Quantum Gravity and the Renormalization Group

Assignment 3 – Nov 08+11

Exercise 6: Superficial degree of divergence

Motivation: In this exercise, we discuss the superficial degree of divergence of a scalar theory that mimics the divergence count of General Relativity. The goal is to reinforce your understanding of why the perturbative quantisation fails.

As we have discussed in the lecture, General Relativity generates new divergences at every loop order. This concretely means that the loop divergences that are created come in a form that *cannot* be absorbed by a renormalisation of the Einstein-Hilbert terms in the action. To understand this in more depth, we will first consider a scalar toy model, and then we'll try to translate what we understand to the case of gravity.

a) Consider a ϕ^4 theory in four-dimensional Minkowski space with a microscopic action

$$S^{\phi^4} = \int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{\lambda_4}{4!}\phi^4\right] . \tag{6.1}$$

At a structural level, i.e. *without explicitly calculating/evaluating any diagram*, argue what the superficial degree of divergence is of the loop diagrams contributing to the renormalisation of the four-scalar-vertex. Argue that at every loop order, you can absorb the divergences in a renormalisation of the coupling λ (and potentially a wave-function renormalisation of the field and a mass term).

b) Now consider ϕ^6 theory, also in four-dimensional Minkowski space, with the microscopic action

$$S^{\phi^6} = \int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{\lambda_6}{6!}\phi^6\right] . \tag{6.2}$$

Once again at a structural level, discuss the superficial degree of divergence for the four- and six-scalar vertex *at one-loop order*. Can this theory be renormalised perturbatively?

c) **[hard question]** Next, consider a scalar field theory with a derivative interaction:

$$S^{\partial\phi} = \int \mathrm{d}^4x \left[\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{\lambda_\partial}{(2!)^2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)\phi^2\right] . \tag{6.3}$$

What is the one-loop divergence structure in this case? Can you absorb all divergences of the four-scalar vertex?

d) **[hard question]** Finally, let us discuss the degree of divergence in General Relativity. In a similar way as for the scalar field theories above, argue what the divergence structure is at one and two loops. For this, consider the divergences of the two-graviton vertex. Can the divergences be absorbed, or do we have to introduce structurally new counterterms? If the latter, how do they look like?

a) Let us first argue what the number of vertices, propagators and loop integrals is for an N-loop Feynman diagram that contributes to the four-point vertex. By construction, an N-loop diagram will have N loop integrals, and thus comes with an integration of the form

$$\int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \ldots \mathrm{d}^4\ell_N \,, \tag{6.4}$$

where ℓ_k is the k-th loop momentum, and we indicated that we introduced an ultraviolet cutoff Λ_{UV} that indicates the divergence when we send it to infinity. Next, an N-loop diagram in our theory will have $N+1$ vertices (this can be shown by induction, convince yourself of this!). This means that we have a total number of $4(N+1)$ legs, four of which are the external legs. This means that there are $4N$ internal legs that have to be connected via propagators. Since each propagator links two legs, we get $2N$ propagators.

What is the superficial degree of divergence of any of these diagrams? For this, we have to understand how loop momenta are distributed over propagators. Each propagator carries a linear combination of at least one loop momentum (potentially more), and optionally external momenta. Note now that we can always assign loop momenta such that they only appear around their own loop, that is, they do not "leak" into other loops. For this, proceed iteratively: pick one propagator and assign it the first loop momentum. Go around the loop and impose momentum conservation at each vertex. This way, the loop momentum cannot appear outside of the loop.

The next step is to realise that for "interesting" diagrams (that is, those that are not bubble diagrams, which are set to zero in dimreg/play no important role in renormalisation), each loop momentum appears in at least two propagators.

