Quantum Gravity and the Renormalization Group

Assignment 1 – Oct 18

Exercise 1: Counting gravitons

Motivation: In this exercise, we estimate the number of gravitons emitted in a binary merger. This gives us an idea whether the detection of single gravitons from such events is feasible.

Consider a binary merger of two solar mass ($m \sim 2 \times 10^{30}$kg) black holes. Estimate the number of gravitons N_h emitted via gravitational waves.
For this, assume that about $r = 5\%$ of the total mass is radiated away in mono-"chromatic" gravitational waves with a frequency of about 100 Hz. Given this number and keeping in mind the huge experimental effort in detecting gravitational waves, do you find it likely that we will detect single gravitons in the near future?

For massless particles like the graviton, we can estimate the energy of a single quantum by the relation

$$E_h = h\,\nu\,. \tag{1.1}$$

The total energy radiated away is

$$E_{\mathrm{GW}} = r \times (2m) \times c^2 \approx \frac{5}{100} \times 2 \times \left(2 \times 10^{30}\mathrm{kg}\right) \times \left(3 \times 10^{8}\mathrm{m/s}\right)^2 = 18 \times 10^{45}\mathrm{J} = N_h \times E_h\,. \tag{1.2}$$

We thus find

$$N_h = \frac{E_{\mathrm{GW}}}{h\,\nu} \approx \frac{18 \times 10^{45}}{(66 \times 10^{-35}) \cdot 100} \approx 3 \times 10^{77}\,. \tag{1.3}$$

It thus seems rather implausible that we will detect single gravitons in such a setup.

Exercise 2: Einstein's equations

Motivation: We review the derivation of Einstein's equations from an action. In this exercise, we will do so with the background field method. The goal is to get some (more) practice with computing perturbations, which will come in very handy very soon.

Einstein's equations are the equations of motion of General Relativity, and can be derived from an action principle. The action of General Relativity is the Einstein-Hilbert action, given by

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \sqrt{-\det g}\,(R - 2\Lambda)\,, \tag{2.I}$$

where G_N is Newtons constant, R is the Ricci scalar and Λ the cosmological constant. Derive Einstein's equations from the requirement of a stationary action,

$$\frac{\delta S[g]}{\delta g_{\mu\nu}(x)} = 0\,. \tag{2.II}$$

There are multiple ways to go about this. We will use one of them that will be helpful in going forward. Let us first expand the determinant of the metric. The starting point is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{2.1}$$

where the metric g is split into an arbitrary background metric $\bar{g}$ and a linear perturbation h. Taking the determinant and suppressing indices, we can write

$$\begin{aligned} \det g = \det(\bar{g} + h) &= \det\left[\bar{g}\left(\mathbb{1} + \bar{g}^{-1}h\right)\right] \\ &= \left[\det \bar{g}\right]\,\left[\det\left(\mathbb{1} + \bar{g}^{-1}h\right)\right] \\ &= \left[\det \bar{g}\right]\,\exp\operatorname{tr}\ln\left(\mathbb{1} + \bar{g}^{-1}h\right) \\ &\simeq \left[\det \bar{g}\right]\,\exp\operatorname{tr}\left(\bar{g}^{-1}h\right) \\ &\simeq \left[\det \bar{g}\right]\,\left[1 + \operatorname{tr}\left(\bar{g}^{-1}h\right)\right] \\ &= \left[\det \bar{g}\right]\,\left[1 + \bar{g}^{\mu\nu}h_{\mu\nu}\right] \,. \end{aligned} \tag{2.2}$$

Taking the square root and expanding again, we have

$$\sqrt{-\det g} \simeq \sqrt{-\det \bar{g}}\,\left[1 + \frac{1}{2}\bar{g}^{\mu\nu}h_{\mu\nu}\right] \equiv \sqrt{-\det \bar{g}}\,\left[1 + \frac{1}{2}h\right] \,. \tag{2.3}$$

Second, we need the Christoffel symbol to compute the Ricci scalar. By definition,

