Quantum Gravity and the Renormalization Group

Assignment 5 – Nov 22+25

Exercise 11: Avoiding Ostrogradski?

Motivation: In the lecture, we discussed the Ostrogradski problem of quadratic gravity. In this exercise, we will try to find ways to avoid the negative aspects without sacrificing the positive aspects of the higher-derivative terms.

The spin-two propagator in quadratic gravity has the structural form

$$\frac{4\lambda}{p^4 - \frac{\lambda M_{\text{Pl}}^2}{2} p^2} \,. \tag{11.1}$$

As we discussed in some detail, the fall-off with p^4 for large momenta (together with the fact that vertices also scale like the fourth power of momentum) make the theory renormalisable, and in particular asymptotically free. However, the partial fraction decomposition of the propagator,

$$\frac{8}{M_{\text{Pl}}^2} \left[\frac{1}{-p^2} - \frac{1}{-p^2 + \frac{\lambda}{2} M_{\text{Pl}}^2} \right] , \tag{11.2}$$

indicates that we have a massive ghost. Let us investigate different potential scenarios in how we could avoid it.

a) First, consider the propagator of a general quartic theory. Does it always propagate a ghost? Why?

b) **[hard question]** Suppose now that we start with a propagator that is the sum of two modes that both come with a positive prefactor, so that we do not propagate a ghost. Try to reconstruct the action that gives rise to such a propagator. Are there any problems with such a theory? Is this a local theory? What is its UV behaviour?

c) **[hard question]** Suppose that instead of quadratic gravity, we now add sixth-order derivative terms. What is propagated in such a theory? Can we avoid ghosts? If yes, what are the conditions for this? What are the renormalisation properties of sixth-derivative gravity?

Exercise 12: Conformal factor instability

Motivation: This exercise discusses the cliffhanger of last week's lecture in some more detail: the spin zero part of the graviton propagator has the wrong sign, that is, it is a ghost. We will now discuss why this is (not?) a problem.

Let use first discuss the sign of the spin zero part in classical GR:

a) Why is the negative sign not a problem *classically*?

Now, consider the Einstein-Hilbert action for *Euclidean* Quantum Gravity, i.e., where the signature of the metric is $(+,+,+,+)$, which is given by

$$S_E = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\,R. \tag{12.1}$$

Consider the conformal transformation that maps $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ with

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{12.2}$$

where $\Omega = \Omega(x)$. Classically, the Einstein-Hilbert action is not conformally invariant. In the path integral, you can view the conformal transformation as a transformation of the integration variable.

b) What is the form of the Einstein-Hilbert action after the conformal transformation, when you express it in terms of the original metric $g_{\mu\nu}$?

c) Discuss the sign of the kinetic term for the conformal mode Ω that you get in b). What is the effect of the conformal mode in the Euclidean path integral? The Euclidean path integral would structurally look like:

$$\int \mathcal{D}g\, e^{-S_E}\,. \tag{12.3}$$