Quantum Gravity and the Renormalization Group

Assignment 6 – Nov 29+Dec 02

Exercise 13: Extra modes in higher derivative gravity

Motivation: In this exercise, we will make the extra degrees of freedom that the curvature-squared terms introduce explicit. The general idea is the following: we first write down an action with an auxiliary (that is, non-dynamical) field, making sure that the equations of motion are equivalent to higher-derivative gravity. We then perform a shift of the metric to bring the action into a form of GR plus "extra stuff". For the R^2-term, we can do this exactly, whereas for the $R^{\mu\nu}R_{\mu\nu}$-term, we will restrict ourselves to the kinetic term of the new field.

Consider the following "weird" action:

$$S^{\text{weird}} = \int \mathrm{d}^4x \, \sqrt{-g} \left[\Phi \, R - \frac{1}{4\alpha} \left(\Phi - 1 \right)^2 \right] . \tag{13.1}$$

Here, α is a constant and Φ is a scalar field.

a) Derive the equations of motion, both for the auxiliary field Φ and for the metric. Plug the solution for the scalar field into the equation of motion for the metric. What is this equation of motion equivalent to? Also, plug the solution into the action. What do you get?

b) Perform a conformal transformation on S^{weird} (as in exercise 12 — don't insert any solution to the equations of motion here!), this time with

$$g_{\mu\nu} \mapsto \Phi^{-1} \, \tilde{g}_{\mu\nu} \, . \tag{13.2}$$

How does the action look like?

c) Finally, reparameterise the scalar field by

$$\Phi = e^{c \, \tilde{\Phi}} \, , \tag{13.3}$$

where c is a suitable constant. Compute the action as a function of $\tilde{g}$ and $\tilde{\Phi}$ and choose the constant c cleverly (you will see what this means). What is this theory? What do we learn from this whole computation?

d) **[hard question]** Try to generalise this to $f(R)$ gravity. Does the same method go through? If so, how does the scalar potential look like?

Now shift your attention to this "super-weird" action:

$$S^{\text{super-weird}} = \int \mathrm{d}^4x \, \sqrt{-g} \left[R + \left(R^{\mu\nu} - \frac{1}{2} R \, g^{\mu\nu} \right) f_{\mu\nu} - \frac{1}{4\beta} \left(f^{\mu\nu} f_{\mu\nu} - f^{\mu}{}_{\mu} f^{\nu}{}_{\nu} \right) \right] . \tag{13.4}$$

Here, $f_{\mu\nu}$ is a symmetric rank-two tensor field, and β is a constant.

e) **[hard question]** Perform the same tasks as in a), but now for the action $S^{\text{super-weird}}$.

f) **[hard question]** Now shift the metric in $S^{\text{super-weird}}$ in the following way:

$$g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} + c \, f_{\mu\nu} \, . \tag{13.5}$$

Doing so, only keep terms up to second order in f, and fix c such that there is no kinetic mixing — that is, ensure that there is no term that is linear in both curvature and f. Once again, what is this theory, and what do we learn from this?

a) The equation of motion for Φ is

$$\frac{\delta S^{\text{weird}}}{\delta \Phi} = 0 = R - \frac{1}{2\alpha}(\Phi - 1)\,. \tag{13.6}$$

This is simply a constraint, and the solution is

$$\Phi = 1 + 2\alpha\, R\,. \tag{13.7}$$

For the metric equation of motion, we can recycle the linearisations computed in exercise 2, namely

$$\begin{aligned}\sqrt{-\det g} &\simeq \sqrt{-\det \bar{g}}\left[1 + \frac{1}{2}h\right]\,,\\ R &\simeq \bar{R} + \bar{D}^\mu \bar{D}^\nu h_{\mu\nu} - \bar{D}^2 h - \bar{R}^{\mu\nu} h_{\mu\nu}\,.\end{aligned} \tag{13.8}$$

Inserting this into the action, taking the term linear in h and performing partial integrations, we get

$$S^{\text{weird}}\Big|_h = \int \mathrm{d}^4x\,\sqrt{-\bar{g}}\,\left[\frac{1}{2}\left(\Phi\,\bar{R} - \frac{1}{4\alpha}\,(\Phi-1)^2\right)\bar{g}^{\mu\nu} + \left(\bar{D}^\mu \bar{D}^\nu \Phi - \bar{g}^{\mu\nu}\bar{D}^2\Phi - \Phi \bar{R}^{\mu\nu}\right)\right] h_{\mu\nu}\,. \tag{13.9}$$

Thus, the metric equation of motion is

$$\frac{1}{2}\left(\Phi\,R - \frac{1}{4\alpha}\,(\Phi-1)^2\right)g^{\mu\nu} + \left(D^\mu D^\nu \Phi - g^{\mu\nu} D^2\Phi - \Phi R^{\mu\nu}\right) = 0\,. \tag{13.10}$$

If we now plug in the solution for Φ, we get

$$\frac{1}{2}\left(R + \alpha R^2 - 4\alpha D^2 R\right) g^{\mu\nu} + 2\alpha D^\mu D^\nu R - R^{\mu\nu} - 2\alpha R R^{\mu\nu} = 0\,. \tag{13.11}$$

These is just the equation of motion that we derived for $R + R^2$ gravity (exercise 4)!

