

Quantum Gravity and the Renormalization Group

Assignment 1 – Oct 20

Exercise 1: Counting gravitons

Motivation: In this exercise, we estimate the number of gravitons emitted in a binary merger. This gives us an idea whether the detection of single gravitons from such events is feasible.

Consider a binary merger of two solar mass ($m \sim 2 \times 10^{30} \text{kg}$) black holes. Estimate the number of gravitons N_h emitted via gravitational waves.

For this, assume that about $r = 5\%$ of the total mass is radiated away in mono-“chromatic” gravitational waves with a frequency of about 100 Hz. Given this number and keeping in mind the huge experimental effort in detecting gravitational waves, do you find it likely that we will detect single gravitons in the near future?

Exercise 2: Einstein’s equations

Motivation: We review the derivation of Einstein’s equations from an action. In this exercise, we will do so with the background field method. The goal is to get some (more) practice with computing perturbations, which will come in very handy very soon.

Einstein’s equations are the equations of motion of General Relativity, and can be derived from an action principle. The action of General Relativity is the Einstein-Hilbert action, given by

$$S_{\text{EH}}[g] = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\det g} (R - 2\Lambda), \quad (2.I)$$

where G_N is Newtons constant, R is the Ricci scalar and Λ the cosmological constant. Derive Einstein’s equations from the requirement of a stationary action,

$$\frac{\delta S[g]}{\delta g_{\mu\nu}(x)} = 0. \quad (2.II)$$

Exercise 3: Linearised Einstein’s equations

Motivation: This short exercises builds upon exercise 2 to derive the wave equation for linear metric perturbations about Minkowski space — a.k.a. gravitational waves.

Linearise Einstein’s field equations about Minkowski space, that is, consider

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0, \quad (3.I)$$

with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (3.II)$$

to linear order in h . Why do we have to set Λ to zero for this?

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Assignment 2 – Oct 27

Exercise 4: Alternative gravitational theories

Motivation: We will now investigate alternative theories for gravity. This has both theoretical and experimental motivations: on the theoretical side, we know that General Relativity breaks down, e.g., at the centre of black holes, and a quantum theory has to take over. On the experimental side, with recent observations of gravitational waves and black hole shadows, we will soon be able to test deviations from General Relativity. We will discuss these theories in some more detail in the lectures soon.

Let us study the simplest extensions of General Relativity by adding terms to the action with four derivatives of the metric, and investigate what impact these have on gravitational waves.

- Find a basis of all independent terms with (in total) four derivatives that are constructed purely in terms of the metric, and that respect diffeomorphism invariance. *Hints:* This should involve curvatures and covariant derivatives only. How many derivatives of the metric does the Riemann tensor have, and what are its symmetries? Also, how many uncontracted indices can terms in the action have?
- Derive the equations of motion from the generalised action that you have found.
- [hard question]** Let us consider one specific (and phenomenologically very important) model, the Starobinsky theory. Its action is that of General Relativity with an R^2 -term added,

$$S^{\text{Staro}} = \int d^4x \sqrt{-\det g} \left[\frac{R - 2\Lambda}{16\pi G_N} + \alpha R^2 \right], \quad (4.I)$$

so it is a special case of the theory above. From the equations of motion that you derived in b), find out if solutions to Einstein's equations are still solutions to the Starobinsky theory. If yes, where is the “new physics” in this theory?

- [hard question]** Back to the full system of b): linearise the equations of motion in Minkowski space. Do gravitational waves propagate differently now? Can these differences be removed by a suitable gauge fixing?

Exercise 5: Evidence for Quantum Gravity?

In the lecture, we discussed an argument for the quantisation of gravity that relies on what the gravitational field would be of a superposition of a massive object at different positions.

In 1981, Page and Geilker wrote a paper about an experiment they conducted ([Phys. Rev. Lett. 47 \(1981\) 979-982](#)). Read the paper and critically comment on whether their experiment realises the above argument.

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Assignment 3 – Nov 03

Exercise 6: Superficial degree of divergence

Motivation: In this exercise, we discuss the superficial degree of divergence of a scalar theory that mimics the divergence count of General Relativity. The goal is to reinforce your understanding of why the perturbative quantisation fails.

As we have discussed in the lecture, General Relativity generates new divergences at every loop order. This concretely means that the loop divergences that are created come in a form that *cannot* be absorbed by a renormalisation of the Einstein-Hilbert terms in the action. To understand this in more depth, we will first consider a scalar toy model, and then we'll try to translate what we understand to the case of gravity.

- a) Consider a ϕ^4 theory in four-dimensional Minkowski space with a microscopic action

$$S^{\phi^4} = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{\lambda_4}{4!} \phi^4 \right]. \quad (6.I)$$

At a structural level, i.e. *without explicitly calculating/evaluating any diagram*, argue what the superficial degree of divergence is of the loop diagrams contributing to the renormalisation of the four-scalar-vertex. Argue that at every loop order, you can absorb the divergences in a renormalisation of the coupling λ (and potentially a wave-function renormalisation of the field and a mass term).

- b) Now consider ϕ^6 theory, also in four-dimensional Minkowski space, with the microscopic action

$$S^{\phi^6} = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{\lambda_6}{6!} \phi^6 \right]. \quad (6.II)$$

Once again at a structural level, discuss the superficial degree of divergence for the four- and six-scalar vertex *at one-loop order*. Can this theory be renormalised perturbatively?

- c) **[hard question]** Next, consider a scalar field theory with a derivative interaction:

$$S^{\partial\phi} = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{\lambda_\partial}{(2!)^2} (\partial_\mu \phi) (\partial^\mu \phi) \phi^2 \right]. \quad (6.III)$$

What is the one-loop divergence structure in this case? Can you absorb all divergences of the four-scalar vertex?

- d) **[hard question]** Finally, let us discuss the degree of divergence in General Relativity. In a similar way as for the scalar field theories above, argue what the divergence structure is at one and two loops. For this, consider the divergences of the two-graviton vertex. Can the divergences be absorbed, or do we have to introduce structurally new counterterms? If the latter, how do they look like?

Exercise 7: *[Presence]* Computing the one-loop divergence in gravity with xAct

Motivation: In this exercise, we will explicitly compute the one-loop divergence in General Relativity with the help of xAct. You do not have to bring your own laptop, the Mathematica file will be sent around afterwards.

Exercise 8: *[Presence]* Counting degrees of freedom

Motivation: In this exercise, we will go into some more detail on how we can count the number of physical degrees of freedom in a gauge theory.

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Assignment 4 – Nov 10

Exercise 9: Approaches to Quantum Gravity

Motivation: To get a small overview of the field of Quantum Gravity, we will discuss some specific approaches and how they (aim to) solve the predictivity problem of perturbative renormalisation. The different approaches are expected to give rise to different physical effects. However, it is not yet understood whether they really do, because most quantum-gravity approaches are quite involved and it is very difficult to extract physical predictions. In addition, because the Planck scale is so far removed from scales that experiments can probe directly, it is difficult to check any predictions.

