{\zeta}\rangle = 0$ for simplicity) gives

$$\mathcal{P}_\zeta(k) = \left(1 - \frac{1}{20}\langle \mathrm{tr}\,\bar{\gamma}^2\rangle \frac{d}{d\ln k} + \frac{1}{2}\langle\bar{\zeta}^2\rangle \frac{d^2}{d(\ln k)^2}\right)\mathcal{P}^0_\zeta(k)$$

  (in agreement with Giddings/Sloth)

- The two terms are of the same order ($\mathrm{tr}\,\bar{\gamma}^2$ is more slow-roll suppressed, but comes with only one derivative in $\ln k$).

Higher correlation functions

- We could try to generalize the 'almost scale-invariant' spectrum by writing

$$\mathcal{P}_{(n)}(k_1 \ldots k_n) \sim k^{3n} \int_{y_1} \cdots \int_{y_n} e^{i(k_1 y_1 + \cdots + k_n y_n)} \langle \zeta(x)\zeta(x+y_1) \cdots \zeta(x+y_n) \rangle$$

- However, it is not clear which particular combination of $k_1 \ldots k_n$ one should use to define the prefactor k^{3n}.
- This is not irrelevant since factors $e^{\bar{\gamma}}$ will get tangled up in this prefactor.
- Hence, we choose to write the general formula for the higher-order analogue of the conventional spectrum $P(k) \sim \mathcal{P}(k)/k^3$.

- However, given these preliminaries, the generalization of our formalism is completely straightforward.
- The IR-safe spectrum is defined as

$$P^0_{(n)}(k_1 \ldots k_n) \sim \int_{z_1} \cdots \int_{z_n} e^{i(k_1 z_1 + \cdots + k_n z_n)} \langle \zeta(x)\zeta(x+y_1)\cdots\zeta(x+y_n)\rangle\,,$$

where

$$y_i = y_i(z, \bar{\zeta}, \bar{\gamma}) = e^{-\bar{\zeta}-\bar{\gamma}/2} z\,.$$

In words:

- Measure the correlation function in terms of invariant distances, characterized by a set of vectors z_i.
- Then Fourier transform (going from z_i to k_i).

- Then, by a straightforward generalization of the previous calculations, one finds

$$P_{(n)}(k_1\,, \ldots\,, k_n) = \left\langle\; e^{3n\bar{\zeta}}\; P^0_{(n)}(e^{-\bar{\zeta}-\bar{\gamma}/2}k_1\,, \ldots\,, e^{-\bar{\zeta}-\bar{\gamma}/2}k_n) \;\right\rangle .$$

- The prefactor $e^{3n\bar{\zeta}}$ comes from the naive scaling $P^0_{(n)} \sim k^{-3n}$.
- This can be directly applied to observables measuring non-Gaussianity, such as f_{NL}.

Example:
Tensor mode effect on f_{NL} in the squeezed limit

- Using 'consistency relations' (Maldacena '02), we find

$$\frac{12}{5} f_{NL}(k_1, k_2) = \frac{\left\langle\ (\hat{k}'_1)^{-3}\ \mathcal{P}^0_\zeta(k'_1)\ \ \frac{d}{d\ln(1/k'_2)}\left((\hat{k}'_2)^{-3}\ \mathcal{P}^0_\zeta(k'_2)\right)\ \right\rangle}{\left\langle\ (\hat{k}'_1)^{-3}\ \mathcal{P}^0_\zeta(k'_1)\ \right\rangle \left\langle\ (\hat{k}'_2)^{-3}\ \mathcal{P}^0_\zeta(k'_2)\ \right\rangle}$$

  where $k' = e^{-\bar{\gamma}/2} k$.

- At leading order in the background $\bar{\gamma}^2$ this gives

$$f_{NL}(k_1, k_2) = \left[1 - \frac{1}{20}\langle\bar{\gamma}^2\rangle \frac{d}{d\ln k}\right] f^0_{NL}(k_1, k_2)\,.$$

Explicit averaging over the background

- We want to calculate quantities of the type $\langle f(\bar{\zeta}(x))\rangle$.
- In principle, we have to average $\bar{\zeta}(x)$ over the (large) observed region of size L.
- However, this is equivalent to an ensemble average of $\bar{\zeta}(0)$ with IR cutoff L.
- Thus, we are dealing with a sum of Gaussian random variables

$$\bar{\zeta}(0) \sim \int_{1/L \ll q \ll k} \frac{(N_\varphi H)(q)}{q^{3/2}} a_q \,,$$

which is again a Gaussian random variable of width

$$\sigma^2 \equiv \langle \bar{\zeta}^2 \rangle \sim \int_{1/L \ll q \ll k} \frac{(N_\varphi H)^2(q)}{q^3} \,.$$

- Thus, all we need is the single integral

$$\frac{1}{\sigma\sqrt{2\pi}} \int d\bar{\zeta} e^{-\bar{\zeta}^2/2\sigma^2} f(\bar{\zeta}) \,.$$

- For example,

$$\mathcal{P}_\zeta(k) = \frac{1}{\sigma\sqrt{2\pi}} \int d\bar{\zeta} e^{-\bar{\zeta}^2/2\sigma^2} \mathcal{P}^0_\zeta(k e^{-\bar{\zeta}}) \,,$$

  where $\mathcal{P}^0_\zeta(k)$ is the (almost scale-invariant) tree-level spectrum $(N_\varphi H)^2$, written as a function of k.

- The generalization to tensor modes, though conceptually straightforward, is complicated by the matrix structure of $\bar{\gamma}$ and the different independent polarizations involved.

Important conceptual comment:

- In fact, the there exists a value k_{max} corresponding to modes that never left the horizon.
- For very large L, and for k sufficiently close to k_{max}, the region where $ke^{-\bar{\zeta}} > k_{max}$ is relevant in the $\bar{\zeta}$-integral.
- We need to assume that the very late observer is intelligent enough to exclude such regions from his averaging.
- Technically, this is implemented as

$$\int_{\bar{\zeta}_{min}=-\ln(k_{max}/k)} d\bar{\zeta}\;\; e^{-\bar{\zeta}^2/2\sigma^2}\mathcal{P}^0_\zeta(ke^{-\bar{\zeta}})$$

- While this is physically harmless, it clearly affects the convergence properties of the $\bar{\zeta}$-expansion

Summary

- An interesting class of IR divergences comes from long-wavelength background modes.
- This effect seen be seen from an (appropriately modified) δN formalism as well as from the 'geometry of the reheating surface'.
- One can define IR-safe correlators.
- One can return to usual correlators and calculate their IR-sensitive corrections (both scalar and tensor) very explicitly.
- The generalization to multiple scalar fields is interesting but (probably) conceptually straightforward.
- Are there observable effects (given our relatively small L)?
- Are there interesting implications for quantum gravity in de Sitter space?