Quantum Gravity and the Renormalization Group

Assignment 5 – Nov 22+25

Exercise 11: Avoiding Ostrogradski?

Motivation: In the lecture, we discussed the Ostrogradski problem of quadratic gravity. In this exercise, we will try to find ways to avoid the negative aspects without sacrificing the positive aspects of the higher-derivative terms.

The spin-two propagator in quadratic gravity has the structural form

$$\frac{4\lambda}{p^4 - \frac{\lambda M_{\mathrm{Pl}}^2}{2} p^2} \,. \tag{11.1}$$

As we discussed in some detail, the fall-off with p^4 for large momenta (together with the fact that vertices also scale like the fourth power of momentum) make the theory renormalisable, and in particular asymptotically free. However, the partial fraction decomposition of the propagator,

$$\frac{8}{M_{\mathrm{Pl}}^2} \left[\frac{1}{-p^2} - \frac{1}{-p^2 + \frac{\lambda}{2} M_{\mathrm{Pl}}^2} \right] , \tag{11.2}$$

indicates that we have a massive ghost. Let us investigate different potential scenarios in how we could avoid it.

a) First, consider the propagator of a general quartic theory. Does it always propagate a ghost? Why?

b) **[hard question]** Suppose now that we start with a propagator that is the sum of two modes that both come with a positive prefactor, so that we do not propagate a ghost. Try to reconstruct the action that gives rise to such a propagator. Are there any problems with such a theory? Is this a local theory? What is its UV behaviour?

c) **[hard question]** Suppose that instead of quadratic gravity, we now add sixth-order derivative terms. What is propagated in such a theory? Can we avoid ghosts? If yes, what are the conditions for this? What are the renormalisation properties of sixth-derivative gravity?

a) Let us consider the most general quartic theory. Such a theory would have a mass term, a term quadratic in the momentum, and a term quartic in the moment, with (in general) arbitrary prefactors. The propagator would thus read

$$G(p^2) = \frac{1}{a\, m^2 + b\, p^2 + c\, p^4} \,. \tag{11.3}$$

Here, a, b, c are arbitrary real numbers. Let us do a partial fraction decomposition. The poles are at

$$p_\pm^2 = \frac{-b \pm \sqrt{b^2 - 4\, a\, c\, m^2}}{2c} \,. \tag{11.4}$$

Let us assume that both poles are real (i.e. that the argument of the square root is positive). Otherwise, we have an imaginary part, which corresponds to a finite width, and thus a finite lifetime of the particles. The residues at these poles are

$$\mathcal{R}_\pm = \pm\frac{1}{\sqrt{b^2 - 4\,a\,c\,m^2}}\,, \tag{11.5}$$

and thus the propagator reads

$$G(p^2) = \frac{1}{\sqrt{b^2 - 4\,a\,c\,m^2}}\left[\frac{1}{p^2 - p_+^2} - \frac{1}{p^2 - p_-^2}\right]. \tag{11.6}$$

We conclude that, completely independent of the coefficients in the action, one of the two modes is ghostly. We can thus not avoid the instability by a clever choice of coefficients, if at the same time we want to have stable particles.

b) What if we impose that the propagator is the sum of two "normal" modes? That is, let us assume a propagator

$$G(p^2) = \frac{A}{-p^2 - m_1^2} + \frac{B}{-p^2 - m_2^2}\,, \qquad A, B > 0\,. \tag{11.7}$$

What goes wrong with this? Many things. First of all, this propagator does *not* have an improved fall-off at high energies:

$$G(p^2) \sim \frac{A+B}{-p^2}\,, \qquad -p^2 \to \infty\,. \tag{11.8}$$

This means that we do not counteract loop divergences with such a propagator. Furthermore, if we invert the propagator to get the second variation of the action, we find

$$S^{(2)}(p^2) = G(p^2)^{-1} = -\frac{(p^2 + m_1^2)(p^2 + m_2^2)}{(A+B)p^2 + Bm_1^2 + Am_2^2}\,. \tag{11.9}$$

This is always a rational function, and thus a non-local theory — except if $m_1^2 = m_2^2$, but then we do not have different modes in the propagator in the first place! So also here, we do not find a path around Ostrogradski.

c) Adding sixth-derivative terms gives a flat propagator of the form (ignoring tensor structure)

$$G(p^2) = \frac{1}{-p^2(1 + a\,p^2 + bp^4)}\,. \tag{11.10}$$

Let us briefly think about which terms give a p^6-behaviour. It is certainly not any term that is cubic in curvature — it must be terms that have two covariant derivatives. This means that we need to add

$$R\,\Box\,R\,, \qquad R^{\mu\nu}\,\Box\,R_{\mu\nu} \tag{11.11}$$

to the action. A similar term with the Riemann tensor is dependent (see extra material).

Back to the propagator. By construction, we have the massless graviton pole, and then two additional poles. By the arguments in a), exactly one of them would come with a minus sign, indicating once again a ghost. Except — if the square root is imaginary. Then, the two extra modes would correspond to unstable that decay, and thus cannot be asymptotic states.

What are the renormalisation properties of this theory? Convince yourself that it is *super-renormalisable* — this means that only *finitely many loop diagrams diverge*! This is because the superficial degree of divergence is $D = (d-6)L + 4 = 4 - 2L$, so there are only divergences at one and two loops.