Read up on one (or more, if you like) approach to Quantum Gravity. What is the respective starting point/idea? How would it solve the issue that in a perturbative quantisation, predictivity breaks down? Is predictivity restored? If so, how? What is the current state of the approach, and what are the major open questions? Pick out of these approaches:

- String Theory
- Loop Quantum Gravity
- Causal Set Theory
- Causal Dynamical Triangulations
- Group Field Theory

Do **not** pick Asymptotic Safety — this will be discussed during this course!

Exercise 10: Effective field theory reasoning and the cosmological constant

Motivation: To get some more feeling about what the scale of new (gravitational) physics could be, we will have a look at the action from the point of view of effective field theory.

Based on the standard effective field theory treatment, what would you expect the size of the cosmological constant to be if the action is written in the form

$$\Gamma = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\det g} (-2\Lambda + R + \dots)? \quad (10.I)$$

Compare to the observational result that the vacuum energy density is $\rho_\Lambda = \frac{\Lambda}{8\pi G} \approx 10^{-47} \text{ GeV}^4$.

Discuss, based on this result, where you would expect the “scale of new physic” to be. Is there actually new physics there?

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Assignment 5 – Nov 17

Exercise 11: Ostrogradsky instabilities

Motivation: Higher-curvature invariants imply higher-order equations of motion. Here, we find out what this means for a toy model – a single higher-derivative harmonic oscillator.

We consider a single particle, which is a harmonic oscillator with a higher-derivative correction controlled by a parameter ϵ , namely

$$\mathcal{L} = -\frac{\epsilon m}{2\omega^2} \ddot{x}^2 + \frac{m}{2} \dot{x}^2 - \frac{m\omega^2}{2} x^2. \quad (11.I)$$

We will now study its solutions, count degrees of freedom, and look out for instabilities.

- a) Show that the Euler-Lagrange equation for the Lagrangian in (11.I) is

$$\frac{\epsilon}{\omega^2} \ddot{x} \dddot{x} + \ddot{x} + \omega^2 x, \quad (11.II)$$

which has the solution

$$x(t) = C_+ \cos(k_+ t) + S_+ \sin(k_+ t) + C_- \cos(k_- t) + S_- \sin(k_- t), \quad (11.III)$$

with the two frequencies

$$k_{\pm} = \omega \sqrt{\frac{1 \mp \sqrt{1 - 4\epsilon}}{2\epsilon}}, \quad (11.IV)$$

and where $C_{\pm}$ and $S_{\pm}$ are integration constants.

Bottom line: We need four pieces of initial data. Think before reading on: How many degrees of freedom usually require four pieces of initial data?

- b) Ostrogradsky defined the canonical phase-space coordinates for second-order derivative Lagrangians as

$$\begin{aligned} X_1 &\equiv x, & P_1 &\equiv \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}}, \\ X_2 &\equiv \dot{x}, & P_2 &\equiv \frac{\partial L}{\partial \ddot{x}}. \end{aligned} \quad (11.V)$$

Show that in this case

$$P_1 = m\dot{x} + \frac{\epsilon m}{\omega^2} \ddot{x}, \quad P_2 = -\frac{\epsilon m}{\omega^2} \ddot{x}. \quad (11.VI)$$

Check that this is equivalent to

$$\ddot{x} = \omega^2 \frac{P_1 - mX_2}{\epsilon m}, \quad \ddot{x} = -\frac{\omega^2 P_2}{\epsilon m}. \quad (11.VII)$$

c) The Hamiltonian is defined as

$$H \equiv \sum_{i=1}^2 P_i \dot{X}_i - L. \quad (11.VIII)$$

Demonstrate that for our model, the Hamiltonian can be expressed as

$$H = P_1 X_2 - \frac{\omega^2}{2\epsilon m} P_2^2 - \frac{m}{2} X_2^2 + \frac{m\omega^2}{2} X_1^2. \quad (11.IX)$$

Think before moving on: What's problematic with this Hamiltonian?

d) Show that, evaluated on the solution in a), the Hamiltonian reads

$$H = \frac{m}{2} \sqrt{1 - 4\epsilon} [k_+^2 (C_+^2 + S_+^2) - k_-^2 (C_-^2 + S_-^2)]. \quad (11.X)$$

What's the difference between the two modes?

e) **[hard question]** The Hamiltonian in (11.X) is not bounded from above and below (why?). Yet, we could solve the differential equation in (11.II) to obtain the solution in (11.III) without instabilities in sight. Why is that? Will this property persist for more complicated models? For those who want to dive into this a bit more, there is a recent [example](#) in the literature.

Exercise 12: Spin projectors

Motivation: A symmetric rank-two field $h_{\mu\nu}$ in flat spacetime carries ten components, but only two correspond to the physical graviton polarisations. The spin-projector formalism provides a clean algebraic decomposition of $h_{\mu\nu}$ into its irreducible components under the little group $SO(3)$. This exercise builds the spin projectors step by step and connects them to the propagating degrees of freedom in Einstein gravity.

We work in four-dimensional Minkowski spacetime with metric $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$. Fourier modes with momentum p_μ can be classified in representations of the little group – projected on and normal to the momentum. Correspondingly, define the transverse and longitudinal projectors

$$\theta_\mu{}^\nu = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2}, \quad \omega_\mu{}^\nu = \frac{p_\mu p^\nu}{p^2}. \quad (12.I)$$

a) Show that θ and ω are orthogonal and idempotent:

$$\theta_\mu{}^\rho \omega_\rho{}^\nu = 0, \quad \theta_\mu{}^\rho \theta_\rho{}^\nu = \theta_\mu{}^\nu, \quad \omega_\mu{}^\rho \omega_\rho{}^\nu = \omega_\mu{}^\nu. \quad (12.II)$$

Based on this result, argue that $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$ are a complete set projectors in the space of four-vectors.

b) Discuss which of the tensors in (12.I) project along the momentum, and which one onto the subspace normal to the momentum. What is the little group, i. e. the symmetry group of the normal subspace?

c) Decompose the four-vector A_μ into irreducible representations of the little group with the help of $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$. Which representations are those?

