↳ How to fix $\bar{\pi}_{\mu\nu}$?

1) gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$

in Fourier space, this entails

$$p^\mu \bar{\pi}_{\mu\nu} = 0$$ ← same "bar" as for $\bar{h}_{\mu\nu}$

these are $\underline{4}$ conditions

2) residual gauge dependence $\partial^2 \varepsilon_\nu = 0$

⇝ superposition of plane waves

focus on mode with some momentum p,

$$\varepsilon_\mu(x) = \epsilon_\mu e^{ip_\alpha x^\alpha} + \epsilon_\mu^* e^{-ip_\alpha x^\alpha}$$

under this gauge transformation,

$$\overline{\pi}_{\mu\nu}' = \overline{\pi}_{\mu\nu} + i\left(p_\mu \epsilon_\nu + p_\nu \epsilon_\mu - \eta_{\mu\nu}\, p\cdot\epsilon\right)$$

(check this)

we now can impose

$$u^\mu \overline{\pi}_{\mu\nu}' = 0 \qquad \text{for a constant vector } u^\mu$$

$$= u^\mu \overline{\pi}_{\mu\nu} + i\left(p\cdot u\, \epsilon_\nu + p_\nu\, u\cdot\epsilon - u_\nu\, p\cdot\epsilon\right)$$

$\underline{\underline{3}}$ eqs. for 4 components of ϵ

→ always has (complex) sol.

→ this fixes $\underline{\underline{3}}$ more components of $\bar{\Pi}_{\mu\nu}$

not 4 because $\underbrace{p^\mu \bar{\Pi}_{\mu\nu} u^\nu = 0}_{\text{linear combination of } \bar{\Pi}_{\mu\nu} u^\nu}$

3) residual gauge part II

we can take the trace of the transformed $\bar{\Pi}_{\mu\nu}$:

$$\bar{\Pi}^{\mu}{}_{\mu}{}' = \bar{\Pi}^{\mu}{}_{\mu} + i(2p\cdot\epsilon - 4p\cdot\epsilon)$$
$$= \bar{\Pi}^{\mu}{}_{\mu} - 2i\,p\cdot\epsilon$$

$\hookrightarrow$ we can always make $\bar{\pi}_{\mu\nu}$ traceless

remember: we had one free component of ϵ left above!

$\rightarrow$ fixes $\underline{1}$ component of $\bar{\pi}_{\mu\nu}$

In summary, the conditions on $\pi_{\mu\nu}$ are:

$$\pi^{\mu}{}_{\mu} = 0 \quad , \quad p^{\mu} \pi_{\mu\nu} = 0 \quad , \quad u^{\mu} \pi_{\mu\nu} = 0$$

traceless

transverse
(to direction of propagation, like for photons)

check that 1), 2), 3) carry over from $\bar{\pi}$ to π!

$\Rightarrow$ in $d=4$, the graviton has two physically distinct polarisations

Excursion: general dimension d ($d-1$ spatial $+ 1$ time)

- $\Pi_{\mu\nu}$ is symmetric $\rightarrow$ $\frac{d(d+1)}{2}$ components
- $p^\mu \Pi_{\mu\nu} = 0$ $\rightarrow$ d conditions
- $u^\mu \Pi_{\mu\nu} = 0$ $\rightarrow$ $d-1$ conditions
- $\Pi^\mu{}_\mu = 0$ $\rightarrow$ 1 condition

$$\Rightarrow \frac{d(d-3)}{2} \text{ independent polarisations}$$

General wisdom: "gauge symmetry always hits twice"

above: d conditions from gauge fixing

but eom has non-trivial kernel

↳ common feature in gauge theories, related

to secondary constraints ↗ sheet 3

↳ in $d \leq 3$: no propagating gravitational waves

→ no two orthogonal directions

in which to oscillate

$\Rightarrow$ $d=4$ is the minimal spacetime dimensionality

for GWs to propagate ↗(why) is our universe 4d?

How do the polarisations look like?