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right) \,. \tag{2.4}$$

It turns out that it is most convenient to first consider the Christoffel symbol with all indices lowered, as it is linear in the metric. We expand the Christoffel symbol by inserting the split of the metric and then converting partial derivatives to background covariant derivatives:

$$\begin{aligned} \Gamma_{\nu\alpha\beta} &= \frac{1}{2}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right) \\ &= \frac{1}{2}\left(\partial_\alpha \bar{g}_{\nu\beta} + \partial_\beta \bar{g}_{\nu\alpha} - \partial_\nu \bar{g}_{\alpha\beta}\right) + \frac{1}{2}\left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta}\right) \\ &= \bar{\Gamma}_{\nu\alpha\beta} + \frac{1}{2}\left(\bar{D}_\alpha h_{\nu\beta} + \bar{D}_\beta h_{\nu\alpha} - \bar{D}_\nu h_{\alpha\beta}\right) + \bar{\Gamma}^{\mu}{}_{\alpha\beta}h_{\mu\nu} \\ &= \bar{\Gamma}^{\mu}{}_{\alpha\beta}\left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) + \frac{1}{2}\left(\bar{D}_\alpha h_{\nu\beta} + \bar{D}_\beta h_{\nu\alpha} - \bar{D}_\nu h_{\alpha\beta}\right) \\ &= \bar{\Gamma}^{\mu}{}_{\alpha\beta}g_{\mu\nu} + \frac{1}{2}\left(\bar{D}_\alpha h_{\nu\beta} + \bar{D}_\beta h_{\nu\alpha} - \bar{D}_\nu h_{\alpha\beta}\right) \,. \end{aligned} \tag{2.5}$$

Here, we have used the general relation between partial and covariant derivatives,

$$\begin{aligned} D_\alpha X^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \partial_\alpha X^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} &+ \sum_{i=1}^{m}\Gamma^{\mu_i}{}_{\alpha\beta}X^{\mu_1\ldots\mu_{i-1}\beta\mu_{i+1}\ldots\mu_m}{}_{\nu_1\ldots\nu_n} \\ &- \sum_{i=1}^{n}\Gamma^{\beta}{}_{\alpha\nu_i}X^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_{i-1}\beta\nu_{i+1}\ldots\nu_n} \,, \end{aligned} \tag{2.6}$$

to transform the partial derivatives that act on h to background covariant derivatives plus the corresponding background Christoffel symbols. From this, we find

$$\Gamma^{\mu}{}_{\alpha\beta} = \bar{\Gamma}^{\mu}{}_{\alpha\beta} + \frac{1}{2}g^{\mu\nu}\left(\bar{D}_\alpha h_{\nu\beta} + \bar{D}_\beta h_{\nu\alpha} - \bar{D}_\nu h_{\alpha\beta}\right) \,. \tag{2.7}$$

To linear order,

$$\Gamma^{\mu}{}_{\alpha\beta} \simeq \bar{\Gamma}^{\mu}{}_{\alpha\beta} + \frac{1}{2}\bar{g}^{\mu\nu}\left(\bar{D}_\alpha h_{\nu\beta} + \bar{D}_\beta h_{\nu\alpha} - \bar{D}_\nu h_{\alpha\beta}\right) . \tag{2.8}$$

For later use, we also evaluate

$$\Gamma^{\mu}{}_{\mu\beta} \simeq \bar{\Gamma}^{\mu}{}_{\mu\beta} + \frac{1}{2}\bar{D}_\beta h \, . \tag{2.9}$$

Clearly, we also need the inverse metric. To linear order, it is easy to verify that

$$g^{\mu\nu} \simeq \bar{g}^{\mu\nu} - h^{\mu\nu} \, , \tag{2.10}$$

where the indices of h are raised with the background metric: $h^{\mu\nu} = \bar{g}^{\mu\alpha} h_{\alpha\beta} \bar{g}^{\beta\nu}$.

We can now construct the linearised Ricci scalar step by step. Recall the Riemann tensor

$$R_{\mu\nu\rho}{}^{\sigma} = \Gamma^{\sigma}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\mu\rho} - \Gamma^{\sigma}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\rho} + \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} - \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} \, , \tag{2.11}$$

and the Ricci tensor

$$R_{\mu\nu} = \Gamma^{\sigma}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\alpha\nu} + \partial_\alpha \Gamma^{\alpha}{}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}{}_{\alpha\nu} \, . \tag{2.12}$$