- d) **Do this exercise before reading on:** A symmetric rank-two tensor $h_{\mu\nu} = h_{\nu\mu}$ like the graviton transforms as the tensor product of two four-vectors under Lorentz transformations. What are the transformation properties of the different objects one can construct from $h_{\mu\nu}$ and combinations of $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$ under the little group?
- e) To project onto the different representations, we need new projectors. Using θ and ω , show that requiring idempotency singles out the projectors

$$\begin{aligned}
\mathcal{P}_{\mu\nu}^{(2)\rho\sigma} &= \frac{1}{2} (\theta_\mu{}^\rho \theta_\nu{}^\sigma + \theta_\mu{}^\sigma \theta_\nu{}^\rho) - \frac{1}{3} \theta_{\mu\nu} \theta^{\rho\sigma}, \\
\mathcal{P}_{\mu\nu}^{(1)\rho\sigma} &= \frac{1}{2} (\theta_\mu{}^\rho \omega_\nu{}^\sigma + \theta_\mu{}^\sigma \omega_\nu{}^\rho + \theta_\nu{}^\rho \omega_\mu{}^\sigma + \theta_\nu{}^\sigma \omega_\mu{}^\rho), \\
\mathcal{P}_{\mu\nu}^{(0,s)\rho\sigma} &= \frac{1}{3} \theta_{\mu\nu} \theta^{\rho\sigma}, \\
\mathcal{P}_{\mu\nu}^{(0,w)\rho\sigma} &= \omega_{\mu\nu} \omega^{\rho\sigma},
\end{aligned} \tag{12.III}$$

for the transverse-traceless, vector, and the two scalar modes (in that order). Show that these operators are an idempotent, orthogonal set of tensors, i.e.

$$\mathcal{P}_{\mu\nu}^{(i)\alpha\beta} \mathcal{P}_{\alpha\beta}^{(j)\rho\sigma} = \delta^{ij} \mathcal{P}_{\mu\nu}^{(i)\rho\sigma}, \quad \sum_i \mathcal{P}_{\mu\nu}^{(i)\rho\sigma} = \frac{1}{2} (\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\sigma \delta_\nu{}^\rho) \equiv \mathbb{1}_{\mu\nu}{}^{\rho\sigma}, \tag{12.IV}$$

where $\mathbb{1}_{\mu\nu}{}^{\rho\sigma}$ denotes the identity in the space of symmetric Lorentz two-tensors.

- f) Verify that the traces of the projectors, $\mathcal{P}_{\mu\nu}^{(i)\mu\nu}$, are 5, 3, 1, 1, respectively. Why is this a good cross check, i.e. what do these numbers mean? Which projector projects onto which representation of the little group?

Bonus: Derive the values of these traces purely from group theory.

- g) Does the above set (12.III) form a basis in the space of rank-(2, 2) tensors that map symmetric rank-two tensors to symmetric rank-two tensors? If not, find the missing element(s). Are these new elements projectors? Compute all possible products of basis elements.
- h) Take the graviton propagator in de Donder gauge from the lecture notes,

$$\mathcal{P}_{\mu\nu\rho\sigma} \simeq \frac{1}{p^2} \left[\mathbb{1}_{\mu\nu\rho\sigma} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right], \tag{12.V}$$

and decompose it into the basis that you have found. What do you learn from this?

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Assignment 6 – Nov 24

Exercise 13: Avoiding Ostrogradski?

Motivation: In the lecture, we discussed the Ostrogradski problem of quadratic gravity. In this exercise, we will try to find ways to avoid the negative aspects without sacrificing the positive aspects of the higher-derivative terms.

The spin-two propagator in quadratic gravity has the structural form

$$\frac{4\lambda}{p^4 - \frac{\lambda M_{\text{Pl}}^2}{2} p^2}. \quad (13.I)$$

As we discussed in some detail, the fall-off with p^4 for large momenta (together with the fact that vertices also scale like the fourth power of momentum) make the theory renormalisable, and in particular asymptotically free. However, the partial fraction decomposition of the propagator,

$$\frac{8}{M_{\text{Pl}}^2} \left[\frac{1}{-p^2} - \frac{1}{-p^2 + \frac{\lambda}{2} M_{\text{Pl}}^2} \right], \quad (13.II)$$

indicates that we have a massive ghost. Let us investigate different potential scenarios in how we could avoid it.

- a) First, consider the propagator of a general quartic theory. Does it always propagate a ghost? Why?
- b) **[hard question]** Suppose now that we start with a propagator that is the sum of two modes that both come with a positive prefactor, so that we do not propagate a ghost. Try to reconstruct the action that gives rise to such a propagator. Are there any problems with such a theory? Is this a local theory? What is its UV behaviour?
- c) **[hard question]** Suppose that instead of quadratic gravity, we now add sixth-order derivative terms. What is propagated in such a theory? Can we avoid ghosts? If yes, what are the conditions for this? What are the renormalisation properties of sixth-derivative gravity?

Exercise 14: Extra modes in higher derivative gravity

Motivation: In this exercise, we will make the extra degrees of freedom that the curvature-squared terms introduce explicit. The general idea is the following: we first write down an action with an auxiliary (that is, non-dynamical) field, making sure that the equations of motion are equivalent to higher-derivative gravity. We then perform a shift of the metric to bring the action into a form of GR plus “extra stuff”. For the R^2 -term, we can do this exactly, whereas for the $R^{\mu\nu}R_{\mu\nu}$ -term, we will restrict ourselves to the kinetic term of the new field.

Consider the following “weird” action:

$$S^{\text{weird}} = \int d^4x \sqrt{-g} \left[\Phi R - \frac{1}{4\alpha} (\Phi - 1)^2 \right]. \quad (14.I)$$

Here, α is a constant and Φ is a scalar field.

- a) Derive the equations of motion, both for the auxiliary field Φ and for the metric. Plug the solution for the scalar field into the equation of motion for the metric. What is this equation of motion equivalent to? Also, plug the solution into the action. What do you get?
- b) Perform a conformal transformation on S^{weird} (don’t insert any solution to the equations of motion here!), this time with

$$g_{\mu\nu} \mapsto \Phi^{-1} \tilde{g}_{\mu\nu}. \quad (14.II)$$

How does the action look like?

- c) Finally, reparameterise the scalar field by

$$\Phi = e^{c\tilde{\Phi}}, \quad (14.III)$$

where c is a suitable constant. Compute the action as a function of $\tilde{g}$ and $\tilde{\Phi}$ and choose the constant c cleverly (you will see what this means). What is this theory? What do we learn from this whole computation?

- d) **[hard question]** Try to generalise this to $f(R)$ gravity. Does the same method go through? If so, how does the scalar potential look like?

Now shift your attention to this “super-weird” action:

$$S^{\text{super-weird}} = \int d^4x \sqrt{-g} \left[R + \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) f_{\mu\nu} - \frac{1}{4\beta} (f^{\mu\nu} f_{\mu\nu} - f^\mu{}_\mu f^\nu{}_\nu) \right]. \quad (14.IV)$$

Here, $f_{\mu\nu}$ is a symmetric rank-two tensor field, and β is a constant.

- e) **[hard question]** Perform the same tasks as in a), but now for the action $S^{\text{super-weird}}$.
- f) **[hard question]** Now shift the metric in $S^{\text{super-weird}}$ in the following way:

$$g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} + c f_{\mu\nu}. \quad (14.V)$$

Doing so, only keep terms up to second order in f , and fix c such that there is no kinetic mixing — that is, ensure that there is no term that is linear in both curvature and f . Once again, what is this theory, and what do we learn from this?