Example: GW that propagates in z-direction

$\hookrightarrow \quad p^\mu = (p, 0, 0, p)$ ↙ can make different choices, this is a specific one

we also choose $u^\mu = (1, 0, 0, 0)$

$$\rightsquigarrow \quad \Pi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_+ & e_\times & 0 \\ 0 & e_\times & -e_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

check this!

e_+ and $e_\times$ are the amplitudes of the two polarisations

↪ metric :

$$ds^2 = dx^\mu dx^\nu (\eta_{\mu\nu} + h_{\mu\nu})$$

$$= -dt^2 + dx^2 \left[1 + e_+ \underbrace{\left(e^{ip(z-t)} + e^{-ip(z-t)} \right)} \right]$$

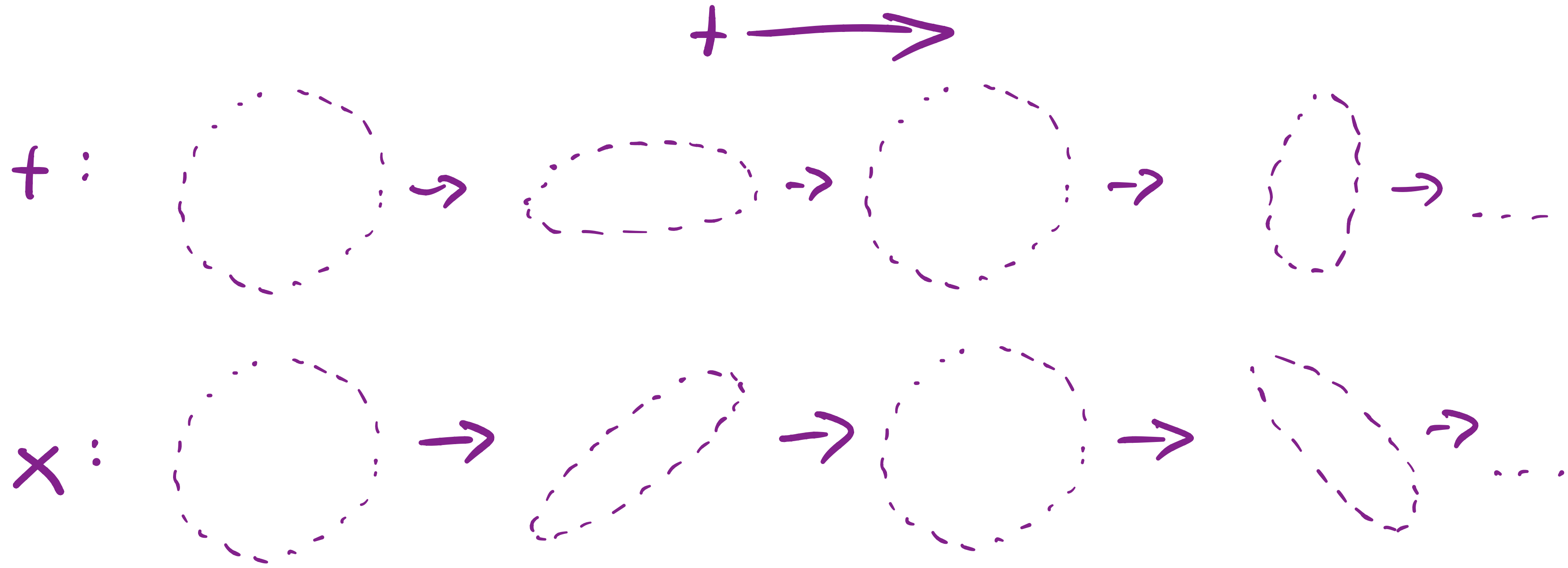
$$+ dy^2 \left[1 - e_+ \underbrace{\left(e^{ip(z-t)} + e^{-ip(z-t)} \right)} \right]$$

$$+ 2 dx\, dy\, e_\times \underbrace{\left(e^{ip(z-t)} + e^{-ip(z-t)} \right)}$$

$$+ dz^2$$

$2\cos[p(z-t)]$

↪ two independent oscillation modes in x-y-plane



what are the dots?

⇝ ring of particles, GW passes through and displaces them in the above pattern

back to quantisation

→ quantise plane waves by promoting coefficients to creation/annihilation operators:

$$h_{\mu\nu}(x) = \sum_{\sigma=\pm} \int d^3\vec{p} \left[a_{\vec{p},\sigma} \Pi_{\mu\nu}(\sigma) e^{ip\cdot x} + c.c. \right]$$

↳ $e_\pm = \frac{1}{\sqrt{2}}(e_+ \pm i e_x)$ ← these are helicity eigenstates = eigenstates of rotation matrix around z-axis

→ $a_{\vec{p},\sigma}$ and $a^*_{\vec{p},\sigma}$ are quantum-mechanical operators with a standard set of commutation relations:

$$[a, a] = [a^\dagger, a^\dagger] = 0 \,, \quad [a_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}] = i\,\delta_{\sigma\sigma'}\,\delta^3(\vec{p}-\vec{p}')$$

→ standard procedure to build Fock space

(vacuum, single graviton states, etc.)

↳ are we done? is this QG yet? of course not!