Finally, to investigate the superficial degree of divergence, we perform an inductive argument and integrate over one loop momentum at a time. Let us first discuss the generic one-loop diagram. This is a self-energy-type diagram, and comes with two propagators. The vertices are momentum-independent, and thus the divergence structure reads

$$\int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \, \frac{1}{\ell_1^2} \frac{1}{(\ell_1+P)^2} \,, \tag{6.5}$$

where P is a combination of external momenta. We can now use the Feynman parameterisation to write this integral as

$$\begin{aligned}
\int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \, \frac{1}{\ell_1^2} \frac{1}{(\ell_1+P)^2} &= \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \int_0^1 \mathrm{d}\alpha_1 \frac{1}{\left[\alpha_1\ell_1^2 + (1-\alpha_1)(\ell_1+P)^2\right]^2} \\
&= \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \int_0^1 \mathrm{d}\alpha_1 \frac{1}{\left[\ell_1^2 + 2(1-\alpha_1)\ell_1\cdot P + (1-\alpha_1)P^2\right]^2} \\
&= \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \int_0^1 \mathrm{d}\alpha_1 \frac{1}{\left[(\ell_1+(1-\alpha_1)P)^2 - (1-\alpha_1)^2P^2 + (1-\alpha_1)P^2\right]^2} \\
&= \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\tilde{\ell}_1 \int_0^1 \mathrm{d}\alpha_1 \frac{1}{\left[\tilde{\ell}_1^2 + \alpha_1(1-\alpha_1)P^2\right]^2} \\
&= \mathrm{d}\Omega \int_0^1 \mathrm{d}\alpha_1 \frac{1}{2}\left[\ln\left(\Lambda_{\mathrm{UV}}^2 + \alpha_1(1-\alpha_1)P^2\right) + \frac{\alpha_1(1-\alpha_1)P^2}{\alpha_1(1-\alpha_1)P^2 + \Lambda_{\mathrm{UV}}^2}\right] \\
&= \frac{1}{2}\mathrm{d}\Omega \left[\ln\frac{\Lambda_{\mathrm{UV}}^2}{P^2} + 1\right] + \mathcal{O}(1/\Lambda_{\mathrm{UV}}^2) \,.
\end{aligned} \tag{6.6}$$

In the first two steps, we simply wrote out the terms and rearranged them suggestively. We then shifted the loop momentum to eliminate the linear term. Next, we exchanged the integrals and performed the loop integral (the factor $d\Omega$ is the angular integral). We then took the leading term as $\Lambda_{\mathrm{UV}} \to \infty$ and performed the remaining integral. Note that one could also perform the integral first and then take the limit – the same result comes out. The bottom line of this computation is that the divergence is logarithmic and not power law. This means that the divergence can be absorbed by a renormalisation of λ_4.

Let us now reduce higher order loop integrals to effective lower order loop integrals. For this, we have to consider that for any given loop, there could be more than two propagators. We also note that generally, propagators depend on more than one loop momentum. Let us now consider a loop with $k+1>2$ propagators. This entails that the corresponding loop integral reads

$$\int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \, \frac{1}{\ell_1^2} \frac{1}{(\ell_1+P_1)^2} \cdots \frac{1}{(\ell_1+P_k)^2} \,, \tag{6.7}$$

where the P_i are linear combinations of external and other loop momenta. We once again use the Feynman parameterisation, shift the loop momentum and swap integrals (note the convention for the Feynman parameter integral in the info box below!):

$$\begin{aligned}
&\int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \, \frac{1}{\ell_1^2} \frac{1}{(\ell_1+P_1)^2} \cdots \frac{1}{(\ell_1+P_k)^2} \\
&= k! \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \int_0^1 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_k \, \frac{1}{\left[\alpha_1\ell_1^2 + \alpha_2(\ell_1+P_1)^2 + \cdots + (1-\alpha_1-\cdots-\alpha_k)\,(\ell_1+P_k)^2\right]^{k+1}} \\
&= k! \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\ell_1 \int_0^1 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_k \, \frac{1}{\left[(\ell_1+\alpha_2 P_1+\dots)^2 + X\right]^{k+1}} \\
&= k! \int_0^1 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_k \int^{\Lambda_{\mathrm{UV}}} \mathrm{d}^4\tilde{\ell}_1 \, \frac{1}{\left[\tilde{\ell}_1^2 + \tilde{X}\right]^{k+1}} \,.
\end{aligned} \tag{6.8}$$

