# Quantum Gravity and the Renormalization Group

Assignment 8 – Dec 13+16

## Exercise 16: Fixed functions – an infinite number of couplings

*Motivation: So far we have studied finite sets of beta functions. However, the non-perturbative RG flow generates infinitely many terms that have to be accounted for. The purpose of this exercise is to learn how to deal with some of the complications that are added in such a case.*

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For this exercise, we will briefly leave the realm of gravity, and instead study a vector field $\Phi^a$ that lives in *three* Euclidean dimensions and has an $O(N)$ symmetry (you can think of this as a special case of a theory with $N$ scalar fields – the "vector" he refers to the $O(N)$ group, not to spacetime; the case $N = 1$ is related to the well-known Ising model). We will look at the following approximation for the effective average action:

$$\Gamma_k \simeq \int \mathrm{d}^3x \left[\frac{1}{2}(\partial_\mu\Phi_a)(\partial^\mu\Phi^a) + V_k(\rho)\right]. \tag{16.1}$$

For convenience, we introduced $\rho = \Phi_a\Phi^a/2$. The arbitrary $k$-dependent potential $V_k$ contains in general infinitely many interaction terms. If you want, you can think of it in terms of a Taylor expansion:

$$V_k(\rho) = \sum_{n\geq 1} c_{n,k}\,\rho^n\,. \tag{16.2}$$

In this, the $c_{n,k}$ are the infinitely many $k$-dependent couplings.

a) What are the mass dimensions of the field $\Phi^a$, the potential $V_k$, and the coupling constants $c_{n,k}$?

b) Given an expression for the RG flow of $V_k(\rho)$, i.e. $k\partial_k V_k(\rho)$, how would the flow of the corresponding dimensionless potential look like? *Hints:* To arrive at this, you could assume the above Taylor expansion, convert the couplings $c_{n,k}$ to their dimensionless counterparts, and then try to go back to the potential. It might also be useful to introduce a dimensionless version of $\rho$. If we call $v_k$ the dimensionless version of $V_k$, and $\bar\rho$ the dimensionless version of $\rho$, you should get a relation of the form

$$k\partial_k V_k(\rho) = a_1\,k\partial_k v_k(\bar\rho) + a_2\,v_k(\bar\rho) + a_3\,\bar\rho\,v_k'(\bar\rho)\,, \tag{16.3}$$

where $a_{1,2,3}$ are field-independent coefficients that you need to determine.

We will now study the large-$N$ limit of this theory, in which an exact solution can be obtained. This allows us to focus on the concepts while keeping analytic control. It will be useful to focus on the derivative of the dimensionless potential, $u(\bar\rho) \equiv v_k'(\bar\rho)$, where we now suppress the index $k$ to simplify notation. This is commonly done in the literature.
Suppose that in such a limit, the $k$-dependence of the dimensionless derivative of the potential is given by

$$k\partial_k u(\bar\rho) = -2u(\bar\rho) + \bar\rho\,u'(\bar\rho) - \frac{1}{2}\frac{u'(\bar\rho)}{(1+u(\bar\rho))^2}\,. \tag{16.4}$$

We will first focus on obtaining the fixed point solution, so we will try to solve the above differential equation for $k\partial_k u = 0$, i.e. the fixed point equation.

c) Let us first do some structural analysis: given that the above fixed point equation for $u$ is a first-order differential equation, how many integration constants do you expect? Does it make sense that a fixed point solution has integration constants? If not, what could fix these?

d) Let us now try to get some more feeling for the solution. We first try a Taylor expansion. Assume that

$$u_*(\bar{\rho}) = \sum_{n=0}^{N} d_{n*}\, \bar{\rho}^n \,. \tag{16.5}$$

Plug this ansatz into the fixed point equation, and compute the fixed point couplings $d_{n*}$ for a reasonable number $N$ by expanding the equation in powers of $\bar{\rho}$ as well (do this however far you want to, maybe something like $N = 4$). You should be able to solve for $N - n_0$ couplings, where $n_0$ is the number of free integration constants found in c).

