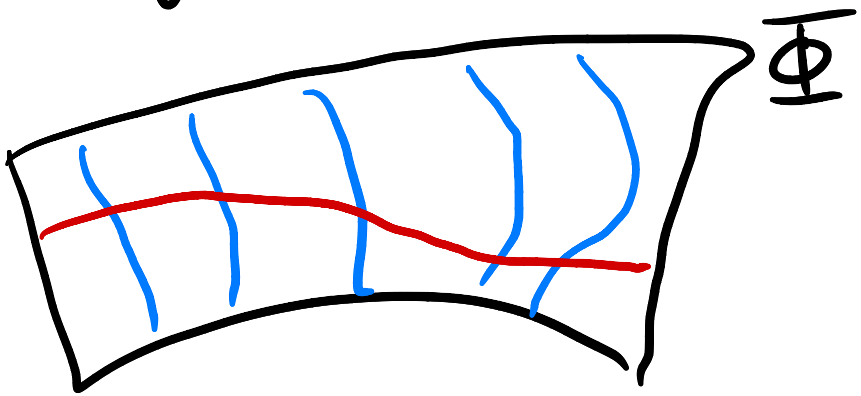
Faddeev-Popov method = how to gauge fix in QFT

general idea: gauge symmetry = redundancy in description

→ "overcounting" of physically equivalent field configurations



Section
→ "cut" each gauge orbit once

gauge orbits = physically equivalent

↑
this is impossible globally in general

→ "Gribov problem"

in path integral: there are "directions" (going along gauge orbit) along which action doesn't change

→ would like to factorise physical fluctuations from gauge garbage

→ $\int \mathcal{D}\Phi_{\text{gauge}} = \infty$

→ absorbed in normalisation

analogy: standard integral

$$I = \int_{-\infty}^{\infty} dx\, dy\, e^{-S(x)}$$

~ physical ~gauge

"$\int dy = \infty = 1$"

→ if we can factorise our measure like this, we could simply drop the integration over gauge orbits

(we can't in practice)

→ alternatively:

this is normalised now

$$\rightarrow \quad I = \int_{-\infty}^{\infty} dx\, dy\, \delta(y)\, e^{-S(x)}$$

we can even choose

$$I = \int_{-\infty}^{\infty} dx\, dy\, \delta(y - f(x))\, e^{-S(x)}$$

for <u>any</u> choice $f(x)$

Suppose now that $y = f(x)$ is the unique solution

of $F(x,y) = 0$

then $$\delta(F(x,y)) = \frac{\delta(y - f(x))}{\left| \det \frac{\delta F}{\delta y} \right|}$$ in principle

$\hookrightarrow$ $$I = \int_{-\infty}^{\infty} dx\, dy \, \det\frac{\delta F}{\delta y} \, \delta(F(x,y)) \, e^{-S(x)}$$

still the same!

we would like these to be additions to S to simplify things

- $F(x,y)$ is the gauge-fixing condition

→ we can represent

$$\delta(F) \propto \lim_{\alpha \to 0} \#(\alpha) e^{-\frac{F^2}{2\alpha}}$$

⇒ gauge-fixing action $S_{gf} \sim -\frac{F^2}{2\alpha}$

- $\det \frac{\delta F}{\delta y}$ can be written as a Grassmann integral

$$\det \frac{\delta F}{\delta y} \simeq \int d\bar{c}\, dc\; e^{-\bar{c} \frac{\delta F}{\delta y} c}$$

→ c and $\bar{c}$ anticommute

⇒ Faddeev-Popov ghost action $S_{FP} \sim \bar{c} \frac{\delta F}{\delta y} c$

apply this to gravity:

- gauge-fixing condition:

gauge parameter

de Donder: $\beta = 1$

$$F_\mu = \bar{D}^\nu h_{\mu\nu} - \frac{1+\beta}{4} \bar{D}_\mu h = \hat{F}_\mu{}^{\alpha\beta} h_{\alpha\beta}$$