To compute the linear perturbation of the Ricci tensor, we could simply insert the expansion of the Christoffel symbol and perform the computation by brute force. Let us try to be a bit more economical. Without perturbations, the Ricci tensor is just the background Ricci tensor, so let us subtract the latter from the former and use their respective definitions:

$$R_{\mu\nu} - \bar{R}_{\mu\nu} = \left(\Gamma^{\sigma}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\sigma\alpha}\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \left(\Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\alpha\nu}\right) + \partial_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \partial_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) . \tag{2.13}$$

We will now use (2.6) to transform the partial derivatives above to background covariant derivatives. For this, recall that the difference of connections transforms as a tensor, so performing this transformation makes sense geometrically. Concretely,

$$\partial_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) = \bar{D}_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) + \bar{\Gamma}^{\beta}{}_{\mu\nu}\left(\Gamma^{\alpha}{}_{\alpha\beta} - \bar{\Gamma}^{\alpha}{}_{\alpha\beta}\right) , \tag{2.14}$$

and

$$\partial_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) = \bar{D}_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \bar{\Gamma}^{\alpha}{}_{\alpha\beta}\left(\Gamma^{\beta}{}_{\mu\nu} - \bar{\Gamma}^{\beta}{}_{\mu\nu}\right) + \bar{\Gamma}^{\beta}{}_{\alpha\mu}\left(\Gamma^{\alpha}{}_{\beta\nu} - \bar{\Gamma}^{\alpha}{}_{\beta\nu}\right) + \bar{\Gamma}^{\beta}{}_{\alpha\nu}\left(\Gamma^{\alpha}{}_{\mu\beta} - \bar{\Gamma}^{\alpha}{}_{\mu\beta}\right) . \tag{2.15}$$

Using this, we can group terms to only get differences of full and background Christoffel symbols:

$$\begin{aligned} R_{\mu\nu} - \bar{R}_{\mu\nu} = {} & \left(\Gamma^{\sigma}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\sigma\alpha}\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \left(\Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\alpha\nu}\right) \\ & + \bar{D}_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \bar{\Gamma}^{\alpha}{}_{\alpha\beta}\left(\Gamma^{\beta}{}_{\mu\nu} - \bar{\Gamma}^{\beta}{}_{\mu\nu}\right) + \bar{\Gamma}^{\beta}{}_{\alpha\mu}\left(\Gamma^{\alpha}{}_{\beta\nu} - \bar{\Gamma}^{\alpha}{}_{\beta\nu}\right) + \bar{\Gamma}^{\beta}{}_{\alpha\nu}\left(\Gamma^{\alpha}{}_{\mu\beta} - \bar{\Gamma}^{\alpha}{}_{\mu\beta}\right) \\ & - \bar{D}_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) - \bar{\Gamma}^{\beta}{}_{\mu\nu}\left(\Gamma^{\alpha}{}_{\alpha\beta} - \bar{\Gamma}^{\alpha}{}_{\alpha\beta}\right) \\ = {} & \left[\Gamma^{\sigma}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\sigma\alpha}\bar{\Gamma}^{\alpha}{}_{\mu\nu} - \left(\Gamma^{\sigma}{}_{\sigma\alpha} - \bar{\Gamma}^{\sigma}{}_{\sigma\alpha}\right)\bar{\Gamma}^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\sigma\alpha}\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right)\right] \\ & - \left[\Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\alpha\nu} - \left(\Gamma^{\alpha}{}_{\mu\sigma} - \bar{\Gamma}^{\alpha}{}_{\mu\sigma}\right)\bar{\Gamma}^{\sigma}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\sigma}\left(\Gamma^{\sigma}{}_{\alpha\nu} - \bar{\Gamma}^{\sigma}{}_{\alpha\nu}\right)\right] \\ & + \bar{D}_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \bar{D}_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) \\ = {} & \left(\Gamma^{\sigma}{}_{\sigma\alpha} - \bar{\Gamma}^{\sigma}{}_{\sigma\alpha}\right)\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \left(\Gamma^{\alpha}{}_{\mu\sigma} - \bar{\Gamma}^{\alpha}{}_{\mu\sigma}\right)\left(\Gamma^{\sigma}{}_{\alpha\nu} - \bar{\Gamma}^{\sigma}{}_{\alpha\nu}\right) \\ & + \bar{D}_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \bar{D}_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) \\ \simeq {} & +\bar{D}_\alpha\left(\Gamma^{\alpha}{}_{\mu\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu}\right) - \bar{D}_\mu\left(\Gamma^{\alpha}{}_{\alpha\nu} - \bar{\Gamma}^{\alpha}{}_{\alpha\nu}\right) . \end{aligned} \tag{2.16}$$

