Quantum Gravity and the Renormalization Group

Assignment 11 – Jan 24+27

Exercise 19: Heat kernel, part 2

Motivation: This is part 2 of the heat kernel. Are you exhausted yet? Good, it's going to become much worse. :)

At the end of the last sheet, you should have obtained the following recursion relation for the coefficients A_n:

$$\left(n - \frac{d}{2} + \frac{1}{2}\left(D^2\sigma(x,y)\right)\right) A_n(x,y) + \left(D^\mu\sigma(x,y)\right)\left(D_\mu A_n(x,y)\right) - D^2 A_{n-1}(x,y) = 0\,, \tag{19.1}$$

with

$$A_0(x,x) \equiv \overline{A_0} = 1\,, \qquad A_{-1}(x,y) = 0\,, \qquad n \geq 0\,. \tag{19.2}$$

From now on, we will drop the position arguments. The goal of this sheet is to compute the coincidence limit of the first heat kernel coefficient, $\overline{A_1}$.

a) Set $n = 1$ in (19.1) and take the coincidence limit to understand which ingredients you need to compute $\overline{A_1}$. Beware: the coincidence limit does *not* commute with covariant derivatives, i.e. $\overline{D^2 A_0} \neq D^2 \overline{A_0}$!

b) One of the ingredients to compute $\overline{A_1}$ is $\overline{D^2 A_0}$. Derive an equation for the latter by acting with D^2 on (19.1) and taking the coincidence limit.

At this point you should get worried about the recursion, but maybe there is some hope after all. Instead of focussing on different derivatives of the A_n, let us switch our focus and try to compute coincidence limits of derivatives of the world function. Recall that

$$\frac{1}{2}(D^\mu\sigma(x,y))(D_\mu\sigma(x,y)) = \sigma(x,y)\,, \tag{19.3}$$

for *any* x, y, i.e. even away from the coincidence limit.

c) **[hard question]** Use (19.3) to compute $\overline{\sigma}$, $\overline{D_\mu\sigma}$, $\overline{D_\mu D_\nu\sigma}$, $\overline{D_\mu D_\nu D_\kappa\sigma}$ and $\overline{D_\mu D_\nu D_\kappa D_\lambda\sigma}$. *Hints:* take successive covariant derivatives of (19.3). Then take the coincidence limit of these equations and solve iteratively. You might have to commute covariant derivatives. Think about what you can pull out of the coincidence limit.

If you succeeded, you should feel relieved now if you look back at the equations for $\overline{A_1}$ and $\overline{D^2 A_0}$.

d) Use your results from c) to compute $\overline{D^2 A_0}$, and from there compute $\overline{A_1}$.

This illustrates the general procedure, and you can follow the same recipe to compute the A_n for larger n. It goes without saying that once again, this should not be done by hand.

Exercise 20: Heat kernel, part 3, or the inverse Laplace transform

Motivation: This is part 3 of the heat kernel – I lied that there would be only two parts. Remember where we started? Good, we have to actually come back and compute the original supertrace.

The starting point of the heat kernel exercises was that we originally wanted to compute

$$\mathrm{STr}\, W(\Delta)\,, \tag{20.1}$$

for some general function W. We spent a lot of time to compute the supertrace for an exponential, but in general we will not deal with only exponential functions, so we still need a recipe to connect the two.

For this, suppose we could write something like

$$W(\Delta) = \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-s\Delta}\,, \tag{20.2}$$

for some new function $\tilde{W}$. Wouldn't this be great? We could simply use this equation and use all previous results:

$$\mathrm{STr}\, W(\Delta) = \mathrm{STr} \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-s\Delta} = \int_0^\infty \mathrm{d}s\, \tilde{W}(s) \underbrace{\mathrm{STr}\, e^{-s\Delta}}_{\text{we did this!}}\,. \tag{20.3}$$

The only thing left to do would be to actually compute $\tilde{W}$ and perform the integrals over s, and we would be done. Also, we assumed that we can exchange the integral with the supertrace, but shhhhhhh.

Let us give some substance to this idea. The integral transform (20.2) is called the *inverse Laplace transform*. You can think of it like this: the original function W is the Laplace transform of some (a priori unknown) function $\tilde{W}$. Of course, there are some conditions on its existence, but let's simply assume for the moment that it exists. In the two previous exercises, we computed the supertrace of the exponential in an expansion in powers of s, so that

$$\mathrm{STr}\, W(\Delta) \sim \int_0^\infty \mathrm{d}s\, \tilde{W}(s) \left(\frac{1}{4\pi s}\right)^{d/2} \sum_{n\geq 0} s^n \int \mathrm{d}^d x\, \sqrt{g}\, \overline{A_n}\,. \tag{20.4}$$

We thus have to deal with integrals over $\tilde{W}$ multiplying either negative (small n) or positive (large n) powers of s. Do we now really have to compute $\tilde{W}$? Actually, no.

a) Negative powers: show that for $n > 0$,

$$\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1}\, W(z)\,. \tag{20.5}$$

This means that one can map these integrals over $\tilde{W}$ to integrals over the original function W! The integral over z has the interpretation of the integral over the loop momentum.

b) Non-negative powers: show that for $n \geq 0$,

$$\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^n = (-1)^n\, W^{(n)}(0)\,, \tag{20.6}$$

that is, these integrals can be mapped to derivatives of the original function at vanishing argument.

c) **[hard question]** Use your combined knowledge to compute

$$\mathrm{STr}\, W(\Delta) \tag{20.7}$$

up to linear order in curvature, in arbitrary dimension, for W taken from Exercise 18. To evaluate the integrals, use the Litim regulator. What exactly did you just compute?