Quantum Gravity and the Renormalization Group

Assignment 7 – Dec 01

Exercise 15: Analysing beta functions

Motivation: The purpose of this exercise is to familiarise yourself with the study of beta functions: finding fixed points, computing critical exponents, and investigating the phase diagram. These are the bread-and-butter (or margarine/olive oil/whatever you fancy) skills when analysing renormalisation group flows.

In the following, we will consider two different sets of beta functions for two couplings G and Λ that depend on a reference scale k . They have mass dimensions -2 and 2 , respectively, so that the dimensionless couplings are given by $g = Gk^2$ and $\lambda = \Lambda k^{-2}$.^a

Consider first the following set of beta functions for the two couplings g, λ :

$$\begin{aligned}\beta_g &= k\partial_k g = 2g - \frac{135g^2}{72\pi - 5g}, \\ \beta_\lambda &= k\partial_k \lambda = -\left(2 + \frac{135g}{72\pi - 5g}\right)\lambda - g\left(\frac{43}{4\pi} - \frac{810}{72\pi - 5g}\right).\end{aligned}\tag{15.I}$$

- Compute all fixed points of this set of beta functions.
- Compute the critical exponents at each of these fixed points. If there is an interacting fixed point, how many relevant parameters does it have? What does that mean for its predictivity?
- We can actually solve the beta functions implicitly. Show in a first step that $g(k)$ implicitly defined via

$$G_N k^2 = \frac{g(k)}{\left(1 - \frac{145}{144\pi}g(k)\right)^{27/29}}\tag{15.II}$$

is a solution to the first beta function, where G_N is a constant. What is the meaning of G_N ? *Hints:* Take a $k\partial_k$ derivative of this equation, solve for β_g , and check that you get back the original beta function. For the meaning of G_N , it might be helpful to expand to leading order in powers of k .

- In a second step, show that $\lambda(k)$ defined by

$$\lambda(k) = \frac{162}{25} - \frac{43}{16\pi}g(k) + \ell(144\pi - 145g(k))^{25/29} - \frac{144\pi}{3625g(k)}\left[87 + 25\ell(144\pi - 145g(k))^{25/29}\right]\tag{15.III}$$

is a solution to the second beta function. Here, the constant ℓ can be written as

$$\ell = -\frac{29}{86400}\left(2^{-13}3^{-21}\pi^{-54}\right)^{1/29}(432\pi + 125G_N\Lambda_0),\tag{15.IV}$$

with Λ_0 being yet another constant. What is the meaning of Λ_0 ? *Hint:* For the meaning of Λ_0 , it might be helpful to expand to leading order in powers of k .

- e) **[hard question]** Expand both solutions in powers of $1/k$ to leading non-trivial order. How does this relate to part b)? Could you have gotten this result more easily?
- f) Instead of using λ , it can be useful to use the dimensionless combination $\tau = G\Lambda = g\lambda$. What is the interpretation of this coupling, and what are the potential IR ($k \rightarrow 0$) values that can be reached from the fixed point(s)?
- g) Sketch the *phase diagram* in terms of g and τ (Mathematica's *StreamPlot* might come in handy, if you don't want to do it by hand). The phase diagram, or RG flow diagram, is a plot of the integral curves of the beta functions.
- Your phase diagram should contain all fixed points and any potential other global features. If there are features other than fixed points, explain them.

Let us now briefly analyse a second set of beta functions:

$$\begin{aligned}\beta_g &= \left(2 - \frac{2}{3\pi} \frac{\lambda^2}{(1-2\lambda)^4} g\right) g, \\ \beta_\lambda &= -\left(2 + \frac{2}{3\pi} \frac{\lambda^2}{(1-2\lambda)^4} g\right) \lambda + \frac{g}{4\pi} \frac{1}{1-2\lambda} \left[1 + \frac{g}{3} \frac{2}{3\pi} \frac{\lambda^2}{(1-2\lambda)^4}\right].\end{aligned}\tag{15.V}$$

- h) Compute all real fixed points and their critical exponents. Is there anything unusual here? If so, try to explain what's going on.
- i) **[hard question]** Sketch the phase diagram. What is the new feature in these beta functions? Which implications does this have for the admissible IR values of G and Λ ? Can the relevant fixed point be used as a UV completion? Why/why not?

^aGuess what G and Λ should represent.

Exercise 16: *[Presence]* Analysing beta functions, part 2

Motivation: Let us study yet another set of beta functions.

Consider two coupling G, G_{C^3} with dimensionless couplings $g = Gk^2$ and $g_{C^3} = G_{C^3}k^4$. For these, look at the following set of beta functions:

$$\begin{aligned}\beta_g &= 2g - \frac{8}{3}g^2, \\ \beta_{g_{C^3}} &= 4g_{C^3} + \frac{g^2}{1890}.\end{aligned}\tag{16.I}$$

Find fixed points and critical exponents. Solve the differential equations to get $g(k)$ and $g_{C^3}(k)$ – you should get two free parameters for the general solution. Fix one of them by requiring $g(k) \sim G_N k^2$ as $k \rightarrow 0$. Can you fix the other constant by requiring that you hit the fixed point as $k \rightarrow \infty$?

Investigate the IR limit ($k \rightarrow 0$) of $g_{C^3}(k)$ – is there anything unusual here? If so, explain!

Quantum Gravity and the Renormalization Group

Assignment 8 – Dec 08

Exercise 17: Fixed functions – an infinite number of couplings

Motivation: So far we have studied finite sets of beta functions. However, the non-perturbative RG flow generates infinitely many terms that have to be accounted for. The purpose of this exercise is to learn how to deal with some of the complications that are added in such a case.

For this exercise, we will briefly leave the realm of gravity, and instead study a vector field Φ^a that lives in *three* Euclidean dimensions and has an $O(N)$ symmetry (you can think of this as a special case of a theory with N scalar fields – the “vector” here refers to the $O(N)$ group, not to spacetime; the case $N = 1$ is related to the well-known Ising model). We will look at the following approximation for the effective average action:

$$\Gamma_k \simeq \int d^3x \left[\frac{1}{2} (\partial_\mu \Phi_a)(\partial^\mu \Phi^a) + V_k(\rho) \right]. \quad (17.I)$$

For convenience, we introduced $\rho = \Phi_a \Phi^a / 2$. The arbitrary k -dependent potential V_k contains in general infinitely many interaction terms. If you want, you can think of it in terms of a Taylor expansion:

$$V_k(\rho) = \sum_{n \geq 1} c_{n,k} \rho^n. \quad (17.II)$$

In this, the $c_{n,k}$ are the infinitely many k -dependent couplings.

