Quantum Gravity and the Renormalization Group

Assignment 9 – Jan 10+13

Exercise 17: Computing an RG flow

Motivation: Stuff is getting real now – we will derive the beta function for the potential that you saw in Exercise 16 to learn how the Wetterich equation works in practice. We will also introduce some general concepts that will come in handy when going back to gravity.

In this exercise, we will learn how to evaluate the right-hand side of the Wetterich equation:

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)^{-1} \cdot k\partial_k \mathfrak{R}_k\right] . \tag{17.34}$$

Here, we used the "supertrace" (STr) to indicate a generalisation of the loop momentum integral that we will explain soon. This will be important later when we look at gravity.
Our system will be the $O(N)$-symmetric vector model in three dimensions. Our ansatz for the effective average action Γ_k is the same as in Exercise 16:

$$\Gamma_k \simeq \int \mathrm{d}^3x \left[\frac{1}{2}(\partial_\mu \Phi_a)(\partial^\mu \Phi^a) + V_k(\rho)\right] . \tag{17.35}$$

Recall that $\rho = \Phi_a\Phi^a/2$. The goal is to plug this ansatz into (17.34) and find the expression for $k\partial_k V_k$.

a) Let us start with the easiest bit – the left-hand side. Evaluate the left-hand side of (17.34) for the ansatz (17.35).

Next, let us construct the right-hand side step by step. We will be very careful in the notation at every step – only once you have understood what is going on, you can become sloppy (like everybody working on it)!

b) The first ingredient that we need is the second variation of the effective average action. Compute

$$\frac{\delta^2 \Gamma_k}{\delta\Phi^a(x)\,\delta\Phi^b(y)} . \tag{17.36}$$

For this, recall that by definition,

$$\frac{\delta\Phi^a(x)}{\delta\Phi^b(y)} = \delta_b{}^a\,\delta(x-y) . \tag{17.37}$$

Here, the delta function is three-dimensional.

Next, we have to choose some form of regularisation. Recall that in the path integral, we added a term to the action in the form of

$$\Delta S_k = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\tilde{\Phi}^a(-p)\,\mathfrak{R}_{k,ab}(p^2)\,\tilde{\Phi}^b(p) . \tag{17.38}$$

This is in momentum space. Often, we actually work in position space, and then this reads

$$\Delta S_k = \frac{1}{2}\int \mathrm{d}^3x\, \Phi^a\, \mathfrak{R}_{k,ab}(-\partial^2)\, \Phi^b\,. \tag{17.39}$$

Note how the regulator $\mathfrak{R}_k$ actually needs to carry ($O(N)$) indices, since the field also carries indices. In our case, we can choose the regulator diagonal in field space,

$$\mathfrak{R}_{k,ab}(-\partial^2) = \delta_{ab}\,\mathcal{R}_k(-\partial^2)\,. \tag{17.40}$$

Now, $\mathcal{R}_k$ is a simple function (that happens to have an operator as its argument). Note the different type setting to make the difference (the literature often does not differentiate this).

c) Compute how the above regulator gets added to the two-point function $\Gamma_k^{(2)}$ by taking the second variation of ΔS_k in position space.

d) In the same language, what do we actually mean by the $k\partial_k\mathfrak{R}_k$ in (17.34)?

The easy part is over now – the next step is to invert the regularised two-point function. First, we discuss some conceptual aspects, then the actual computation.

e) The two-point function found in b) depends on two points x and y. You should have found however that it is local, that is, it can be written as

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)(x,y) = \mathcal{O}(x)\,\delta(x-y)\,. \tag{17.41}$$

Here, $\mathcal{O}(x)$ is some differential operator. The "local" refers to the delta function. Now, its inverse – the propagator $G_k(x,y)$ – is defined by

$$\int \mathrm{d}^3y\, G_k(x,y)\,\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)(y,z) = \delta(x-z)\,. \tag{17.42}$$

From this definition, argue that the propagator must also be local in the above sense, and in particular formally of the structure

$$G_k(x,y) = \left[\mathcal{O}(x)\right]^{-1}\,\delta(x-y)\,. \tag{17.43}$$

Our next task is thus to find $\mathcal{O}^{-1}$. For this, we need an *expansion scheme*, since we will not be able to do this without approximations. To decide on what we really need, consider again a). We really only need terms that contribute to the potential – terms with derivatives acting on fields contribute to other couplings. An expansion in derivatives like this is called . . . wait for it . . . *derivative expansion*. In going forward, we will thus assume that we can neglect $\partial_\mu\Phi^a \simeq 0$.