⇝ interaction

this is the point where we will

need some QFT for the first time

to make a QFT out of this, we need to compute

loop corrections ⇝ need to know the Feynman rules

strategy: minimise pain and use deDonder gauge

$\rightsquigarrow$ expand Einstein-Hilbert Lagrangian to higher orders in h

$$\left(\mathcal{L} = \frac{\sqrt{-g}}{16\pi G_N} R \right)$$

$$\hookrightarrow \quad \mathcal{L} \simeq \overbrace{-\frac{1}{2} \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} + \frac{1}{4} (\partial_\mu h)^2}^{\rightsquigarrow \text{ propagator}} + \overbrace{V_3^{\mu\nu\rho\sigma\kappa\lambda}}^{\text{3-graviton vertex}} h_{\mu\nu} h_{\rho\sigma} h_{\kappa\lambda} + \dots$$

up to rescaling, effectively $h_{\mu\nu} \to \sqrt{32\pi G_N}\, h_{\mu\nu}$

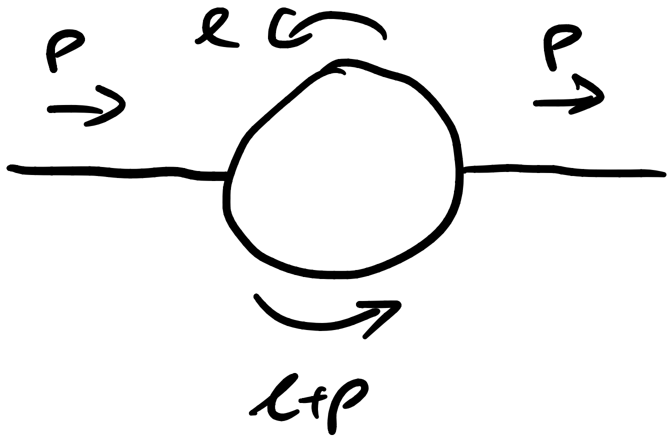
$\rightarrow$ this is useful as it canonically normalises $h_{\mu\nu}$ and makes it that the limit $G_N \to 0$ gives the free theory

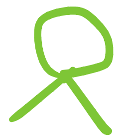
$\rightarrow$ similar to gauge theory: can put coupling in front of F^2 term, or into the gauge field

<u>side note</u>: in a proper treatment, we have to use the Faddeev-Popov method to implement the gauge fixing; for now we do not need it as it doesn't change the power counting, but you will meet it later

↳ Feynman rules for GR

we will discuss the loop correction to the propagator, i.e. the relevant diagram is





there is in principle also a tadpole diagram, but in dimensional regularisation it vanishes

↳ we need the graviton propagator and the 3-graviton vertex

in the following we will neglect factors of "i" as they do not change the scaling with momentum

Propagator: = inverse two-point function

$$[P^{-1}]^{\mu\nu\rho\sigma}(p) \propto p^2 \left[\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} \right]$$

in a more general gauge, there will also be terms of the form $(p^\mu p^\nu \eta^{\rho\sigma} + \eta^{\mu\nu} p^\rho p^\sigma)$, $(p^\mu \eta^{\nu\rho} p^\sigma + p^\nu \eta^{\mu\rho} p^\sigma$ ④

③ $+ p^\mu \eta^{\nu\sigma} p^\rho + p^\nu \eta^{\mu\sigma} p^\rho)$,

and if higher curvature terms are there as well, $p^\mu p^\nu p^\rho p^\sigma$ ⑤

$\Rightarrow$ these ⑤ elements form a basis for two-point functions of a symmetric field

to compute the propagator, we have to find the inverse – but what is the identity?

$h_{\mu\nu}$ is a <u>symmetric</u> tensor

$$\Rightarrow \quad \mathbb{1}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = h_{\mu\nu}$$

maps symmetric tensors onto themselves

$$\Rightarrow \quad \mathbb{1}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\sigma \delta_\nu{}^\rho\right)$$

$$\rightarrow \quad P_{\alpha\beta\mu\nu} \left[P^{-1}\right]^{\mu\nu\rho\sigma} \overset{!}{=} \mathbb{1}_{\alpha\beta}{}^{\rho\sigma}$$

in general, you would now proceed by spanning P in the above basis ① – ⑤

in our gauge, we only need ① and ②

(why?)