$\Rightarrow$ gauge-fixing action

using $\bar{g}$ is a choice here

$\rightarrow$ gauge-fixing only affects h^2 i.e. the propagator, and not the interactions

$$S_{gf} \propto \frac{1}{2\alpha} \int d^4x \sqrt{\bar{g}}\, F_\mu \bar{g}^{\mu\nu} F_\nu$$

gauge parameter

- FP ghosts:

$$\text{"}\frac{\delta F}{\delta g}\text{"} = \text{how gauge-fixing condition changes under a gauge transformation}$$

$\hookrightarrow \hat{F}_{\mu}^{\ \alpha\beta} \mathcal{L}_{\nu} g_{\alpha\beta}$

no bar

check intermediate steps!

$\Rightarrow$ FP ghost action

$$S_{FP} \propto \int d^4x \sqrt{g}\, \bar{c}^{\mu} \hat{F}_{\mu}^{\ \alpha\beta} (D_{\alpha} c_{\beta} + D_{\beta} c_{\alpha})$$

$\hookrightarrow$ FRG in gravity:

Grassmann = − sign in STr

- $\underline{3}$ dynamical fields: $h, c, \bar{c}$

$\Rightarrow$ $\Gamma_k^{(2)}$ is a 3x3 matrix

$\rightarrow$ STr traces over this

"only details missing"

$\nearrow$ sheets 10+11 for STr

$\Rightarrow$ $\bar{c}, c$ also need a regulator

$\Delta S_k^{FP} \sim \bar{c} \cdot R_k^{\bar{c}c} \cdot c$

- $\Gamma_k^{tot} = \Gamma_k[g] + S_{gf}[\bar{g}, h] + S_{FP}[\bar{c}, c, \bar{g}, h]$

$\Uparrow$ you wish... there is also $\hat{\Gamma}_k[\bar{g}, h]$ ...

a very brief glimpse on the consequence of quantum diffeomorphism breaking: more later

- fact I: regulator and gauge-fixing break QDiff
- fact II: all terms that respect symmetries are generated by the RG flow

$\Rightarrow$ since QDiff is broken, "anything goes" (almost)

$\hookrightarrow \quad \Gamma_k = \Gamma_k[\bar{g}, h]$

but: we still have BG Diff

$\Rightarrow$ covariance w.r.t. $\bar{g}$

$$\hookrightarrow \quad \Gamma_k[\bar{g}, h] \supset \int d^4x \sqrt{\bar{g}} \Big[\frac{1}{16\pi G_k}(2\Lambda_k - \bar{R})$$
$$+ \alpha \bar{R}^2 + \beta \bar{R}_{\mu\nu} \bar{R}^{\mu\nu}$$
$$+ \gamma h + \delta h_{\mu\nu} \bar{R}^{\mu\nu}$$
$$+ \varepsilon h_{\mu\nu} \bar{D}^2 h^{\mu\nu} + \dots \Big]$$

derivative w.r.t. h!

$\rightarrow$ flow equation: $k\partial_k \Gamma_k[\bar{g}, h] = \frac{1}{2} \mathrm{STr}\Big[(\Gamma_k^{(0,2)} + R_k)^{-1} k\partial_k R_k \Big]$

plus ghosts, but they are not a big problem

<u>Constraint</u>: as $k \to 0$, $\Gamma_k[\bar{g}, h] \longrightarrow \Gamma[\bar{g} + h]$

$+$ subtleties

non-trivial!

gauge-fixing

Does AS work? A selection of results

← biased!