In the last step, we used that each of the brackets is of order h, so products of the brackets are already of quadratic order. We thus find the linearised Ricci tensor

$$\begin{aligned} R_{\mu\nu} &\simeq \bar{R}_{\mu\nu} + \bar{D}_\alpha \left(\Gamma^\alpha{}_{\mu\nu} - \bar{\Gamma}^\alpha{}_{\mu\nu}\right) - \bar{D}_\mu \left(\Gamma^\alpha{}_{\alpha\nu} - \bar{\Gamma}^\alpha{}_{\alpha\nu}\right) \\ &\simeq \bar{R}_{\mu\nu} + \frac{1}{2}\bar{D}^\beta \left(\bar{D}_\mu h_{\beta\nu} + \bar{D}_\nu h_{\beta\mu} - \bar{D}_\beta h_{\mu\nu}\right) - \frac{1}{2}\bar{D}_\mu \bar{D}_\nu h \,. \end{aligned} \tag{2.17}$$

For the Ricci scalar, we then have

$$\begin{aligned} R = g^{\mu\nu} R_{\mu\nu} &\simeq \bar{g}^{\mu\nu} \left[\frac{1}{2}\bar{D}^\beta \left(\bar{D}_\mu h_{\beta\nu} + \bar{D}_\nu h_{\beta\mu} - \bar{D}_\beta h_{\mu\nu}\right) - \frac{1}{2}\bar{D}_\mu \bar{D}_\nu h\right] - h^{\mu\nu} \bar{R}_{\mu\nu} \\ &= \bar{D}^\mu \bar{D}^\nu h_{\mu\nu} - \bar{D}^2 h - \bar{R}^{\mu\nu} h_{\mu\nu} \,. \end{aligned} \tag{2.18}$$

Putting everything together, we find

$$S_{\mathrm{EH}}[\bar{g} + h] \simeq \frac{1}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{-\det \bar{g}} \left[\frac{1}{2}\bar{g}^{\mu\nu} h_{\mu\nu} (\bar{R} - 2\Lambda) + \bar{D}^\mu \bar{D}^\nu h_{\mu\nu} - \bar{D}^2 h - \bar{R}^{\mu\nu} h_{\mu\nu}\right] . \tag{2.19}$$

Throwing out total derivatives, we can read off Einstein's equations:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} (R - 2\Lambda) = 0 \,. \tag{2.20}$$

Extra material 1: (Covariant) Partial integration

Let us briefly discuss why we can throw out total **covariant** derivatives. The reason is essentially that we have to take into account the determinant of the metric. By the divergence theorem, the integral over a divergence of a vector field v is a surface term (which we neglect):

$$\int \mathrm{d}^4x\, \partial_\alpha v^\alpha \simeq 0\,.$$

How do typical vector fields look like? Certainly, they will involve $\sqrt{-\det g}$. Since we are interested in a partial integration formula, let us furthermore consider the product of two tensors:

$$v^\alpha = \sqrt{-\det g}\, X^{\mu_1 \dots \mu_M}{}_{\nu_1 \dots \nu_N}\, Y^{\alpha\nu_1 \dots \nu_N}{}_{\mu_1 \dots \mu_M}\,.$$

Note that Y has one index more than X, and all other indices are contracted. Inserting this into the above divergence formula and using the product rule, we get

$$0 \simeq \int \mathrm{d}^4x \sqrt{-\det g}\, \left[\frac{\partial_\alpha \sqrt{-\det g}}{\sqrt{-\det g}} X^{\mu_1 \dots \mu_M}{}_{\nu_1 \dots \nu_N}\, Y^{\alpha\nu_1 \dots \nu_N}{}_{\mu_1 \dots \mu_M} + \partial_\alpha \left(X^{\mu_1 \dots \mu_M}{}_{\nu_1 \dots \nu_N}\, Y^{\alpha\nu_1 \dots \nu_N}{}_{\mu_1 \dots \mu_M}\right)\right]\,.$$

The first term gives a Christoffel symbol (try to prove this!):

$$\frac{\partial_\alpha \sqrt{-\det g}}{\sqrt{-\det g}} = \Gamma^\beta{}_{\alpha\beta}\,.$$