Here we abbreviated the square of the linear combination of the P_i by X, and the modification of it due to the loop momentum shift by $\tilde{X}$. Now since $k>1$ by assumption, this integral is finite when we send $\Lambda_{\mathrm{UV}} \to \infty$. This entails that only the lower integral limit contributes, and the above integral evaluates to

$$k! \frac{\mathrm{d}\Omega}{2k(k-1)} \int_0^1 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_k \, \tilde{X}^{1-k} \,. \tag{6.9}$$

The detailed dependence on the Feynman parameters and remaining momenta is not important – we can use this expression with a generalised Feynman parameterisation. The key point is that by performing one loop integral, we have reduced the total number of propagators by two. To finish the argument, we can now come back to our initial counting: an N-loop diagram contains $2N$ propagators. Performing $N-1$ loop integrals successively as above, we end up with the same divergence structure as for the one-loop diagram. This means that we can absorb all loop divergences by renormalising λ_4.

Convince yourself that all remaining divergences appear only in the two-point function, and that these can be absorbed by a mass term and a wave-function renormalisation factor in front of the kinetic term (plus an overall constant term, which has no physical meaning in the absence of gravity).

Extra material 3: Recap — Feynman parameterisation

Let us do a quick recap of the Feynman parameterisation (and a small generalisation). It is a clever rewriting of a product of n propagators into a nested integral over the n-th power of a single propagator, which is easier to perform. The underlying trick is the Schwinger parameterisation of the propagator. For any $X > 0$, we have

$$\frac{1}{X} = \int_0^\infty \mathrm{d}s\, e^{-sX} \,. \tag{6.10}$$

Let us do this for a product of n propagators:

$$\frac{1}{X_1 \cdots X_n} = \int_0^\infty \mathrm{d}s_1 \cdots \mathrm{d}s_n\, e^{-(s_1 X_1 + \cdots + s_n X_n)} \,. \tag{6.11}$$

Let us now shift the integration variables in the following way:

$$\alpha = s_1 + \cdots + s_n \,, \qquad \alpha_{i<n} = \frac{s_i}{s_1 + \cdots + s_n} \,. \tag{6.12}$$

This gives

$$\frac{1}{X_1 \cdots X_n} = \int_0^1 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{n-1} \int_0^\infty \mathrm{d}\alpha\, \alpha^{n-1}\, e^{-\alpha(\alpha_1 X_1 + \cdots + \alpha_{n-1} X_{n-1} + (1-\alpha_1 - \cdots - \alpha_{n-1}) X_n)} \,. \tag{6.13}$$

Note here that the integral over the α_i is constrained so that their sum is less than one, $\sum \alpha_i \leq 1$! We can perform the integration over α, and get

$$\frac{1}{X_1 \cdots X_n} = (n-1)! \int_0^1 \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{n-1} \frac{1}{[\alpha_1 X_1 + \cdots + \alpha_{n-1} X_{n-1} + (1 - \alpha_1 - \cdots - \alpha_{n-1}) X_n]^n} \,. \tag{6.14}$$

This can also be written as (also resolving the above-mentioned constraint on the integration range)

$$\frac{1}{X_1 \cdots X_n} = (n-1)! \int_0^1 \mathrm{d}\alpha_1 \cdots \int_0^1 \mathrm{d}\alpha_n \frac{\delta(1 - \alpha_1 - \cdots - \alpha_n)}{[\alpha_1 X_1 + \cdots + \alpha_{n-1} X_{n-1} + \alpha_n X_n]^n} \,. \tag{6.15}$$