**[hard question]** Think about what could be done to fix the integration constants. *Hint:* A commonly used technique is to set the highest retained coupling to zero, thus imposing another condition on the fixed point couplings. One then increases $N$ systematically and checks for the convergence of any candidate solution.

e) **[hard question]** The fixed point equation admits an implicit solution: instead of $u_*(\bar{\rho})$, one can solve for $\bar{\rho}(u_*)$. Show that indeed,

$$\bar{\rho} = q\,\sqrt{u_*} + \frac{3}{4} + \frac{3}{4}\sqrt{u_*}\,\arctan\sqrt{u_*} - \frac{1}{4}\frac{1}{1+u_*} \tag{16.6}$$

is the general solution, where $q$ is an integration constant. *Hint:* Assume from the start that $u_*(\bar{\rho})$ can be inverted so that you can write $\bar{\rho}(u_*)$. Then express $u_*'(\bar{\rho})$ in terms of $\bar{\rho}'(u_*)$ and solve the fixed point equation for $\bar{\rho}(u_*)$.

f) Let us now try to fix the integration constant $q$. For this, we can argue in the following way: the function $u_*(\bar{\rho})$ should be real and well-defined for all $\bar{\rho} \geq 0$. Does this already fix the integration constant? Also plot the resulting function $u_*(\bar{\rho})$ for this value of $q$.

g) Check that your polynomial solution from d) is consistent with the solution in e). Also compute the right value of your integration constant in the Taylor solution from f).

Now that we have a fixed point function, we can compute the critical exponents! How can we do so? Let us take a step back and remember how we compute critical exponents for a finite set of beta functions. We expand the beta functions to linear order about the fixed point:

$$\vec{\beta}(\vec{d}) \simeq \underbrace{\vec{\beta}(\vec{d}_*)}_{=0} + \left.\frac{\partial\vec{\beta}}{\partial\vec{d}}\right|_{\vec{d}=\vec{d}_*} (\vec{d} - \vec{d}_*)\,, \tag{16.7}$$

where we combined our above couplings $d$ in a vector, $\vec{d} = \{d_0, d_1, \dots\}$. The solution to this linearised differential equation is

$$\vec{d} \simeq \vec{d}_* + \sum_i c_i\, \vec{e}_i\, k^{-\theta_i}\,, \tag{16.8}$$

where $c_i$ are integration constants, and the $\vec{e}_i$ are the eigenvectors of the stability matrix with eigenvalue $\theta_i$.

Let us try to generalise this to the case of a full function. For this, we insert the linearised solution into $u$:

$$u(\bar{\rho}) = \vec{d} \cdot \{1, \bar{\rho}, \bar{\rho}^2, \dots\} = \underbrace{\vec{d}_* \cdot \{1, \bar{\rho}, \bar{\rho}^2, \dots\}}_{u_*(\bar{\rho})} + \sum_i c_i \left[ \vec{e}_i \cdot \{1, \bar{\rho}, \bar{\rho}^2, \dots\} \right] k^{-\theta_i} \,. \tag{16.9}$$

Here, we simply wrote the sum as a vector product of the coupling vector with the vector of all powers of $\bar{\rho}$. What is this expression? The first terms is clearly just the fixed function $u_*$. The second is a sum of different *eigenfunctions* that are in one-to-one correspondence with the eigenvectors of the stability matrix. This motivates the following way to compute critical exponents. Assume that

$$u(\bar{\rho}) \simeq u_*(\bar{\rho}) + \delta u(\bar{\rho})\, k^{-\theta} \,. \tag{16.10}$$

We plug this into the RG equation (16.4) and expand to linear order in perturbations $\delta u$. This gives a *linear* differential equation for the perturbations $\delta u$, and $\theta$ plays the role of an eigenvalue of a differential operator. We will try to put this into practice now.