Note that the other term, where the partial derivative acts on the product of X and Y, is a vector. If we now assume that neither of the two factors contain the determinant of the metric, we can convert the partial derivative to a covariant derivative via (2.6). The single Christoffel symbol then exactly cancels the one originating from the derivative acting on the determinant, so that

$$0 \simeq \int \mathrm{d}^4x \sqrt{-\det g}\, D_\alpha \left(X^{\mu_1 \dots \mu_M}{}_{\nu_1 \dots \nu_N}\, Y^{\alpha\nu_1 \dots \nu_N}{}_{\mu_1 \dots \mu_M}\right)\,, \tag{2.21}$$

or equivalently,

$$\begin{aligned} &\int \mathrm{d}^4x \sqrt{-\det g}\, \left(D_\alpha X^{\mu_1 \dots \mu_M}{}_{\nu_1 \dots \nu_N}\right) Y^{\alpha\nu_1 \dots \nu_N}{}_{\mu_1 \dots \mu_M} \\ &\simeq -\int \mathrm{d}^4x \sqrt{-\det g}\, X^{\mu_1 \dots \mu_M}{}_{\nu_1 \dots \nu_N}\, \left(D_\alpha Y^{\alpha\nu_1 \dots \nu_N}{}_{\mu_1 \dots \mu_M}\right)\,. \end{aligned} \tag{2.22}$$

We can thus perform partial integrations with covariant derivatives – they simply take care of the extra term that appears when we would do partial integrations with partial derivatives, when the partial derivatives acts on the determinant of the metric.

Convince yourself that for the above argument, it was crucial that the whole expression had no open indices, that is, it was a scalar. For example, show explicitly that

$$\int \mathrm{d}^4x\, \sqrt{-\det g}\, D_\mu R \tag{2.23}$$

is generally not a total derivative.

Exercise 3: Linearised Einstein's equations

Motivation: This short exercises builds upon exercise 2 to derive the wave equation for linear metric perturbations about Minkowski space — a.k.a. gravitational waves.

Linearise Einstein's field equations about Minkowski space, that is, consider

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0\,, \tag{3.I}$$

with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.II}$$

to linear order in h. Why do we have to set Λ to zero for this?

We can recycle some formulas from exercise 2. First note that since $\bar{\Gamma}^{\mu}{}_{\alpha\beta} = 0$ for the Minkowski metric, the only non-vanishing terms in the expression for the Ricci tensor are the those linear in the connection. This means

$$\begin{aligned} R_{\mu\nu} &\simeq \partial_\alpha \left[\frac{1}{2}\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right)\right] - \partial_\mu \left[\frac{1}{2}\partial_\nu h\right] \\ &= \frac{1}{2}\left[\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h\right]\,. \end{aligned} \tag{3.1}$$

The only contribution from the Ricci scalar comes from the contraction of the above with the Minkowski metric. This entails

$$R \simeq \partial^\mu \partial^\nu h_{\mu\nu} - \partial^2 h\,. \tag{3.2}$$

Combining the two expressions, we find

$$\begin{aligned} 0 &= R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \\ &\simeq \frac{1}{2}\left[\partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu}\left(\partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^2 h\right)\right]\,. \end{aligned} \tag{3.3}$$

As in the lecture, we now introduce the trace-subtracted perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\,. \tag{3.4}$$

Inserting this and collecting terms gives an equation purely in terms of $\bar{h}_{\mu\nu}$:

$$0 = -\partial^2 \bar{h}_{\mu\nu} + \left[\frac{1}{2}\delta_\mu{}^\beta \partial_\nu + \frac{1}{2}\delta_\nu{}^\beta \partial_\mu - \eta_{\mu\nu}\partial^\beta\right]\partial^\alpha \bar{h}_{\alpha\beta}\,. \tag{3.5}$$

If we now impose the gauge condition

$$\partial^\alpha \bar{h}_{\alpha\beta} = 0\,, \tag{3.6}$$

we arrive at a wave equation for $\bar{h}_{\mu\nu}$:

$$\Box \bar{h}_{\mu\nu} = 0\,. \tag{3.7}$$

We have to set the cosmological constant to zero because otherwise, the zero-th order of Einstein's equations is not fulfilled.