A related formula can be derived that is useful for the inductive argument that we have used:

$$\frac{1}{X_1^{c_1} \cdots X_n^{c_n}} = \frac{\Gamma(c_1 + \cdots + c_n)}{\Gamma(c_1) \cdots \Gamma(c_n)} \int_0^1 \mathrm{d}\alpha_1 \cdots \int_0^1 \mathrm{d}\alpha_n \frac{\delta(1 - \alpha_1 - \cdots - \alpha_n) \alpha_1^{c_1 - 1} \cdots \alpha_n^{c_n - 1}}{[\alpha_1 X_1 + \cdots + \alpha_{n-1} X_{n-1} + \alpha_n X_n]^{c_1 + \cdots + c_n}} \,. \tag{6.16}$$

This equation can be proven in a similar way as the above formula, using that

$$\frac{1}{X^n} = \int_0^\infty \mathrm{d}s\, s^{n-1} e^{-sX} = \frac{(-1)^{n-1}}{\Gamma(n)} \frac{\partial^{n-1}}{\partial X^{n-1}} \int_0^\infty \mathrm{d}s\, e^{-sX} \,. \tag{6.17}$$

b) At one-loop order, there are no "interesting" diagrams – the only diagram that exists is a bubble diagram that contributes to the four-point function, which vanishes in dimreg. Is this theory renormalisable then? Actually not. For this, we can consider the two-loop diagram that contributes to the six-point function (it is a self-energy-type diagram as for ϕ^4 at one loop). The structure is

$$\int^{\Lambda_{\rm UV}} \mathrm{d}^4\ell_1 \mathrm{d}^4\ell_2 \, \frac{1}{\ell_1^2} \frac{1}{\ell_2^2} \frac{1}{(\ell_1+\ell_2+P)^2}\,, \tag{6.18}$$

where P is the sum of some external momenta. You can now go through the same steps as above, and you will find that this integral has two kinds of divergences: a quadratic divergence $\sim \Lambda_{\rm UV}^2$ and a logarithmic divergence $\sim P^2 \ln \Lambda_{\rm UV}$. These indicate that the theory cannot be perturbatively renormalised without introducing a new term of the form $\phi^3\partial^2\phi^3$.

c) Compared to standard ϕ^4-theory, the vertex has changed and receives some momentum dependence. Before we derive the Feynman rule, let us rewrite the interaction term:

$$(\partial_\mu\phi)\,(\partial^\mu\phi)\,\phi^2 = \frac{1}{3}\,(\partial_\mu\phi)\,\partial^\mu\phi^3 = -\frac{1}{3}\left(\partial^2\phi\right)\,\phi^3 + \frac{1}{3}\partial^\mu\left[(\partial_\mu\phi)\,\phi^3\right]\,. \tag{6.19}$$

We can neglect the last term since it is a total derivative. We thus conclude that the four-vertex is proportional to

$$\lambda_\partial\left(p_1^2+p_2^2+p_3^2+p_4^2\right)\,, \tag{6.20}$$

where the p_i are the (all-incoming) momenta of the four fields.

We now consider once again the self-energy-type diagram contributing to the four-point function. This time, the diagram is proportional to

$$\lambda_\partial^2\int^{\Lambda_{\rm UV}} \mathrm{d}^4\ell\,\left(p_1^2+p_2^2+\ell^2+(\ell+p_1+p_2)^2\right)\,\left(p_3^2+p_4^2+\ell^2+(\ell+p_3+p_4)^2\right)\,\frac{1}{\ell^2}\frac{1}{(\ell+p_1+p_2)^2}\,. \tag{6.21}$$

This diagram is badly divergent: repeating the same calculation as above (or having an informed guess based on the mass dimension), the divergent part must structurally be of the form

$$\Lambda_{\rm UV}^4 + p^2\,\Lambda_{\rm UV}^2 + p^4\,\ln\Lambda_{\rm UV}\,, \tag{6.22}$$

where p generically stands for one of the external momenta (or a combination thereof). This can clearly not be absorbed in the original coupling, so once again we have to deal with a perturbatively non-renormalisable theory.