Similarly, plugging the solution for Φ into S^{weird}, we find

$$S^{\text{weird}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\left[(1+2\alpha\,R)\,R - \frac{1}{4\alpha}\,(2\alpha\,R)^2\right] = \int \mathrm{d}^4x\,\sqrt{-g}\,\left[R + \alpha\,R^2\right]\,. \tag{13.12}$$

The weird action is thus simply a reformulation of the Starobinsky theory. Note that, for simplicity, we have used Planck units here ($16\pi G_N = 1$).

b) Here, we can recycle the results of the previous exercise by identifying $\Phi = \Omega^2$. This gives the action:

$$\begin{aligned}\tilde{S}^{\text{weird}} &= \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\,\left[\tilde{R} - \frac{3}{2\Phi^2}D^\mu\Phi D_\mu\Phi - \frac{1}{4\alpha}\left(1 - \frac{1}{\Phi}\right)^2\right]\\ &= \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\,\left[\tilde{R} - \frac{3}{2}D^\mu \ln\Phi D_\mu \ln\Phi - \frac{1}{4\alpha}\left(1 - \frac{1}{\Phi}\right)^2\right]\,.\end{aligned} \tag{13.13}$$

c) The suitable constant is $c = 1/\sqrt{3}$, and we get

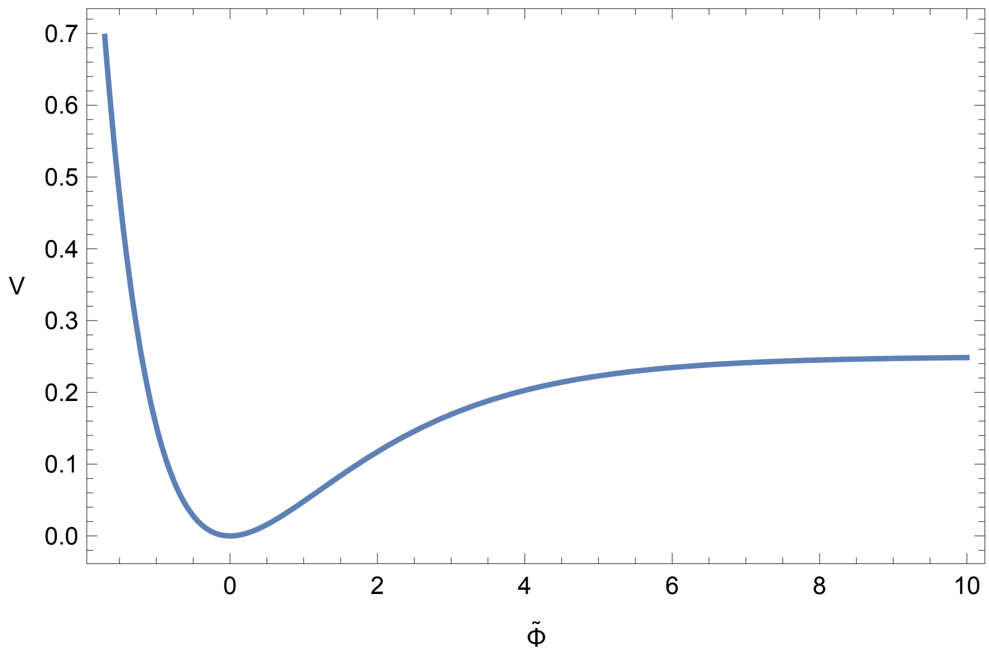
$$\tilde{S}^{\text{weird}} = \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\,\left[\tilde{R} - \frac{1}{2}D^\mu\tilde{\Phi} D_\mu\tilde{\Phi} - \frac{1}{4\alpha}\left(1 - \frac{1}{e^{\tilde{\Phi}/\sqrt{3}}}\right)^2\right] \tag{13.14}$$

This is a scalar-tensor theory: gravity coupled to a scalar field with a potential

$$V(\tilde{\Phi}) = \frac{1}{4\alpha}\left(1 - \frac{1}{e^{\tilde{\Phi}/\sqrt{3}}}\right)^2 . \tag{13.15}$$

We have thus shown that $R + R^2$ gravity is equivalent to GR coupled to a scalar field with this specific potential! In other words, we have made the extra scalar degree of freedom (that originates from the R^2 term) visible.