h) **[hard question]** Show that the eigenvalue equation for the linear perturbation $\delta u$ reads

$$-\theta \delta u(\bar{\rho}) = -2\delta u(\bar{\rho}) + \bar{\rho}\, \delta u'(\bar{\rho}) + \frac{u_*'(\bar{\rho})}{(1+u_*(\bar{\rho}))^3} \delta u(\bar{\rho}) - \frac{1}{2} \frac{1}{(1+u_*(\bar{\rho}))^2} \delta u'(\bar{\rho}) \,. \tag{16.11}$$

i) **[hard question]** Use the fixed point equation to replace $u_*'(\bar{\rho})$ in terms of $u_*(\bar{\rho})$ and $\bar{\rho}$, and then perform a variable transform with the implicit fixed point solution $\bar{\rho}(u_*)$ to derive an equation for $\delta u(u_*)$.

j) **[hard question]** Solve this differential equation for $\delta u(u_*)$. This should look rather ugly. To make it nicer, recall that $\delta u$ is actually the $\bar{\rho}$-derivative of the corresponding perturbation of the potential, $\delta v$. From this, derive the perturbations $\delta v(u_*)$. You should get

$$\delta v(u_*) = c\, u_*^{\frac{3-\theta}{2}} \,, \tag{16.12}$$

with $c$ being (yet another) normalisation constant. Can you find an argument that would restrict (reasonable) perturbations which would also fix the "allowed" critical exponents?

a) The mass dimension of the action is still zero, but now the measure $\mathrm{d}^3x$ has dimension $-3$. From the kinetic term, we then get

$$2 + 2[\Phi] = 3 \qquad \Rightarrow [\Phi] = \frac{1}{2} \,. \tag{16.13}$$

The potential also has mass dimension three,

$$[V] = 3 \,. \tag{16.14}$$

Lastly, for the couplings, we have

$$[c_{n,k}] + n\,[\rho] = [c_{n,k}] + 2n\,[\Phi] = [c_{n,k}] + n = 3 \qquad [c_{n,k}] = 3 - n \,. \tag{16.15}$$

b) Let us take the logarithmic $k$-derivative of the dimensionful potential:

$$k\partial_k V_k(\rho) = \sum_{n\geq 1} (k\partial_k c_{n,k})\, \rho^n \,. \tag{16.16}$$

For the next step, we introduce dimensionless couplings,

$$c_{n,k} = \bar{c}_{n,k}\, k^{3-n} \,. \tag{16.17}$$

Inserting this above, we have

$$k\partial_k V_k(\rho) = \sum_{n\geq 1} \left[(3-n)\bar{c}_{n,k} + (k\partial_k \bar{c}_{n,k})\right]\, k^{3-n}\, \rho^n \,. \tag{16.18}$$

Let us next distribute the factors of $k$ to make the field dimensionless. With the result of a), we introduce the dimensionless $\bar{\rho}$ via

$$\rho = \bar{\rho}\, k \,. \tag{16.19}$$

Recall that the mass dimension of the potential is 3, so we will pull out a factor of $k^3$:

$$k\partial_k V_k(\rho) = k^3 \sum_{n\geq 1} \left[(3-n)\bar{c}_{n,k} + (k\partial_k \bar{c}_{n,k})\right]\, \bar{\rho}^n \,. \tag{16.20}$$

Finally, the dimensionless potential is given by (this follows from the same procedure as we just did: making both the couplings and the field dimensionless)

$$V_k(\rho) = v_k(\bar{\rho})\, k^3 = k^3 \sum_{n\geq 1} \bar{c}_{n,k}\, \bar{\rho}^n \,. \tag{16.21}$$

With this, we can try to understand how to relate the logarithmic $k$-derivative of the two. Looking at (16.20), we can see that we can write

$$\begin{aligned} k\partial_k V_k(\rho) &= 3\, k^3 \underbrace{\sum_{n\geq 1} \bar{c}_{n,k}\, \bar{\rho}^n}_{=v_k(\bar{\rho})} - k^3 \underbrace{\sum_{n\geq 1} n\, \bar{c}_{n,k}\, \bar{\rho}^n}_{=\bar{\rho}\, v_k'(\bar{\rho})} + k^3 \underbrace{\sum_{n\geq 1} (k\partial_k \bar{c}_{n,k})\, \bar{\rho}^n}_{=k\partial_k v_k(\bar{\rho})} \\ &= k\partial_k v_k(\bar{\rho}) + 3 v_k(\bar{\rho}) - \bar{\rho}\, v_k'(\bar{\rho}) \,. \end{aligned} \tag{16.22}$$