d) Now the fun part starts – gravity. But actually, the argument is very much similar to part c), with small modifications. Starting with the action for General Relativity, we find that *all* n-graviton vertices have two momenta. If we then consider a standard self-energy diagram at one-loop (and neglect all the complications that come from indices, but which are irrelevant for the argument), we find a very similar divergence structure as in c), namely

$$\int^{\Lambda_{\rm UV}} \mathrm{d}^4\ell\,(c_1p^2+c_2\ell^2+c_3(\ell+p)^2)^2\,\frac{1}{\ell^2}\frac{1}{(\ell+p)^2}\,. \tag{6.23}$$

Here, the c_i are constants that we shall not compute now. What is the divergence structure? Literally the same as in c)! The only difference is that our action has terms up to quadratic order in momenta, so the "only" bad divergence is the p^4 term. Such a term can be absorbed by terms quadratic in curvature.

What happens at two loops? Practically the same, with one more propagator and one more loop integral. One diagram at two loops is the same as the one-loop diagram, with an extra line joining the vertices:

$$\int^{\Lambda_{\rm UV}} \mathrm{d}^4\ell_1 \mathrm{d}^4\ell_2 \,(d_1 p^2 + d_2 \ell_1^2 + \ldots)^2 \,\frac{1}{\ell_1^2}\frac{1}{\ell_2^2}\frac{1}{(\ell_1+\ell_2+p)^2}\,. \tag{6.24}$$

Here, we didn't write out the full structure of the five-point vertices involved. In short, we added a loop integral, so ℓ^4, but only one propagator, ℓ^{-2}, so that the overall divergence increases by two powers of momenta. This makes it necessary to introduce yet more terms, this time with either three curvatures (this is for the three-point function), or two curvatures and two derivatives. Basically, this is the reason why you are in this course, and not in a course "QFT III: Quantum Gravity".

Exercise 7: *[Presence]* Computing the one-loop divergence in gravity with xAct

Motivation: In this exercise, we will explicitly compute the one-loop divergence in General Relativity with the help of xAct. You do not have to bring your own laptop, the Mathematica file will be sent around afterwards.

Most of the explanations are in the Mathematica notebook. Some things that we use are collected here. The two-point diagrams that we evaluated have terms where the external indices are carried by loop momenta, which we first have to simplify. If we have a single loop momentum that has an open index (and all other indices are carried by either metrics or external momenta, that we can pull out of the integral), we can write

$$\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell_\mu = A_1(p^2)p_\mu\,. \tag{7.1}$$

There are simply no other terms with one index that this integral could produce. To compute $A_1(p^2)$, we contract the above equation with p, divide by p^2, and get:

$$A_1(p^2) = \frac{1}{p^2}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell\cdot p\,. \tag{7.2}$$

In a somewhat sloppy way, we can thus identify the replacement rule

$$\ell_\mu \mapsto p_\mu\,\frac{\ell\cdot p}{p^2}\,. \tag{7.3}$$

This is to be understood that it can only be used if there is a *single* loop momentum with an open index!

Going on with two loop momenta with open indices. This time, it can be proportional to either the metric, or two external momenta:

$$\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell_\mu\ell_\nu = B_1(p^2)\,\eta_{\mu\nu} + B_2(p^2)\,p_\mu p_\nu\,. \tag{7.4}$$

This time, we can contract the equation with either a metric, or two external momenta, and we get a linear system of equations for $B_{1,2}(p^2)$:

$$\begin{aligned}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2 &= 4B_1(p^2) + B_2(p^2)\,p^2\,,\\ \int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^2 &= B_1(p^2)\,p^2 + B_2(p^2)\,p^4\,.\end{aligned} \tag{7.5}$$