- What are the mass dimensions of the field Φ^a , the potential V_k , and the coupling constants $c_{n,k}$?
- Given an expression for the RG flow of $V_k(\rho)$, i.e. $k \partial_k V_k(\rho)$, how would the flow of the corresponding dimensionless potential look like? *Hints:* To arrive at this, you could assume the above Taylor expansion, convert the couplings $c_{n,k}$ to their dimensionless counterparts, and then try to go back to the potential. It might also be useful to introduce a dimensionless version of ρ . If we call v_k the dimensionless version of V_k , and $\bar{\rho}$ the dimensionless version of ρ , you should get a relation of the form

$$k \partial_k V_k(\rho) = a_1 k \partial_k v_k(\bar{\rho}) + a_2 v_k(\bar{\rho}) + a_3 \bar{\rho} v'_k(\bar{\rho}), \quad (17.III)$$

where $a_{1,2,3}$ are field-independent coefficients that you need to determine.

We will now study the large- N limit of this theory, in which an exact solution can be obtained. This allows us to focus on the concepts while keeping analytic control. It will be useful to focus on the derivative of the dimensionless potential, $u(\bar{\rho}) \equiv v'_k(\bar{\rho})$, where we now suppress the index k to simplify notation. This is commonly done in the literature.

Suppose that in such a limit, the k -dependence of the dimensionless derivative of the potential is given by

$$k \partial_k u(\bar{\rho}) = -2u(\bar{\rho}) + \bar{\rho} u'(\bar{\rho}) - \frac{1}{2} \frac{u'(\bar{\rho})}{(1 + u(\bar{\rho}))^2}. \quad (17.IV)$$

We will first focus on obtaining the fixed point solution, so we will try to solve the above differential equation for $k \partial_k u = 0$, i.e. the fixed point equation.

- c) Let us first do some structural analysis: given that the above fixed point equation for u is a first-order differential equation, how many integration constants do you expect? Does it make sense that a fixed point solution has integration constants? If not, what could fix these?
- d) Let us now try to get some more feeling for the solution. We first try a Taylor expansion. Assume that

$$u_*(\bar{\rho}) = \sum_{n=0}^N d_{n*} \bar{\rho}^n. \quad (17.V)$$

Plug this ansatz into the fixed point equation, and compute the fixed point couplings d_{n*} for a reasonable number N by expanding the equation in powers of $\bar{\rho}$ as well (do this however far you want to, maybe something like $N = 4$). You should be able to solve for $N - n_0$ couplings, where n_0 is the number of free integration constants found in c).

[hard question] Think about what could be done to fix the integration constants. *Hint:* A commonly used technique is to set the highest retained coupling to zero, thus imposing another condition on the fixed point couplings. One then increases N systematically and checks for the convergence of any candidate solution.

- e) **[hard question]** The fixed point equation admits an implicit solution: instead of $u_*(\bar{\rho})$, one can solve for $\bar{\rho}(u_*)$. Show that indeed,

$$\bar{\rho} = q \sqrt{u_*} + \frac{3}{4} + \frac{3}{4} \sqrt{u_*} \arctan \sqrt{u_*} - \frac{1}{4} \frac{1}{1 + u_*} \quad (17.VI)$$

is the general solution, where q is an integration constant. *Hint:* Assume from the start that $u_*(\bar{\rho})$ can be inverted so that you can write $\bar{\rho}(u_*)$. Then express $u'_*(\bar{\rho})$ in terms of $\bar{\rho}'(u_*)$ and solve the fixed point equation for $\bar{\rho}(u_*)$.

- f) Let us now try to fix the integration constant q . For this, we can argue in the following way: the function $u_*(\bar{\rho})$ should be real and well-defined for all $\bar{\rho} \geq 0$. Does this already fix the integration constant? Also plot the resulting function $u_*(\bar{\rho})$ for this value of q .
- g) Check that your polynomial solution from d) is consistent with the solution in e). Also compute the right value of your integration constant in the Taylor solution from f).

Now that we have a fixed point function, we can compute the critical exponents! How can we do so? Let us take a step back and remember how we compute critical exponents for a finite set of beta functions. We expand the beta functions to linear order about the fixed point:

$$\vec{\beta}(\vec{d}) \simeq \underbrace{\vec{\beta}(\vec{d}_*)}_{=0} + \left. \frac{\partial \vec{\beta}}{\partial \vec{d}} \right|_{\vec{d}=\vec{d}_*} (\vec{d} - \vec{d}_*), \quad (17.VII)$$

where we combined our above couplings d in a vector, $\vec{d} = \{d_0, d_1, \dots\}$. The solution to this linearised differential equation is

$$\vec{d} \simeq \vec{d}_* + \sum_i c_i \vec{e}_i k^{-\theta_i}, \quad (17.VIII)$$

where c_i are integration constants, and the $\vec{e}_i$ are the eigenvectors of the stability matrix with eigenvalue θ_i .

Let us try to generalise this to the case of a full function. For this, we insert the linearised solution into u :

$$u(\bar{\rho}) = \vec{d} \cdot \{1, \bar{\rho}, \bar{\rho}^2, \dots\} = \underbrace{\vec{d}_* \cdot \{1, \bar{\rho}, \bar{\rho}^2, \dots\}}_{u_*(\bar{\rho})} + \sum_i c_i \left[\vec{e}_i \cdot \{1, \bar{\rho}, \bar{\rho}^2, \dots\} \right] k^{-\theta_i}. \quad (17.IX)$$

Here, we simply wrote the sum as a vector product of the coupling vector with the vector of all powers of $\bar{\rho}$. What is this expression? The first terms is clearly just the fixed function u_* . The second is a sum of different *eigenfunctions* that are in one-to-one correspondence with the eigenvectors of the stability matrix. This motivates the following way to compute critical exponents. Assume that

$$u(\bar{\rho}) \simeq u_*(\bar{\rho}) + \delta u(\bar{\rho}) k^{-\theta}. \quad (17.X)$$

We plug this into the RG equation (17.IV) and expand to linear order in perturbations δu . This gives a *linear* differential equation for the perturbations δu , and θ plays the role of an eigenvalue of a differential operator. We will try to put this into practice now.

h) **[hard question]** Show that the eigenvalue equation for the linear perturbation δu reads

$$-\theta \delta u(\bar{\rho}) = -2\delta u(\bar{\rho}) + \bar{\rho} \delta u'(\bar{\rho}) + \frac{u'_*(\bar{\rho})}{(1 + u_*(\bar{\rho}))^3} \delta u(\bar{\rho}) - \frac{1}{2} \frac{1}{(1 + u_*(\bar{\rho}))^2} \delta u'(\bar{\rho}). \quad (17.XI)$$

- i) **[hard question]** Use the fixed point equation to replace $u'_*(\bar{\rho})$ in terms of $u_*(\bar{\rho})$ and $\bar{\rho}$, and then perform a variable transform with the implicit fixed point solution $\bar{\rho}(u_*)$ to derive an equation for $\delta u(u_*)$.
- j) **[hard question]** Solve this differential equation for $\delta u(u_*)$. This should look rather ugly. To make it nicer, recall that δu is actually the $\bar{\rho}$ -derivative of the corresponding perturbation of the potential, δv . From this, derive the perturbations $\delta v(u_*)$. You should get

$$\delta v(u_*) = c u_*^{\frac{3-\theta}{2}}, \quad (17.XII)$$

with c being (yet another) normalisation constant. Can you find an argument that would restrict (reasonable) perturbations which would also fix the “allowed” critical exponents?