Extra material 4: Box Riemann

Warning: don't do the following by hand! There is an interesting identity that ~~simplifies~~ *complicates* our life. It starts with the differential Bianchi identity:

$$D_{[\alpha}R_{\mu\nu]\rho\sigma} = 0\,. \tag{11.12}$$

Let us act with another covariant derivative on this (and multiply by 3) to form

$$3D^{\alpha}D_{[\alpha}R_{\mu\nu]\rho\sigma} = D^2 R_{\mu\nu\rho\sigma} + D^{\alpha}D_{\mu}R_{\nu\alpha\rho\sigma} - D^{\alpha}D_{\nu}R_{\mu\alpha\rho\sigma} = 0\,. \tag{11.13}$$

Maybe you can already see where this is going. We have two terms that are almost divergences of the Riemann tensor, that is, terms which (via the contracted Bianchi identity) can be replaced by covariant derivatives of the Ricci tensor. To bring the two terms into this form, we have to commute the covariant derivatives, which gives terms that are quadratic in curvature (and without covariant derivatives). This tells you that we can replace any occurrence of $D^2R_{\mu\nu\rho\sigma}$ by terms that are either quadratic in curvatures, or covariant derivatives of either the Ricci tensor or the Ricci scalar. If we then contract this relation with another Riemann tensor, integrate over the spacetime, and perform partial integrations, we find

$$\int \mathrm{d}^4x\,\sqrt{-g}\,\left[R^{\mu\nu\rho\sigma}\,D^2\,R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}\,D^2\,R_{\mu\nu} + R\,D^2\,R\right] = \mathcal{O}(\mathcal{R}^3)\,. \tag{11.14}$$

Bonus exercise: with the help of xAct, verify this relation, and compute the right-hand side of this relation, and **[hard question]** the boundary terms needed for the unintegrated relation.

Exercise 12: Conformal factor instability

Motivation: This exercise discusses the cliffhanger of last week's lecture in some more detail: the spin zero part of the graviton propagator has the wrong sign, that is, it is a ghost. We will now discuss why this is (not?) a problem.

Let use first discuss the sign of the spin zero part in classical GR:

a) Why is the negative sign not a problem *classically*?

Now, consider the Einstein-Hilbert action for *Euclidean* Quantum Gravity, i.e., where the signature of the metric is $(+,+,+,+)$, which is given by

$$S_E = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\,R. \tag{12.1}$$

Consider the conformal transformation that maps $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ with

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{12.2}$$

where $\Omega = \Omega(x)$. Classically, the Einstein-Hilbert action is not conformally invariant. In the path integral, you can view the conformal transformation as a transformation of the integration variable.

b) What is the form of the Einstein-Hilbert action after the conformal transformation, when you express it in terms of the original metric $g_{\mu\nu}$?

c) Discuss the sign of the kinetic term for the conformal mode Ω that you get in b). What is the effect of the conformal mode in the Euclidean path integral? The Euclidean path integral would structurally look like:

$$\int \mathcal{D}g\, e^{-S_E}\,. \tag{12.3}$$

a) The underlying reason is gauge invariance: the spin zero is an off-shell degree of freedom, but does not survive on-shell. It is there to remove the residual gauge degrees of freedom that are also in the spin two part. At the very end, the spin zero part is like the Faddeev-Popov ghosts — they also come with a minus sign (that's why we call them ghosts), but they are there to cancel unphysical modes.

b) The determinant transforms multiplicatively:

$$\det g = \Omega^{-8}\, \det \tilde{g}\,. \tag{12.4}$$

Let us not be too silly and try to compute the transformation of the Ricci scalar by hand. Rather, xAct comes to the rescue (see notebook). The result is:

$$R = \Omega^2\, \tilde{R} + 6\Omega\tilde{D}^2\Omega - 12\tilde{D}_\mu\Omega\tilde{D}^\mu\Omega\,. \tag{12.5}$$

Combining this and performing a partial integration, we get the transformed action:

$$\begin{aligned} S_E[\tilde{g}] &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{\tilde{g}}\,\frac{1}{\Omega^4}\left[\Omega^2\tilde{R} + 6\Omega\tilde{D}^2\Omega - 12\tilde{D}_\mu\Omega\tilde{D}^\mu\Omega\right] \\ &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{\tilde{g}}\left[\Omega^{-2}\tilde{R} + \frac{6}{\Omega^4}\tilde{D}_\mu\Omega\tilde{D}^\mu\Omega\right] \\ &= -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{\tilde{g}}\left[\Omega^{-2}\tilde{R} + 6\tilde{D}_\mu\Omega^{-1}\tilde{D}^\mu\Omega^{-1}\right]\,. \end{aligned} \tag{12.6}$$

c) The sign of the conformal mode (or rather, Ω^{-1}, but the inverse is insubstantial as it was a choice in which direction we transform) is negative, that is, large kinetic energy is not suppressed in the Euclidean path integral. To see this, note that the flat two-point function is ∂^2, which is a non-positive operator. This issue is known as the conformal factor instability. It is simply a reflection of the ghostly nature of the spin zero mode. However, in Euclidean computations/simulations of gravity, this mode can destabilise the system.