Comparing to the hint, we can thus read off

$$a_1 = 1 \,, \qquad a_2 = 3 \,, \qquad a_3 = -1 \,. \tag{16.23}$$

c) This is a first-order differential equation, so we would expect one integration constant. This seems sus, since if we would really have a free integration constant, we would find a fixed line instead of a fixed point. In other words, there would be an *exactly* marginal direction (which is unusual). Eventually this will be fixed by the global existence of the solution.

d) This is best done in Mathematica. It is easiest to leave $d_{0*}$ free, since all higher-order couplings can be uniquely solved in terms of it. Specifically, the first few couplings read

$$\begin{aligned} d_{1*} &= -4 d_{0*} (1 + d_{0*})^2 \,, \\ d_{2*} &= 4 d_{0*} (1 + d_{0*})^3 (1 + 5 d_{0*}) \,, \\ d_{3*} &= -32 d_{0*}^2 (1 + d_{0*})^4 (1 + 3 d_{0*}) \,, \\ d_{4*} &= 32 d_{0*}^3 (1 + d_{0*})^5 (5 + 13 d_{0*}) \,, \\ &\vdots \end{aligned} \tag{16.24}$$

As written in the hint, if we wouldn't have access to an exact solution, we would now systematically increase $N$, set the highest coefficient to zero (which is a polynomial with increasing degree as $N$ increases), and check for solutions that are stable under this extension of the approximation.

As a side effect, you can immediately realise that we also generate many *fiducial fixed points* via this procedure. These are artefacts of the approximation. Identifying valid fixed points and distinguishing them from fiducial ones can, in practice, be a headache. No general procedures are known that would solve this issue. There is even some evidence that in some systems (and with a different kind of renormalisation group flow equation than we are using, the Dyson-Schwinger equation), this procedure doesn't converge to the true solution, see Bender, Karapoulitidis, Klevansky, Phys. Rev. Lett. 130 (2023) 10, 101602. So be vigilant.

e) We can simply use that if $u_*(\bar{\rho})$ is invertible, then

$$u_*'(\bar{\rho}) = \frac{\mathrm{d}u_*}{\mathrm{d}\bar{\rho}} = \frac{1}{\bar{\rho}'(u_*)} \,. \tag{16.25}$$

With this, we can write the original fixed point equation as

$$0 = -2u_* + \frac{\bar{\rho}(u_*)}{\bar{\rho}'(u_*)} - \frac{1}{2}\frac{1}{\bar{\rho}'(u_*)}\frac{1}{(1+u_*)^2} \,. \tag{16.26}$$

This can be brought into a standard form,

$$0 = -2u_*\,\bar{\rho}'(u_*) + \bar{\rho}(u_*) - \frac{1}{2}\frac{1}{(1+u_*)^2} \,. \tag{16.27}$$

We then pull out the integrating factor,

$$0 = -2u_*^{3/2}\,\frac{\mathrm{d}}{\mathrm{d}u_*}\left(\frac{\bar{\rho}(u_*)}{\sqrt{u_*}}\right) - \frac{1}{2}\frac{1}{(1+u_*)^2} \,. \tag{16.28}$$

This can be solved straight-forwardly, and gives the quoted solution.

f) There are different ways of going about this. One way is the following. Let us first check what we get when we set $u_* = 0$. In this case,