Quantum Gravity and the Renormalization Group

Assignment 9 – Dec 15

Exercise 18: Computing an RG flow

Motivation: Stuff is getting real now – we will derive the beta function for the potential that you saw in Exercise 17 to learn how the Wetterich equation works in practice. We will also introduce some general concepts that will come in handy when going back to gravity.

In this exercise, we will learn how to evaluate the right-hand side of the Wetterich equation:

$$k\partial_k\Gamma_k = \frac{1}{2}\text{STr} \left[\left(\Gamma_k^{(2)} + \mathfrak{R}_k \right)^{-1} \cdot k\partial_k\mathfrak{R}_k \right]. \quad (18.I)$$

Here, we used the “supertrace” (STr) to indicate a generalisation of the loop momentum integral that we will explain soon. This will be important later when we look at gravity.

Our system will be the $O(N)$ -symmetric vector model in three dimensions. Our ansatz for the effective average action Γ_k is the same as in Exercise 17:

$$\Gamma_k \simeq \int d^3x \left[\frac{1}{2}(\partial_\mu\Phi_a)(\partial^\mu\Phi^a) + V_k(\rho) \right]. \quad (18.II)$$

Recall that $\rho = \Phi_a\Phi^a/2$. The goal is to plug this ansatz into (18.I) and find the expression for $k\partial_k V_k$.

- a) Let us start with the easiest bit – the left-hand side. Evaluate the left-hand side of (18.I) for the ansatz (18.II).

Next, let us construct the right-hand side step by step. We will be very careful in the notation at every step – only once you have understood what is going on, you can become sloppy (like everybody working on it)!

- b) The first ingredient that we need is the second variation of the effective average action. Compute

$$\frac{\delta^2\Gamma_k}{\delta\Phi^a(x)\delta\Phi^b(y)}. \quad (18.III)$$

For this, recall that by definition,

$$\frac{\delta\Phi^a(x)}{\delta\Phi^b(y)} = \delta_b^a \delta(x-y). \quad (18.IV)$$

Here, the delta function is three-dimensional.

Next, we have to choose some form of regularisation. Recall that in the path integral, we added a term to the action in the form of

$$\Delta S_k = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \tilde{\Phi}^a(-p) \mathfrak{R}_{k,ab}(p^2) \tilde{\Phi}^b(p). \quad (18.V)$$

This is in momentum space. Often, we actually work in position space, and then this reads

$$\Delta S_k = \frac{1}{2} \int d^3x \Phi^a \mathfrak{R}_{k,ab}(-\partial^2) \Phi^b. \quad (18.VI)$$

Note how the regulator $\mathfrak{R}_k$ actually needs to carry $(O(N))$ indices, since the field also carries indices. In our case, we can choose the regulator diagonal in field space,

$$\mathfrak{R}_{k,ab}(-\partial^2) = \delta_{ab} \mathcal{R}_k(-\partial^2). \quad (18.VII)$$

Now, $\mathcal{R}_k$ is a simple function (that happens to have an operator as its argument). Note the different type setting to make the difference (the literature often does not differentiate this).

- c) Compute how the above regulator gets added to the two-point function $\Gamma_k^{(2)}$ by taking the second variation of ΔS_k in position space.
- d) In the same language, what do we actually mean by the $k\partial_k\mathfrak{R}_k$ in (18.I)?

The easy part is over now – the next step is to invert the regularised two-point function. First, we discuss some conceptual aspects, then the actual computation.

- e) The two-point function found in b) depends on two points x and y . You should have found however that it is local, that is, it can be written as

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k \right) (x, y) = \mathcal{O}(x) \delta(x - y). \quad (18.VIII)$$

Here, $\mathcal{O}(x)$ is some differential operator. The “local” refers to the delta function. Now, its inverse – the propagator $G_k(x, y)$ – is defined by

$$\int d^3y G_k(x, y) \left(\Gamma_k^{(2)} + \mathfrak{R}_k \right) (y, z) = \delta(x - z). \quad (18.IX)$$

From this definition, argue that the propagator must also be local in the above sense, and in particular formally of the structure

$$G_k(x, y) = [\mathcal{O}(x)]^{-1} \delta(x - y). \quad (18.X)$$

Our next task is thus to find $\mathcal{O}^{-1}$. For this, we need an *expansion scheme*, since we will not be able to do this without approximations. To decide on what we really need, consider again a). We really only need terms that contribute to the potential – terms with derivatives acting on fields contribute to other couplings. An expansion in derivatives like this is called ... wait for it ... *derivative expansion*. In going forward, we will thus assume that we can neglect $\partial_\mu \Phi^a \simeq 0$.

- f) In the lowest order derivative expansion where we set $\partial_\mu \Phi^a \simeq 0$ (that is, Φ^a is approximately constant, $\Phi^a(x) \simeq \Phi^a$), compute the regularised propagator. *Hints*: Write the regularised two-point function in the form

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k \right)_{ab} (x, y) = \left\{ A(-\partial^2, \Phi^2) \left(\delta_{ab} - \frac{\Phi_a \Phi_b}{\Phi^2} \right) + B(-\partial^2, \Phi^2) \frac{\Phi_a \Phi_b}{\Phi^2} \right\} \delta(x - y), \quad (18.XI)$$

and determine the functions A, B . Do the two operators multiplying A and B look somewhat familiar? (Try to take their sum and products!) Conclude that you can also write the propagator as

$$G^{ab}(x, y) = \left\{ \left(\delta^{ab} - \frac{\Phi^a \Phi^b}{\Phi^2} \right) C(-\partial^2, \Phi^2) + \frac{\Phi^a \Phi^b}{\Phi^2} D(-\partial^2, \Phi^2) \right\} \delta(x - y), \quad (18.XII)$$

and compute C and D from A and B .

The last step is to make sense out of the STr. First of all, you might have noticed the $\cdot$ that I slipped into (18.I). Since both the propagator and the (k -derivative of the) regulator depend on two points, what we once again mean by this is

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k \right)^{-1} \cdot k \partial_k \mathfrak{R}_k \equiv \int d^3 y \left(\Gamma_k^{(2)} + \mathfrak{R}_k \right)^{-1}(x, y) k \partial_k \mathfrak{R}_k(y, z). \quad (18.XIII)$$

g) Reinstate the $O(N)$ indices and compute (18.XIII) by using the results of the previous parts.