f) In the lowest order derivative expansion where we set $\partial_\mu\Phi^a \simeq 0$ (that is, Φ^a is approximately constant, $\Phi^a(x) \simeq \Phi^a$), compute the regularised propagator. *Hints:* Write the regularised two-point function in the form

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)_{ab}(x,y) = \left\{A(-\partial^2,\Phi^2)\left(\delta_{ab} - \frac{\Phi_a\Phi_b}{\Phi^2}\right) + B(-\partial^2,\Phi^2)\frac{\Phi_a\Phi_b}{\Phi^2}\right\}\delta(x-y)\,, \tag{17.44}$$

and determine the functions A, B. Do the two operators multiplying A and B look somewhat familiar? (Try to take their sum and products!) Conclude that you can also write the propagator as

$$G^{ab}(x,y) = \left\{ \left(\delta^{ab} - \frac{\Phi^a \Phi^b}{\Phi^2}\right) C(-\partial^2, \Phi^2) + \frac{\Phi^a \Phi^b}{\Phi^2} D(-\partial^2, \Phi^2) \right\} \delta(x-y)\,, \tag{17.45}$$

and compute C and D from A and B.

The last step is to make sense out of the STr. First of all, you might have noticed the $\cdot$ that I slipped into (17.34). Since both the propagator and the (k-derivative of the) regulator depend on two points, what we once again mean by this is

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)^{-1} \cdot k\partial_k \mathfrak{R}_k \equiv \int \mathrm{d}^3 y \, \left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)^{-1} (x,y)\; k\partial_k \mathfrak{R}_k(y,z)\,. \tag{17.46}$$

g) Reinstate the $O(N)$ indices and compute (17.46) by using the results of the previous parts.

Last but not least, we define the supertrace via the integral over the coincidence limit, plus taking any "standard" index trace (tr):

$$\mathrm{STr}\, F(x,y) = \int \mathrm{d}^3 x \,\mathrm{tr} \lim_{y\to x} F(x,y)\,. \tag{17.47}$$

This of course is more complicated in gravity, where all hell breaks lose, but this is a problem for future-you, not present-you.

h) Use the definition of the supertrace to finally evaluate the right-hand side of the flow equation! *Hints:* It could be helpful to represent the delta function in terms of its Fourier representation,

$$\delta(x-y) = \int \frac{\mathrm{d}^3 \ell}{(2\pi)^3}\, e^{-i\,\ell\cdot(x-y)}\,. \tag{17.48}$$

Use this to convert any $-\partial^2$ into momenta, and take the continuum limit. Finally, the tr is the trace (aka contraction) over the two remaining $O(N)$ indices. Recall for this that ${\delta_a}^a = N$.

Don't perform the momentum integrals yet (you can't anyway without specifying the shape of the regulator). Rather, your result should look like

$$k\partial_k V_k(\rho) = (N-1) I_1(\rho) + I_2(\rho)\,, \tag{17.49}$$

where $I_{1,2}$ are two different integrals.

i) To evaluate the integrals, use the Litim regulator,

$$\mathcal{R}_k(z) = (k^2 - z)\,\theta\left(1 - \frac{z}{k^2}\right)\,, \tag{17.50}$$

where θ is the Heaviside function.

j) As the very last step, let us reproduce (16.25). For this, we rescale the field and the potential via

$$\Phi^a \mapsto c\,\sqrt{N}\,\hat{\Phi}^a\,, \qquad V(\rho) \mapsto c^2\,N\,\hat{V}(\hat{\rho})\,, \tag{17.51}$$

for a suitable constant c. Then take a ρ-derivative of the flow equation (since (16.25) is the flow for the derivative of the potential) and take the limit $N \to \infty$. What is the right value for c to achieve a match?

k) **[hard question]** Think about systematic extensions of our approximation. You can take the following questions as an orientation:

- Where does the statement come from that the Wetterich equation generates all terms compatible with symmetries?
- Related to this, how can we systematically extend our approximation? For example, how would we take into account terms that include factors of $\partial_\mu \Phi^a$, and which beta functions would we be able to compute by keeping these terms?
- Do you think everything is going through in the same way with gravity? If not, what could go wrong?