↳ ansatz :

$$P_{\alpha\beta\mu\nu} = c_1 \cdot \overbrace{\frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu}\right)}^{\mathbb{1}_{\alpha\beta\mu\nu}} + c_2\, \eta_{\alpha\beta}\eta_{\mu\nu}$$

compute the product:

$$(P \cdot P^{-1})_{\alpha\beta}{}^{\rho\sigma} = c_1 \left[p^2 \mathbb{1}_{\alpha\beta}{}^{\rho\sigma} - \frac{p^2}{2} \eta_{\alpha\beta} \eta^{\rho\sigma} \right]$$

$$+ c_2 \left[p^2 \eta_{\alpha\beta} \eta^{\rho\sigma} - \frac{p^2}{2} \eta_{\alpha\beta} \underbrace{\eta_{\mu\nu} \eta^{\mu\nu}}_{=4} \eta^{\rho\sigma} \right]$$

$$= \underset{\overset{!}{=}\, 1}{c_1} p^2 \mathbb{1}_{\alpha\beta}{}^{\rho\sigma}$$

$$+ p^2 \underbrace{\left[-\frac{c_1}{2} + c_2 - 2 c_2 \right]}_{-\frac{c_1}{2} - c_2 \overset{!}{=} 0} \eta_{\alpha\beta} \eta^{\rho\sigma}$$

$$\Rightarrow c_1 = 1/p^2$$

$$\Rightarrow c_2 = -\frac{c_1}{2} = -\frac{1}{2p^2}$$

$$\Rightarrow \quad \boxed{P_{\alpha\beta\mu\nu} = \frac{1}{p^2}\left[\mathbb{1}_{\alpha\beta\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\mu\nu}\right]}$$

note! this is up to the proportionality factor that we dropped for the two-point fct. (essentially $\sim i$)

propagator in de Donder gauge

3-graviton vertex: this is where things start to be painful

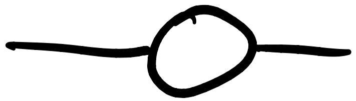
↗ tutorial 3 on Nov 3

→ general structure of V_3: combinations of two momenta (out of the three) and metrics, each with 6 indices

↑ why two?

→ with momentum conservation ($p_1+p_2+p_3=0$), there are 161 terms (not the most compact way of writing this)

relevant info: $V_3 \sim$ "p^2" ← meaning all possible combinations

↳ can evaluate  now

learn in the tutorial how to not waste hours on this

↳ doing so and performing the loop integral, we encounter divergences of the form

$$[\text{div. prefactor}] \times \underbrace{[\partial^\mu \partial^\nu h_{\mu\nu} - \partial^2 h]^2}$$

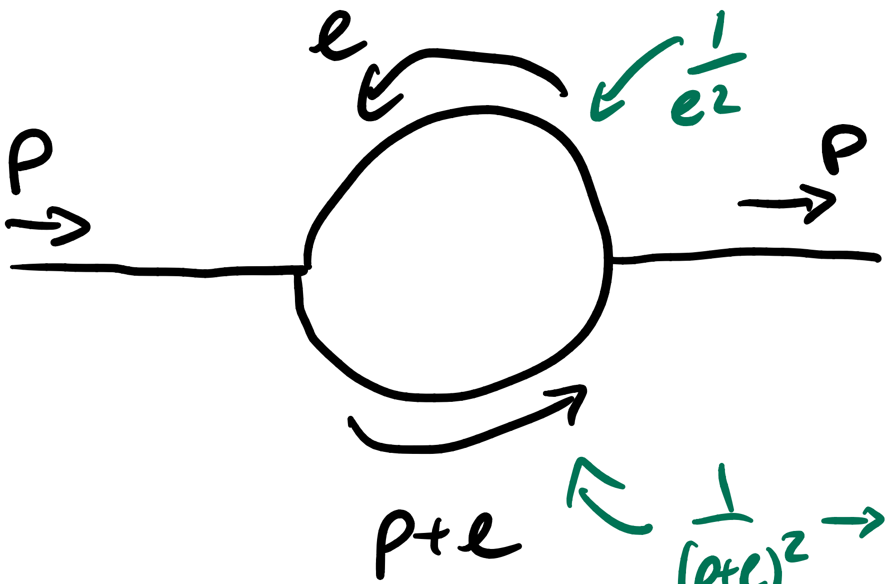
and other terms

this is the expansion of R^2 to second order in h

⇝ how do these terms arise and what do they mean?