Handbook of QG
Lectures in QG
+ references therein

for now, we will forget about the issues related to broken QDiff

$$\Rightarrow \text{start with } \Gamma_k[g] + S_{gf}[\bar{g},h] + S_{FP}[\bar{g},h] + \Delta S_k[\bar{g},h],$$

take second h-derivative, and set $h=0$

background field approximation

we will check later how good or bad this is

this approximation simplifies computations significantly, so historically this has been widely used

hope: QDiff breaking of regulator is small

Einstein-Hilbert truncation

↳ EH truncation

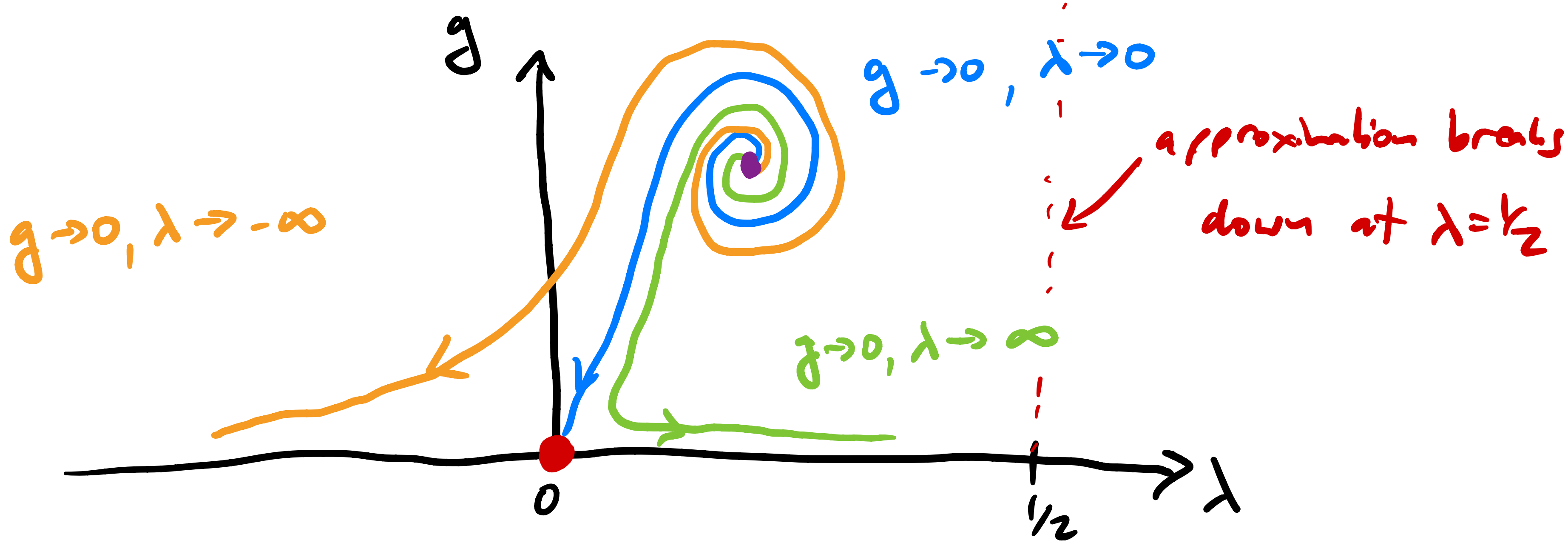
Reuter hep-th/9605030

↑ the OG paper

simplest-possible approximation:

$$\Gamma_k = \frac{1}{16\pi G_{N,k}} \int d^4x \sqrt{g} \left[2\Lambda_k - R \right]$$

→ compute β functions for $g_k = G_{N,k} k^2$, $\lambda_k = \Lambda_k k^{-2}$



→ there is a physically relevant fixed point

checked stability extensively

e.g. gauge and parametrisation dependence

Gies, Knorr, Lippoldt ↗ sheet 7

1507.08859

→ both g and λ are relevant, critical exponents

are $\theta_{1,2} \sim 2-4$

precise value depends on technical choices → approx. not good yet

⇝ very close to mass dimension

⇝ small anomalous dimension (= quantum correction to scaling)

⇒ promising, but only the first step!