Last but not least, we define the supertrace via the integral over the coincidence limit, plus taking any “standard” index trace (tr):

$$\text{STr } F(x, y) = \int d^3 x \text{tr} \lim_{y \rightarrow x} F(x, y). \quad (18.XIV)$$

This of course is more complicated in gravity, where all hell breaks loose, but this is a problem for future-you, not present-you.

h) Use the definition of the supertrace to finally evaluate the right-hand side of the flow equation! *Hints:* It could be helpful to represent the delta function in terms of its Fourier representation,

$$\delta(x - y) = \int \frac{d^3 \ell}{(2\pi)^3} e^{-i \ell \cdot (x - y)}. \quad (18.XV)$$

Use this to convert any $-\partial^2$ into momenta, and take the continuum limit. Finally, the tr is the trace (aka contraction) over the two remaining $O(N)$ indices. Recall for this that $\delta_a^a = N$.

Don’t perform the momentum integrals yet (you can’t anyway without specifying the shape of the regulator). Rather, your result should look like

$$k \partial_k V_k(\rho) = (N - 1) I_1(\rho) + I_2(\rho), \quad (18.XVI)$$

where $I_{1,2}$ are two different integrals.

i) To evaluate the integrals, use the Litim regulator,

$$\mathcal{R}_k(z) = (k^2 - z) \theta \left(1 - \frac{z}{k^2} \right), \quad (18.XVII)$$

where θ is the Heaviside function.

- j) As the very last step, let us reproduce (17.IV). For this, we rescale the field and the potential via

$$\Phi^a \mapsto c \sqrt{N} \hat{\Phi}^a, \quad V(\rho) \mapsto c^2 N \hat{V}(\hat{\rho}), \quad (18.XVIII)$$

for a suitable constant c . Then take a ρ -derivative of the flow equation (since (17.IV) is the flow for the derivative of the potential) and take the limit $N \rightarrow \infty$. What is the right value for c to achieve a match?

- k) **[hard question]** Think about systematic extensions of our approximation. You can take the following questions as an orientation:

- Where does the statement come from that the Wetterich equation generates all terms compatible with symmetries?
- Related to this, how can we systematically extend our approximation? For example, how would we take into account terms that include factors of $\partial_\mu \Phi^a$, and which beta functions would we be able to compute by keeping these terms?
- Do you think everything is going through in the same way with gravity? If not, what could go wrong?

Quantum Gravity and the Renormalization Group

Assignment 10 – Jan 12

Exercise 19: Heat kernel, part 1

Motivation: The purpose of this exercise (split into two parts) is to generalise the integral over the loop momentum to curved spacetimes. In other words, we will define the supertrace on a curved manifold. This is necessary to compute beta functions in gravity.

In this exercise, we will define the supertrace with the help of a simple example. Consider a “free” (in quotation marks, since nothing is free in life and/or when gravity is present) scalar field ϕ in dimension d with the action

$$\Gamma_{\phi,k} = \int d^d x \sqrt{g} \left[\frac{1}{2} (D_\mu \phi)(D^\mu \phi) \right]. \quad (19.I)$$

Our goal is to compute the contribution of the quantum fluctuations of this scalar field to the beta functions of Newton’s coupling and the cosmological constant, while neglecting metric quantum fluctuations. In a Feynman diagram picture, this would correspond to having all internal lines being scalar lines. By diffeomorphism invariance, we expect that whatever these quantum fluctuations generate, the RG flow should be parametrisable in the form

$$\frac{1}{2} \text{STr} [\dots] = \int d^d x \sqrt{g} [c_0 + c_1 R + c_2 R^2 + c_3 R_{\mu\nu} R^{\mu\nu} + \dots]. \quad (19.II)$$

Concretely, our task is to compute the numerical coefficients c_i . To ease the notation, from now on we strip away all delta functions and only work with the operators, and we will work in position space.

- a) Follow similar steps as in Exercise 18 and formally compute the argument of the supertrace. You should get something like

$$\frac{1}{2} \text{STr} W(\Delta) \quad (19.III)$$

for the right-hand side of the flow equation, for a suitable function W and where $\Delta = -D^2 = g^{\mu\nu} D_\mu D_\nu$.

We will now make a “small” detour – instead of computing the supertrace of a general function, let us first compute the supertrace of a very particular function:

$$\text{STr} e^{-s\Delta} \equiv \text{STr} H(s). \quad (19.IV)$$

The matrix elements of $H(s)$ can be defined in a position basis,

$$H(x, y; s) = \langle y | H(s) | x \rangle. \quad (19.V)$$

If we were able to compute these matrix elements, the supertrace would simply correspond to the sum over the eigenvalues. This information will come in handy later.

b) Argue that the matrix elements $H(x, y; s)$ fulfill the *heat equation*

$$\partial_s H(x, y; s) = -\Delta_x H(x, y; s), \quad (19.VI)$$

with initial condition

$$H(x, y; 0) = \delta(x - y). \quad (19.VII)$$

Above, the subscript x on Δ_x indicates that the operator acts with respect to the position x , and we will always take this convention, omitting the subscript. What is the meaning of the variable s ?

We can now finally define the supertrace:

$$\text{STr } e^{-s\Delta} = \text{tr} \int d^d x \sqrt{g} \langle x | e^{-s\Delta} | x \rangle = \text{tr} \int d^d x \sqrt{g} H(x, x; s). \quad (19.VIII)$$

Here, tr indicates the trace over discrete indices (e.g. spacetime indices, or the $O(N)$ indices from Exercise 18). The expression $H(x, x; s)$ is called the *coincidence limit* of the heat kernel (since both points coincide). This is often indicated with an overbar, $H(x, x; s) \equiv \overline{H}(s)$.

c) Time for some sanity checks. Show that for a flat spacetime where $\Delta = -\partial^2$, the heat kernel elements read

$$H^{\text{flat}}(x, y; s) = \left(\frac{1}{4\pi s} \right)^{d/2} e^{-\frac{(x-y)^2}{4s}}. \quad (19.IX)$$

Next, make sure that our definition of the supertrace actually reproduces the standard loop momentum integral in flat spacetime. That is, show that

$$\overline{H}^{\text{flat}}(s) = \int \frac{d^d p}{(2\pi)^d} e^{-s p^2} \quad (19.X)$$

by explicit computation. The right-hand side here should ring a bell (Exercise 18 again) and comes from (19.VIII) using that STr is essentially a loop momentum integral, just that the integrand is the exponential instead of the propagator times the derivative of the regulator.

It is time to get serious now. How does the heat kernel $H(x, y; s)$ look like in an arbitrary, curved spacetime? Let us take some inspiration from the flat heat kernel (19.IX). The first thing we have to “upgrade” is the coordinate difference in the exponential, $(x - y)^2$. What would a suitable generalisation look like? If you guessed the geodesic distance, you are correct! For reasons, we define the *world function* $\sigma(x, y)$ to be *half* of the square of the geodesic distance. The world function has a funny property, namely

$$\frac{1}{2}(D_\mu \sigma(x, y))(D^\mu \sigma(x, y)) = \sigma(x, y). \quad (19.XI)$$

Recall that all derivatives act on the x -coordinate. Convince yourself that this relation makes sense in flat spacetime.