Solving this, we find

$$\begin{aligned} B_1(p^2) &= \frac{1}{3}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2 - \frac{1}{3p^2}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^2\,, \\ B_2(p^2) &= -\frac{1}{3p^2}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2 + \frac{4}{3p^4}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^2\,. \end{aligned} \tag{7.6}$$

We can again extract a sloppy rule from that:

$$\ell_\mu\ell_\nu \mapsto \frac{1}{3}\ell^2\left(1-\frac{(\ell\cdot p)^2}{\ell^2p^2}\right)\eta_{\mu\nu} - \frac{1}{3}\frac{\ell^2}{p^2}\left(1-4\frac{(\ell\cdot p)^2}{\ell^2p^2}\right)p_\mu p_\nu\,. \tag{7.7}$$

Once again caution, this rule is only to be used if there are exactly two loop momenta with open indices!

On to three loop momenta. What are allowed structures? Note that the expression is completely symmetric in the indices. This means that once again only two structures can appear:

$$\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell_\mu\ell_\nu\ell_\rho = C_1(p^2)\,\eta_{(\mu\nu}p_{\rho)} + C_2(p^2)\,p_\mu p_\nu p_\rho\,. \tag{7.8}$$

We can contract with one metric and one momentum, or three momenta. Skipping intermediate steps, the result is

$$\begin{aligned} C_1(p^2) &= \frac{1}{p^2}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2\,(\ell\cdot p) - \frac{1}{p^4}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^3\,, \\ C_2(p^2) &= -\frac{1}{p^4}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2\,(\ell\cdot p) + \frac{2}{p^6}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^3\,. \end{aligned} \tag{7.9}$$

The sloppy rule for three momenta is

$$\ell_\mu\ell_\nu\ell_\rho \mapsto \left[\frac{\ell^2}{p^2}(\ell\cdot p)\left(1-\frac{(\ell\cdot p)^2}{\ell^2p^2}\right)\right]\eta_{(\mu\nu}p_{\rho)} - \left[\frac{\ell^2}{p^4}(\ell\cdot p)\left(1-2\frac{(\ell\cdot p)^2}{\ell^2p^2}\right)\right]p_\mu p_\nu p_\rho\,. \tag{7.10}$$

Last but not least, we have four loop momenta. Once again, the expression is completely symmetric, but now we can have three terms:

$$\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell_\mu\ell_\nu\ell_\rho\ell_\sigma = D_1(p^2)\,\eta_{(\mu\nu}\eta_{\rho\sigma)} + D_2(p^2)\,\eta_{(\mu\nu}p_\rho p_{\sigma)} + D_3(p^2)p_\mu p_\nu p_\rho p_\sigma\,. \tag{7.11}$$

This time we can form three contractions. The solution is given by

$$\begin{aligned} D_1(p^2) &= \frac{1}{5}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^4 - \frac{2}{5p^2}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2(\ell\cdot p)^2 + \frac{1}{5p^4}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^4\,, \\ D_2(p^2) &= -\frac{2}{5p^2}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^4 + \frac{14}{5p^4}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2(\ell\cdot p)^2 - \frac{12}{5p^6}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^4\,, \\ D_3(p^2) &= \frac{1}{5p^4}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^4 - \frac{12}{5p^6}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,\ell^2(\ell\cdot p)^2 + \frac{16}{5p^8}\int \mathrm{d}^4\ell\, \mathcal{I}(p^2,\ell^2,p\cdot\ell)\,(\ell\cdot p)^4\,. \end{aligned} \tag{7.12}$$