$$\bar{\rho}(0) = \frac{1}{2} \,. \tag{16.29}$$

Said differently,

$$u_*(1/2) = 0 \,. \tag{16.30}$$

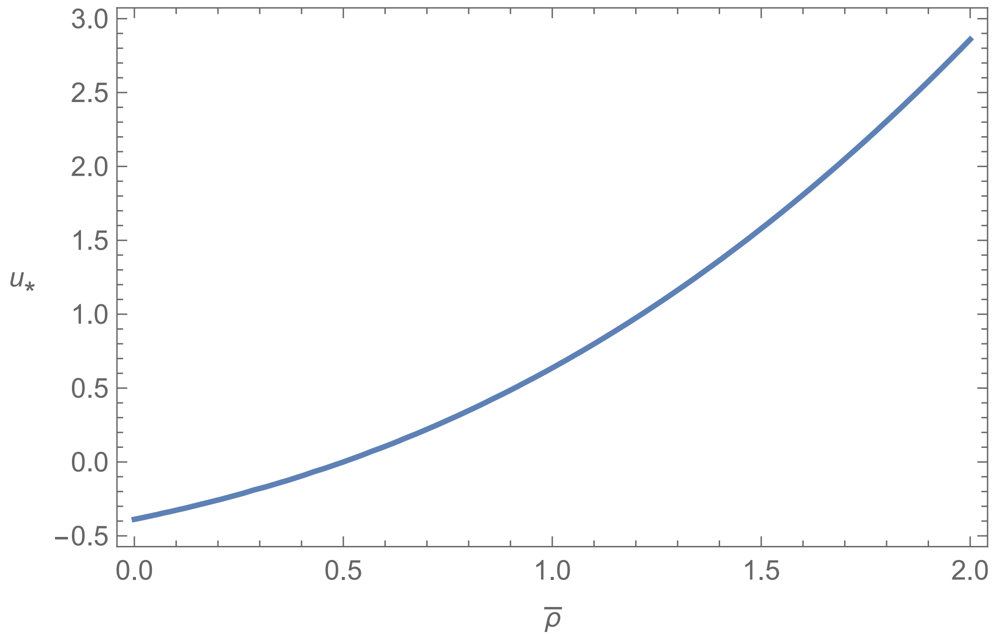
We thus find that $u_*$ vanishes at a finite, positive value of $\bar{\rho}$. Since we want a solution for all non-negative $\bar{\rho}$, we also need that the derivative of the potential exists and is finite at this point. This in turn implies that also the $u_*$-derivative of $\bar{\rho}$ at $u_* = 0$ is finite. Expand the derivative of the solution, $\bar{\rho}'(u_*)$, about $u_* = 0$, we however find

$$\bar{\rho}'(u_*) \sim \frac{1}{2}\frac{q}{\sqrt{u_*}} + 1 + \dots \,. \tag{16.31}$$

Requiring finiteness as $u_* \to 0$ then yields

$$q = 0 \,. \tag{16.32}$$

The solution is a visually rather boring function. Pro tip: You can easily plot $u_*(\bar{\rho})$ with the help of *ContourPlot*.




g) To check your solution, you can insert the Taylor expansion of $u_*$ into the expression for $\bar{\rho}(u_*)$ and then expand in powers of $\bar{\rho}$. This should be a true equation order by order in $\bar{\rho}$. You will find that all of these equations are fulfilled simultaneously if

$$2 + 3d_0 + 3\sqrt{d_0}\,\arctan\sqrt{d_0}(1 + d_0) = 0\,, \tag{16.33}$$

whose relevant numerical solution is

$$d_0 \approx -0.388346718912782825\,. \tag{16.34}$$

h) We plug the expansion (16.10) into (16.4). The $k$-independent term vanishes by definition of the fixed point condition. To linear order in the perturbation $k$, we then get on the left-hand side

$$k\partial_k u(\bar{\rho}) \simeq -\theta\,\delta u(\bar{\rho})\,k^{-\theta}\,. \tag{16.35}$$

On the right-hand side, we have

$$\left[-2\delta u(\bar{\rho}) + \bar{\rho}\,\delta u'(\bar{\rho}) + \frac{u_*'(\bar{\rho})}{(1+u_*(\bar{\rho}))^3}\delta u(\bar{\rho}) - \frac{1}{2}\frac{1}{(1+u_*(\bar{\rho}))^2}\delta u'(\bar{\rho})\right]\,k^{-\theta}\,. \tag{16.36}$$

