Quantum Gravity and the Renormalization Group

Assignment 6 – Nov 29+Dec 02

Exercise 13: Extra modes in higher derivative gravity

Motivation: In this exercise, we will make the extra degrees of freedom that the curvature-squared terms introduce explicit. The general idea is the following: we first write down an action with an auxiliary (that is, non-dynamical) field, making sure that the equations of motion are equivalent to higher-derivative gravity. We then perform a shift of the metric to bring the action into a form of GR plus "extra stuff". For the R^2-term, we can do this exactly, whereas for the $R^{\mu\nu}R_{\mu\nu}$-term, we will restrict ourselves to the kinetic term of the new field.

Consider the following "weird" action:

$$S^{\text{weird}} = \int \mathrm{d}^4x \, \sqrt{-g} \left[\Phi \, R - \frac{1}{4\alpha} \left(\Phi - 1 \right)^2 \right] . \tag{13.1}$$

Here, α is a constant and Φ is a scalar field.

a) Derive the equations of motion, both for the auxiliary field Φ and for the metric. Plug the solution for the scalar field into the equation of motion for the metric. What is this equation of motion equivalent to? Also, plug the solution into the action. What do you get?

b) Perform a conformal transformation on S^{weird} (as in exercise 12 — don't insert any solution to the equations of motion here!), this time with

$$g_{\mu\nu} \mapsto \Phi^{-1} \, \tilde{g}_{\mu\nu} \, . \tag{13.2}$$

How does the action look like?

c) Finally, reparameterise the scalar field by

$$\Phi = e^{c \, \tilde{\Phi}} \, , \tag{13.3}$$

where c is a suitable constant. Compute the action as a function of $\tilde{g}$ and $\tilde{\Phi}$ and choose the constant c cleverly (you will see what this means). What is this theory? What do we learn from this whole computation?

d) **[hard question]** Try to generalise this to $f(R)$ gravity. Does the same method go through? If so, how does the scalar potential look like?

Now shift your attention to this "super-weird" action:

$$S^{\text{super-weird}} = \int \mathrm{d}^4x \, \sqrt{-g} \left[R + \left(R^{\mu\nu} - \frac{1}{2} R \, g^{\mu\nu} \right) f_{\mu\nu} - \frac{1}{4\beta} \left(f^{\mu\nu} f_{\mu\nu} - f^{\mu}{}_{\mu} f^{\nu}{}_{\nu} \right) \right] . \tag{13.4}$$

Here, $f_{\mu\nu}$ is a symmetric rank-two tensor field, and β is a constant.

e) **[hard question]** Perform the same tasks as in a), but now for the action $S^{\text{super-weird}}$.

f) **[hard question]** Now shift the metric in $S^{\text{super-weird}}$ in the following way:

$$g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} + c \, f_{\mu\nu} \, . \tag{13.5}$$

Doing so, only keep terms up to second order in f, and fix c such that there is no kinetic mixing — that is, ensure that there is no term that is linear in both curvature and f. Once again, what is this theory, and what do we learn from this?