Finally, the sloppy rule for four loop momenta is

$$\begin{aligned} \ell_\mu\ell_\nu\ell_\rho\ell_\sigma \mapsto & \left[\frac{1}{5}\ell^4\left(1-\frac{(\ell\cdot p)^2}{\ell^2p^2}\right)^2\right]\eta_{(\mu\nu}\eta_{\rho\sigma)} - \left[\frac{2}{5}\frac{\ell^4}{p^2}\left(1-7\frac{(\ell\cdot p)^2}{\ell^2p^2}+6\frac{(\ell\cdot p)^4}{\ell^4p^4}\right)\right]\eta_{(\mu\nu}p_\rho p_{\sigma)} \\ & + \left[\frac{1}{5}\frac{\ell^4}{p^4}\left(1-12\frac{(\ell\cdot p)^2}{\ell^2p^2}+16\frac{(\ell\cdot p)^4}{\ell^4p^4}\right)\right]p_\mu p_\nu p_\rho p_\sigma\,. \end{aligned} \tag{7.13}$$

Exercise 8: *[Presence]* Counting degrees of freedom

Motivation: In this exercise, we will go into some more detail on how we can count the number of physical degrees of freedom in a gauge theory.

There is the slogan "The gauge symmetry always hits twice." — meaning that usually (but not always), each gauge symmetry produces two Hamiltonian constraints. In electrodynamics, there is a single gauge condition, e.g. $\partial^\mu A_\mu = 0$, and it reduces the four components of A_μ down to two polarisations. In the same way in gravity, the metric has 10 components, the gauge condition has four components, reducing things once again to two polarisations. We will discuss a very simple mechanical toy model to illustrate how the extra constraint can appear. The example is taken from [A. Golovnev, Universe 9 (2023) 2, 101].

Consider the Lagrange function

$$L = \frac{1}{2}\left(\dot{x} + y\right)^2 , \tag{8.1}$$

with dynamical variables x and y. A priori, we would think that this model has two (mechanical) degrees of freedom. However, the Lagrangian is invariant under the "gauge" symmetry

$$x \mapsto x + u \,, \qquad y \mapsto y - \dot{u} \,. \tag{8.2}$$

So naively, we should be left with one degree of freedom, right? Wrong – the gauge symmetry "hits twice", meaning it reduces the system to zero degrees of freedom, or a trivial theory. How do we see that?

Let us go to the Hamiltonian picture. We first compute the momenta:

$$\pi_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} + y \,, \qquad \pi_y = \frac{\partial L}{\partial \dot{y}} = 0 \,. \tag{8.3}$$

Aha – the conjugate momentum π_y vanishes, that is, we have a primary constraint. In defining the Hamiltonian, we thus have to add it with a Lagrange multiplier λ:

$$H = \frac{1}{2}\pi_x^2 - y\,\pi_x + \lambda\,\pi_y \,. \tag{8.4}$$

We now get four equations of motion and a constraint equation:

$$\begin{aligned} \dot{x} &= \pi_x - y \,, \\ \dot{y} &= \lambda \,, \\ \dot{\pi}_x &= 0 \,, \\ \dot{\pi}_y &= \pi_x \,, \\ \pi_y &= 0 \,. \end{aligned} \tag{8.5}$$

We can see that the primary constraint $\pi_y = 0$ induces a secondary constraint $\pi_x = 0$ (this is the "hitting twice")! What is the solution to this dynamical system? Well, given any arbitrary $x(t)$, y is given by a constraint equation $y = -\dot{x}$, so we do not have to specify any initial data – the system is trivial and has no dynamical degrees of freedom. In the Lagrangian picture, the physical combination of variables $\dot{x} + y$ is fully constrained to vanish on any solution of the equations of motion.

A similar story carries over to actual gauge theories like electrodynamics and gravity. For example in electrodynamics, the Lagrangian does not depend on $\dot{A}_0$, which gives a primary constraint much as for the above model. This then induces once again a secondary constraint, reducing the number of degrees of freedom to two. In gravity, one would often use so-called ADM variables (basically a $3+1$-split of the metric) to carry out the analysis.

A standard reference for all this is [M. Henneaux, C. Teitelboim, Quantization of gauge systems].