Beyond EH – approximation strategies

to check the existence & properties of this FP further, we have to <u>systematically</u> extend the truncation

① derivative expansion $=$ expansion in mass dimension

expand in powers of derivatives (in our case: of $g_{\mu\nu}$)

∂^0 : "1" aka cosmological constant

∂^2 : R

∂^4 : R^2, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, E ← Gauss-Bonnet

∂^6 : $R\Delta R$, $R_{\mu\nu}\Delta R^{\mu\nu}$, R^3, $R R_{\mu\nu}R^{\mu\nu}$, $R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu$,
$R\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $R^{\mu\nu}R^{\rho\sigma}C_{\mu\rho\nu\sigma}$, $C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\kappa\lambda}C_{\kappa\lambda}{}^{\mu\nu}$

+ total derivatives e.g. ΔR

in $d>4$: two more invariants

∂^8: 28 terms + 15 in higher dimensions

Fulling, King, Wybourne, Cummins CQG 9, 1151

② special backgrounds

use a specific background metric $\bar{g}_{\mu\nu}$ where

→ a lot is known about the STr of Δ

→ only a manageable subset of terms survives

example: sphere / hyperboloid ~ dS/AdS in Lorentzian signature

$$\rightarrow \bar{C}_{\mu\nu\rho\sigma} = 0, \quad \bar{R}_{\mu\nu} = \frac{1}{d}\bar{g}_{\mu\nu}\bar{R}, \quad \bar{D}_\mu \bar{R} = 0$$

↪ "only" $f_4(\bar{R})$ survives

⇝ in spirit similar to potential $V(\phi)$ of scalar field ϕ & evaluating on constant ϕ, $\partial_\mu \phi = 0$ ↗ sheet 8

③ curvature expansion

expand in powers of curvature but not in derivatives

this is tricky to define since $[D_\mu, D_\nu]\xi^\rho = R_{\mu\nu}{}^\rho{}_\sigma \xi^\sigma$

and fun identities like extra material 4, sheet 6

proper definition: expansion about flat spacetime

R^0: 1

R^1: R

"form factors" no $f(\Delta)$ due to fun identities

R^2: $R\, f_R(\Delta) R$, $R_{\mu\nu}\, f_{Ric}(\Delta) R^{\mu\nu}$, E

R^3: Barvinsky, Gusev, Zhytnikov, Vilkovisky 0911.1168 ~200 pages of goodness

why expansion about flat background:

if we expand $\bar{g}_{\mu\nu} = \delta_{\mu\nu} + \bar{h}_{\mu\nu}$ up to $\bar{h}^4$, we only get contributions of up to R^4

e.g. $R f_R(\Delta) R \sim (\partial\partial h) f_R(\partial^2)(\partial\partial h) + \mathcal{O}(h^3)$

① Derivative expansion

- first computations beyond EH also used ② and added R^2

Lauscher, Reuter
hep-th/0205062

- general picture: similar to EH

i.e. there is a FP with compatible crit. exp.

but: R^2 is relevant

3/3, oh no?

θ_3 is large $(\sim 10\ldots30)$

oh no!

- only recently first result for <u>full</u> ∂^4 basis

Knorr 2104.11336

a) only C^2-term: irrelevant ☺

b) only R^2-term: <u>irrelevant</u> but tachyonic scalar mode

c) both: highly non-canonical θ's

↳ approximation not under control

FP is <u>very</u> close to region where approx. breaks down

· some results at ∂^6 order

EH + C^3 → Gies, Knorr, Lippoldt, Saueressig 1601.01800

full basis + field redefinitions → Baldazzi, Falls, Kluth, Knorr 2312.03831

→ C^3 - term is safely irrelevant ($\theta \sim -10 \sim 20$)

recall perturbative discussion: this is the relevant two-loop counterterm that couldn't be removed by field redefinitions

⇒ AS is <u>better</u> than pert. theory!

general problem for DE: "fiducial" ghosts

⇝ similar to quadratic gravity, but here believed to be truncation artefact:

- extending order of DE introduces new poles at each order → nonsense physically
- relevant information: order → ∞

↳ which poles in the propagator survive?

possible mechanism: residue of fake poles → 0

Platania, Wetterich 2009.06637

complete answers needs complete propagators

② special backgrounds

- general idea: expand $\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \bar{h}_{\mu\nu}$

$\neq h_{\mu\nu}$!