The potential is interesting for cosmology, as it implements the so-called slow-roll condition: the potential has a minimum at $\tilde{\Phi} = 0$, it grows quickly for negative $\tilde{\Phi}$, and flattens out for positive $\tilde{\Phi}$. For inflation modelling, this is interesting as the scalar field can slowly roll down the potential (giving an exponential expansion), and then settling in the minimum (the end of inflation). Below is a plot of the potential for $\alpha = 1$.



d) Let us consider a generalisation of the weird action by introducing a general potential for the scalar field:

$$\hat{S} = \int \mathrm{d}^4x\,\sqrt{-g}\,\left[\Phi\, R - V(\Phi)\right] . \tag{13.16}$$

The equations of motion are

$$\begin{aligned} 0 &= R - V'(\Phi)\,, \\ 0 &= \frac{1}{2}\left(\Phi\, R - V(\Phi)\right) g^{\mu\nu} + \left(D^\mu D^\nu - g^{\mu\nu} D^2 - R^{\mu\nu}\right)\,\Phi\,. \end{aligned} \tag{13.17}$$

Let us compare this to the equations of motion of an $f(R)$ theory. For this, we have

$$\begin{aligned} S^{f(R)} &= \int \mathrm{d}^4x\,\sqrt{-g}\,f(R) \\ &\simeq \int \mathrm{d}^4x\,\sqrt{-\bar{g}}\,\left[1 + \frac{1}{2}h\right]\,\left[f(\bar{R}) + f'(\bar{R})\left(\bar{D}^\mu\bar{D}^\nu h_{\mu\nu} - \bar{D}^2 h - \bar{R}^{\mu\nu}h_{\mu\nu}\right)\right] \\ &\simeq \int \mathrm{d}^4x\,\sqrt{-\bar{g}}\left[f(\bar{R}) + \left\{\frac{1}{2}f(\bar{R})\,\bar{g}^{\mu\nu} - f'(\bar{R})\bar{R}^{\mu\nu} + \left(\left(\bar{D}^\mu\bar{D}^\nu - \bar{g}^{\mu\nu}\bar{D}^2\right)f'(\bar{R})\right)\right\} h_{\mu\nu}\right] . \end{aligned} \tag{13.18}$$

This gives the equation of motion

$$0 = \frac{1}{2} f(R)\, g^{\mu\nu} + \left(D^\mu D^\nu - g^{\mu\nu} D^2 - R^{\mu\nu}\right) f'(R)\,. \tag{13.19}$$

Comparing with the above, we would need

$$\Phi = f'(R)\,, \qquad f(R) = \Phi\, R - V(\Phi)\,. \tag{13.20}$$

This looks like a Legendre transform! Indeed, Φ and R are the conjugate variables, and the potential is the Legendre transform of the function f. This also means that the equivalence only holds under the assumptions of the Legendre transform (convexity etc.). Going forward, we will assume that these assumptions are met.

We can do another conformal transformation of the metric. Skipping the intermediate steps (which mirror the previous parts), we finally arrive at

$$\tilde{\tilde{S}} = \int \mathrm{d}^4x\, \sqrt{-\tilde{g}}\, \left[\tilde{R} - \frac{1}{2} D^\mu \tilde{\Phi} D_\mu \tilde{\Phi} - \tilde{V}(\tilde{\Phi})\right]\,, \tag{13.21}$$

where

$$\tilde{V}(x) = e^{-\frac{2}{\sqrt{3}}x}\, V\left(e^{x/\sqrt{3}}\right)\,. \tag{13.22}$$

e) Fun incoming. The equation of motion for $f_{\mu\nu}$ reads

$$0 = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} - \frac{1}{2\beta}\left(f_{\mu\nu} - f\, g_{\mu\nu}\right)\,. \tag{13.23}$$

We want to solve this for $f_{\mu\nu}$, but the trace comes into way, so let us first consider the traced equation of motion:

$$0 = -R + \frac{3}{2\beta} f \quad \Rightarrow \quad f = \frac{2\beta}{3} R\,. \tag{13.24}$$

Inserting this and solving for the full tensor, we find

$$f_{\mu\nu} = 2\beta\left(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}\right)\,. \tag{13.25}$$

To insert this into the action, we have to compute

$$f_{\mu\nu} f^{\mu\nu} = 4\beta^2\left(R^{\mu\nu} R_{\mu\nu} - \frac{2}{9} R^2\right)\,, \qquad f^2 = 4\beta^2\, \frac{R^2}{9}\,. \tag{13.26}$$

We thus get

$$S^{\text{super-weird}} = \int \mathrm{d}^4x\, \sqrt{-g}\, \left[R + \beta\left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2\right)\right]\,. \tag{13.27}$$

Now, you might wonder why there is an R^2-term. The answer is that the above combination is equivalent to (half of) the square of the Weyl tensor upon subtracting a multiple of the Gauss-Bonnet term. With a lengthy computation (*cough* use xAct if you want to do that *cough*), one finds that (in flat spacetime) R^2 only contributes to spin zero, whereas C^2 only contributes to spin two. The more you know.

Feel free to check that deriving the metric equations of motion and inserting the solution for $f_{\mu\nu}$, you get the equations of motion of this action.

f) If by now, you are still doing this by hand, it is your fault ;) The theory gives rise to a kinetic term with the wrong sign and a mass term for the spin two (transverse traceless) part of $f_{\mu\nu}$. This is related to the so-called Fierz-Pauli action for the massive spin-two field, in case you want to read more about it.