Equating both sides, we get the claimed equation,

$$-\theta\,\delta u(\bar{\rho}) = -2\delta u(\bar{\rho}) + \bar{\rho}\,\delta u'(\bar{\rho}) + \frac{u_*'(\bar{\rho})}{(1+u_*(\bar{\rho}))^3}\delta u(\bar{\rho}) - \frac{1}{2}\frac{1}{(1+u_*(\bar{\rho}))^2}\delta u'(\bar{\rho})\,. \tag{16.37}$$

i) We can take a $\bar{\rho}$-derivative of the fixed point solution (16.6) (with $q = 0$), and get

$$1 = \frac{1}{8}\left[3\frac{\arctan\sqrt{u_*(\bar{\rho})}}{\sqrt{u_*(\bar{\rho})}} + \frac{5+3u_*(\bar{\rho})}{(1+u_*(\bar{\rho}))^2}\right]\,u_*'(\bar{\rho})\,, \tag{16.38}$$

which we can solve for the derivative,

$$u_*'(\bar{\rho}) = \frac{8}{3\frac{\arctan\sqrt{u_*(\bar{\rho})}}{\sqrt{u_*(\bar{\rho})}} + \frac{5+3u_*(\bar{\rho})}{(1+u_*(\bar{\rho}))^2}}\,. \tag{16.39}$$

If we then also use the chain rule to express

$$\frac{\mathrm{d}\delta u}{\mathrm{d}\bar{\rho}} = \frac{\mathrm{d}\delta u}{\mathrm{d}u_*}\frac{\mathrm{d}u_*}{\mathrm{d}\bar{\rho}} = \frac{\delta u'(u_*)}{\bar{\rho}'(u_*)}\,, \tag{16.40}$$

in finitely many steps we can write the equation for the perturbation as

$$2u_*\,\delta u'(u_*) + \left[\theta - 2\frac{\sqrt{u_*}(1+u_*(8+3u_*)) + 3(1+u_*)^3 \arctan\sqrt{u_*}}{\sqrt{u_*}(1+u_*)(5+3u_*) + 3(1+u_*)^3 \arctan\sqrt{u_*}}\right]\delta u(u_*) = 0\,. \tag{16.41}$$

j) The general solution for $\delta u(u_*)$ reads

$$\delta u(u_*) = \hat{c}\,\frac{u_*^{1-\frac{\theta}{2}}(1+u_*)^2}{\sqrt{u_*}(5+3u_*) + 3(1+u_*)^2 \arctan\sqrt{u_*}}\,, \tag{16.42}$$

where $\hat{c}$ is the integration constant. If we write this suggestively as

$$\delta u(u_*) = \hat{c}\,u_*^{\frac{1-\theta}{2}}\,\frac{1}{3\frac{\arctan\sqrt{u_*}}{\sqrt{u_*}} + \frac{5+3u_*}{(1+u_*)^2}}\,, \tag{16.43}$$

and compare to previous computations, we see that there is a hidden $\bar{\rho}$-derivative here. Indeed,

$$\delta u(u_*) = \frac{\mathrm{d}\delta v(u_*)}{\mathrm{d}\bar{\rho}} = \frac{\delta v'(u_*)}{\bar{\rho}'(u_*)}\,. \tag{16.44}$$

We thus have

$$\delta v'(u_*) = \tilde{c}\,u_*^{\frac{1-\theta}{2}}\,, \tag{16.45}$$

with a suitable constant $\tilde{c}$, or

$$\delta v(u_*) = c\,u_*^{\frac{3-\theta}{2}}\,, \tag{16.46}$$

with yet another (arbitrary) constant $c$.

We want the perturbations to be analytic in $\bar{\rho}$, at least for small $\bar{\rho}$. Since there is a bijective map between non-negative $\bar{\rho}$ and $u_*$, this indicates that the exponent should be a non-negative integer. This entails the spectrum of critical exponents

$$\theta_i = 3 - 2i\,. \tag{16.47}$$

As you can see, almost all of these are irrelevant! By the way, the eigenfunction with $\theta = 3$ is just a constant – i.e., we shift the potential by a constant value. This is usually not counted as a relevant parameter outside gravity, as it corresponds to the cosmological constant, which is inessential in non-gravitational theories as it doesn't feed back into the renormalisation group flow.

A final remark: you might wonder about the integration constant $c$ in the perturbation. This is nothing else than the equivalent of the normalisation of the eigenvector of the stability matrix when considering finitely many couplings.