So far so good, but this might not be the only upgrade needed. Let us thus make the ansatz

$$H(x, y; s) = \left(\frac{1}{4\pi s} \right)^{d/2} e^{-\frac{\sigma(x, y)}{2s}} \Omega(x, y; s), \quad (19.XII)$$

where Ω parameterises the information on the curvature of the manifold. In particular, we have the boundary condition that $\Omega = 1$ whenever the manifold is flat.

- d) What is the coincidence limit of the world function σ , that is, what is $\bar{\sigma} = \sigma(x, x)$?
- e) What are the mass dimensions of s and Ω ? Use this to argue with which powers of s curvatures and covariant derivatives in Ω have to come. From this, can you make the boundary condition $\Omega = 1$ when the spacetime is flat more precise?

We will now use the heat equation to compute Ω .

- f) Insert the ansatz for the heat kernel into the heat equation to derive a differential equation for Ω .
- g) Make a power series ansatz for Ω in powers of s , i.e. assume that

$$\Omega(x, y; s) \sim \sum_{n \geq 0} s^n A_n(x, y), \quad s \rightarrow 0. \quad (19.XIII)$$

Using this ansatz, derive a recursion relation for the coefficients A_n (this should still be a differential equation). What does the boundary condition from e) translate to in terms of the A_n ?

This completes the setup of the computation. In part 2 of this exercise, we will solve the recursion relation to some low order.

Quantum Gravity and the Renormalization Group

Assignment 11 – Jan 19

Exercise 20: Heat kernel, part 2

Motivation: This is part 2 of the heat kernel. Are you exhausted yet? Good, it's going to become much worse. :)

At the end of the last sheet, you should have obtained the following recursion relation for the coefficients A_n :

$$\left(n - \frac{d}{2} + \frac{1}{2} (D^2 \sigma(x, y)) \right) A_n(x, y) + (D^\mu \sigma(x, y)) (D_\mu A_n(x, y)) - D^2 A_{n-1}(x, y) = 0, \quad (20.I)$$

with

$$A_0(x, x) \equiv \overline{A_0} = 1, \quad A_{-1}(x, y) = 0, \quad n \geq 0. \quad (20.II)$$

From now on, we will drop the position arguments. The goal of this exercise is to compute the coincidence limit of the first heat kernel coefficient, $\overline{A_1}$.

- Set $n = 1$ in (20.I) and take the coincidence limit to understand which ingredients you need to compute $\overline{A_1}$. Beware: the coincidence limit does *not* commute with covariant derivatives, i.e. $\overline{D^2 A_0} \neq D^2 \overline{A_0}$!
- One of the ingredients to compute $\overline{A_1}$ is $\overline{D^2 A_0}$. Derive an equation for the latter by acting with D^2 on (20.I) and taking the coincidence limit.

At this point you should get worried about the recursion, but maybe there is some hope after all. Instead of focussing on different derivatives of the A_n , let us switch our focus and try to compute coincidence limits of derivatives of the world function. Recall that

$$\frac{1}{2} (D^\mu \sigma(x, y)) (D_\mu \sigma(x, y)) = \sigma(x, y), \quad (20.III)$$

for *any* x, y , i.e. even away from the coincidence limit.

- [hard question]** Use (20.III) to compute $\overline{\sigma}$, $\overline{D_\mu \sigma}$, $\overline{D_\mu D_\nu \sigma}$, $\overline{D_\mu D_\nu D_\kappa \sigma}$ and $\overline{D_\mu D_\nu D_\kappa D_\lambda \sigma}$. *Hints:* take successive covariant derivatives of (20.III). Then take the coincidence limit of these equations and solve iteratively. You might have to commute covariant derivatives. Think about what you can pull out of the coincidence limit.

If you succeeded, you should feel relieved now if you look back at the equations for $\overline{A_1}$ and $\overline{D^2 A_0}$.

- Use your results from c) to compute $\overline{D^2 A_0}$, and from there compute $\overline{A_1}$.

This illustrates the general procedure, and you can follow the same recipe to compute the A_n for larger n . It goes without saying that once again, this should not be done by hand.

Exercise 21: Heat kernel, part 3, or the inverse Laplace transform

Motivation: This is part 3 of the heat kernel – I lied that there would be only two parts. Remember where we started? Good, we have to actually come back and compute the original supertrace.

The starting point of the heat kernel exercises was that we originally wanted to compute

$$\text{STr } W(\Delta), \quad (21.I)$$

for some general function W . We spent a lot of time to compute the supertrace for an exponential, but in general we will not deal with only exponential functions, so we still need a recipe to connect the two.

For this, suppose we could write something like

$$W(\Delta) = \int_0^\infty ds \tilde{W}(s) e^{-s\Delta}, \quad (21.II)$$

for some new function $\tilde{W}$. Wouldn't this be great? We could simply use this equation and use all previous results:

$$\text{STr } W(\Delta) = \text{STr} \int_0^\infty ds \tilde{W}(s) e^{-s\Delta} = \int_0^\infty ds \tilde{W}(s) \underbrace{\text{STr } e^{-s\Delta}}_{\text{we did this!}}. \quad (21.III)$$

The only thing left to do would be to actually compute $\tilde{W}$ and perform the integrals over s , and we would be done. Also, we assumed that we can exchange the integral with the supertrace, but shhhhhhh.

Let us give some substance to this idea. The integral transform (21.II) is called the *inverse Laplace transform*. You can think of it like this: the original function W is the Laplace transform of some (a priori unknown) function $\tilde{W}$. Of course, there are some conditions on its existence, but let's simply assume for the moment that it exists. In the two previous exercises, we computed the supertrace of the exponential in an expansion in powers of s , so that

$$\text{STr } W(\Delta) \sim \int_0^\infty ds \tilde{W}(s) \left(\frac{1}{4\pi s} \right)^{d/2} \sum_{n \geq 0} s^n \int d^d x \sqrt{g} \overline{A}_n. \quad (21.IV)$$

We thus have to deal with integrals over $\tilde{W}$ multiplying either negative (small n) or positive (large n) powers of s . Do we now really have to compute $\tilde{W}$? Actually, no.

a) Negative powers: show that for $n > 0$,

$$\int_0^\infty ds \tilde{W}(s) s^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(z). \quad (21.V)$$

This means that one can map these integrals over $\tilde{W}$ to integrals over the original function W ! The integral over z has the interpretation of the integral over the loop momentum.

b) Non-negative powers: show that for $n \geq 0$,

$$\int_0^\infty ds \tilde{W}(s) s^n = (-1)^n W^{(n)}(0), \quad (21.VI)$$

that is, these integrals can be mapped to derivatives of the original function at vanishing argument.

c) **[hard question]** Use your combined knowledge to compute

$$\text{STr } W(\Delta) \tag{21.VII}$$

up to linear order in curvature, in arbitrary dimension, for W taken from Exercise 19. To evaluate the integrals, use the Litim regulator. What exactly did you just compute?