① the divergence

for this, we count powers of the loop momentum ℓ



$\frac{1}{\ell^2}$

$\frac{1}{(p+\ell)^2} \xrightarrow[\ell\to\infty]{} \frac{1}{\ell^2}$

vertices could either be p^2, $p\cdot\ell$ or ℓ^2

⇒ in the least divergent case, $\sim \int d^4\ell \left(\frac{1}{\ell^2}\right)^2 p^4$ at large ℓ

→ logarithmic divergence × p^4

(there are others as well)

② the tensor structure

the resulting terms must be proportional to expansions of curvature invariants — otherwise, diffeomorphism invariance would be broken

at fourth order in derivatives — p^4 above — candidates are

$\nearrow$ sheet 2

$$R^2,\ R_{\mu\nu}R^{\mu\nu},\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},\ D^2R$$

total derivative

$\rightsquigarrow$ can be neglected

in $d=4$, the combination

$$\mathcal{E} = \sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

is a total derivative – it is called the

Gauss-Bonnet invariant

⇝ its integral $\int d^4x\, \mathcal{E}$ is a topological invariant and measures the number of "handles" of spacetime

↳ only $\underline{2}$ terms are linearly independent and can contribute to the action (in the absence of boundary terms): R^2 and $R_{\mu\nu}R^{\mu\nu}$ this is a choice

Consequence: to absorb these divergences, $\sim \int d^4x \, \mathcal{L}$

counterterms have to be added to the action

$$\rightsquigarrow \quad \int d^4x \sqrt{g} \left[a R^2 + b R_{\mu\nu} R^{\mu\nu} \right]$$

these are **not** of the form of the original

Einstein-Hilbert action

$\Rightarrow$ two new couplings appear in the action

fun fact: in the absence of matter, one can perform a field redefinition

$g_{\mu\nu} \to g_{\mu\nu} + c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R$ so that for suitable c_1, c_2,

the counterterms vanish

to see this, we plug in this shift into the action and expand to linear order in c_1, c_2 [this is enough because we look at the one-loop terms]

$$\rightsquigarrow \quad S[g_{\mu\nu} + c_1 R_{\mu\nu} + c_2 R g_{\mu\nu}]$$

$$\simeq S[g_{\mu\nu}] + \int d^4x \, \underbrace{\frac{\delta S[g]}{\delta g_{\mu\nu}}}_{-\sqrt{-g}\, G_{\mu\nu}} (c_1 R_{\mu\nu} + c_2 R g_{\mu\nu})$$

$$\simeq S[g] + \int d^4x \sqrt{-g} \left[c_1 R_{\mu\nu} R^{\mu\nu} + \left(\frac{c_1}{2} + c_2 \right) R^2 \right]$$

(up to boundary terms)

this pattern of new divergences persists at every loop order: at n loops, we need terms with $2n+2$ derivatives note: at $n=2$, this includes both cubic curvature terms as well as quadratic curvature terms with two derivatives

→ the latter are not all total derivatives

e.g. $R D^2 R$

sad fact: starting at $n=2$, you cannot absorb the divergences with a (local) field redefinition

for $n=2$ the offending term in $d=4$ is

$$R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu\nu}$$

this two-loop counterterm was first computed by Goroff & Sagnotti in '85/'86

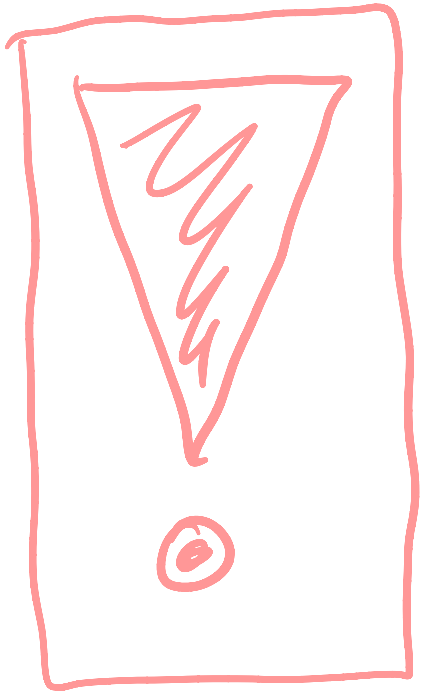
Phys. Lett. B 160 (1985) 81-86

Nucl. Phys. B 266 (1986) 709-736

⇝ think about this, given the argumentation we used at one-loop order — is it clear that the above term cannot be absorbed?

⇝ the problem (likely) repeats at higher loop orders — why?

likely, because the prefactor of a divergence could be zero; the point of the Goroff-Sagnotti computation was to show it is not conclusion: the perturbative quantisation of the Einstein-Hilbert action leads to a non-predictive theory which has infinitely many free couplings



the finite prefactors of the new terms, once divergences have been absorbed

$\Rightarrow$ perturbative non-renormalisability

= breakdown of predictivity