"nice" metric

- so far only $\tilde{g} \sim$ sphere / hyperboloid, $\bar{h} = 0$

$\hookrightarrow f(R)$

- mostly polynomial truncations

$$f(R) = \sum_{n=0}^{N} f_n R^n$$

$N = 34$

Falls, Litim, Nikolakopoulos, Rahmede 1301.4191 good news:

- $N=2$ is a glitch, $N>2$ stabilises results
- 3 relevant directions
- near-Gaussian scaling

$$\theta_n \approx 4.1 - 2.2n \qquad \text{as} \quad n \to \infty$$

→ remember some lectures ago? near-diagonal stability matrix?

⇒ this suggests that mass dimension is a good ordering principle at the FP! non-trivial!

but if true, then the DE would be the better expansion $\rightsquigarrow$ include all monomials hard problem!

$\rightarrow$ pattern to be confirmed, but promising

mixing of different terms at same order in DE might make it more complicated

③ curvature expansion

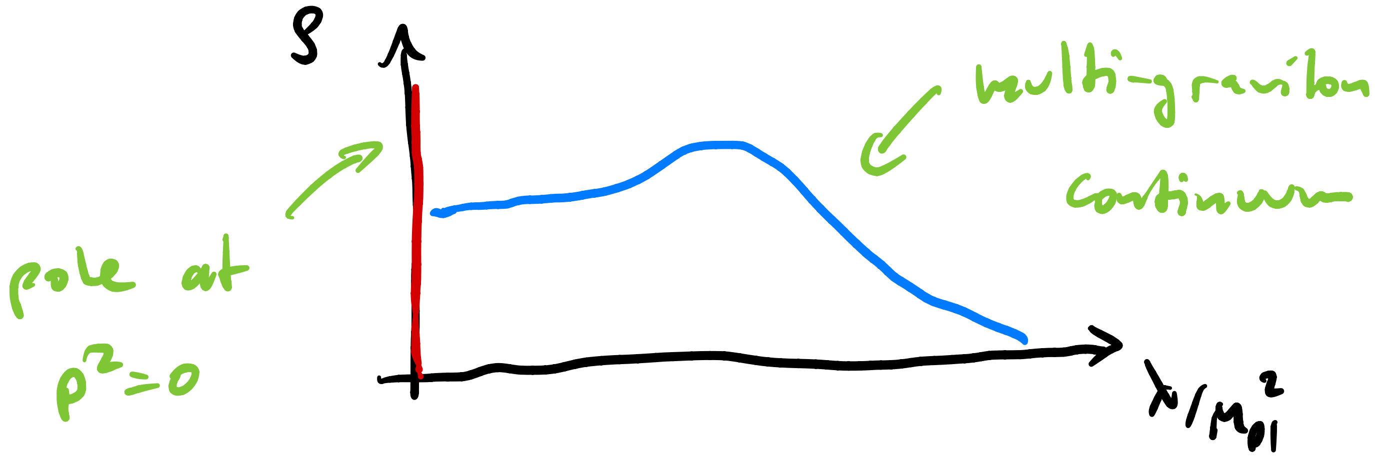
→ can resolve momentum dependence of propagator

↪ # poles !

→ important for scattering amplitudes

→ key result: no extra poles only at $p^2=0$

Lorentzian computations exist by now, e.g. graviton spectral function



Fehre, Litim, Pawlowski, Reichert 2111.13232

Summary:

- there <u>is</u> a FP
- likely ≈ 3 relevant directions
- no extra modes

  = quantised GR!

  <u>not</u> quadratic gravity

  or other options

- many thing still to be figured out, e.g. unitarity, causality, ...
- many more results exist