Quantum Gravity and the Renormalization Group

Assignment 10 – Jan 17+20

Exercise 18: Heat kernel, part 1

Motivation: The purpose of this exercise (split into two parts) is to generalise the integral over the loop momentum to curved spacetimes. In other words, we will define the supertrace on a curved manifold. This is necessary to compute beta functions in gravity.

In this exercise, we will define the supertrace with the help of a simple example. Consider a "free" (in quotation marks, since nothing is free when gravity is present) scalar field ϕ in dimension d with the action

$$\Gamma_{\phi,k} = \int \mathrm{d}^d x \sqrt{g} \left[\frac{1}{2}(D_\mu \phi)(D^\mu \phi) \right] . \tag{18.1}$$

Our goal is to compute the contribution of the quantum fluctuations of this scalar field to the beta functions of Newton's coupling and the cosmological constant, while neglecting metric quantum fluctuations. In a Feynman picture, this would correspond to having all internal lines being scalar lines. By diffeomorphism invariance, we expect that whatever these quantum fluctuations generate, the RG flow should be parametrisable in the form

$$\frac{1}{2}\mathrm{STr}\left[\dots \right] = \int \mathrm{d}^d x \sqrt{g} \left[c_0 + c_1 R + c_2 R^2 + c_3 R_{\mu\nu} R^{\mu\nu} + \dots \right] . \tag{18.2}$$

Concretely, our task is to compute the numerical coefficients c_i. To ease the notation, from now on we strip away all delta functions and only work with the operators, and we will work in position space.

a) Follow similar steps as in Exercise 17 and formally compute the argument of the supertrace. You should get something like

$$\frac{1}{2}\mathrm{STr}\, W(\Delta) \tag{18.3}$$

for the right-hand side of the flow equation, for a suitable function W and where $\Delta = -D^2 = g^{\mu\nu} D_\mu D_\nu$.

We will now make a small detour – instead of computing the supertrace of a general function, let us first compute the supertrace of a very particular function:

$$\mathrm{STr}\, e^{-s\Delta} \equiv \mathrm{STr}\, H(s) . \tag{18.4}$$

The matrix elements of $H(s)$ can be defined in a position basis,

$$H(x, y; s) = \langle y | H(s) | x \rangle . \tag{18.5}$$

If we were able to compute these matrix elements, the supertrace would simply correspond to the sum over the eigenvalues. This information will come in handy later.

b) Argue that the matrix elements $H(x,y;s)$ fulfill the *heat equation*

$$\partial_s H(x,y;s) = -\Delta_x H(x,y;s)\,, \tag{18.6}$$

with initial condition

$$H(x,y;0) = \delta(x-y)\,. \tag{18.7}$$

Above, the subscript x on Δ_x indicates that the operator acts with respect to the position x, and we will always take this convention, omitting the subscript. What is the meaning of the variable s?

We can now finally define the supertrace:

$$\boxed{\mathrm{STr}\, e^{-s\Delta} = \mathrm{tr}\int \mathrm{d}^d x\,\sqrt{g}\langle x|e^{-s\Delta}|x\rangle = \mathrm{tr}\int \mathrm{d}^d x\,\sqrt{g}\,H(x,x;s)\,.} \tag{18.8}$$

Here, tr indicates the trace over discrete indices (e.g. spacetime indices, or the $O(N)$ indices from Exercise 17). The expression $H(x,x;s)$ is called the *coincidence limit* of the heat kernel (since both points coincide). This is often indicated with an overbar, $H(x,x;s) \equiv \overline{H}(s)$.

c) Time for some sanity checks. Show that for a flat spacetime where $\Delta = -\partial^2$, the heat kernel elements read

$$H^{\text{flat}}(x,y;s) = \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{(x-y)^2}{4s}}\,. \tag{18.9}$$

Next, make sure that our definition of the supertrace actually reproduces the standard loop momentum integral in flat spacetime. That is, show that

$$\overline{H}^{\text{flat}}(s) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{-s\,p^2} \tag{18.10}$$

by explicit computation. The right-hand side here should ring a bell (Exercise 17 again) and comes from (18.8) using that STr is essentially a loop momentum integral, just that the integrand is the exponential instead of the propagator times the derivative of the regulator.

It is time to get serious now. How does the heat kernel $H(x,y;s)$ look like in an arbitrary, curved spacetime? Let us take some inspiration from the flat heat kernel (18.9). The first thing we have to “upgrade” is the coordinate difference in the exponential, $(x-y)^2$. What would a suitable generalisation look like? If you guessed the geodesic distance, you are correct! For reasons, we define the *world function* $\sigma(x,y)$ to be *half* of the square of the geodesic distance. The world function has a funny property, namely

$$\frac{1}{2}(D_\mu \sigma(x,y))(D^\mu \sigma(x,y)) = \sigma(x,y)\,. \tag{18.11}$$

Recall that all derivatives act on the x-coordinate. Convince yourself that this relation makes sense in flat spacetime.

So far so good, but this might not be the only upgrade needed. Let us thus make the ansatz

$$H(x,y;s) = \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}}\,\Omega(x,y;s)\,, \tag{18.12}$$

where Ω parameterises the information on the curvature of the manifold. In particular, we have the boundary condition that $\Omega = 1$ whenever the manifold is flat.

d) What is the coincidence limit of the world function σ, that is, what is $\overline{\sigma} = \sigma(x, x)$?

e) What are the mass dimensions of s and Ω? Use this to argue with which powers of s curvatures and covariant derivatives in Ω have to come. From this, can you make the boundary condition $\Omega = 1$ when the spacetime is flat more precise?

We will now use the heat equation to compute Ω.

f) Insert the ansatz for the heat kernel into the heat equation to derive a differential equation for Ω.

g) Make a power series ansatz for Ω in powers of s, i.e. assume that

$$\Omega(x, y; s) \sim \sum_{n \geq 0} s^n A_n(x, y) \,, \qquad s \to 0 \,. \tag{18.13}$$

Using this ansatz, derive a recursion relation for the coefficients A_n (this should still be a differential equation). What does the boundary condition from e) translate to in terms of the A_n?

This completes the setup of the computation. In part 2 of this exercise, we will solve the recursion relation to some low order.