let us argue for the above claim with $2n+2$ derivatives at n loops

⇝ compute superficial degree of divergence D in $\underline{d}$ dimensions

= expected (worst) divergence, barring any cancellations (e.g. through contractions of indices)

to count the divergence, we introduce a cutoff in the integrals over loop momenta, Λ_{UV}: $\int d^d \ell \longrightarrow \int^{\Lambda_{UV}} d^d \ell$

recall: propagators scale like $1/p^2$; vertices scale like p^2 } in GR

G in a diagram with P propagators, L loops, and V vertices, we will have

$$\left[\frac{1}{\ell^2}\right]^P \left[\int d^d\ell\right]^L \left[\ell^2\right]^V \sim \left[\ell\right]^{\overbrace{-2P+dL+2V}^{=D}}$$

$$\Rightarrow \quad D = -2P + dL + 2V$$

we can simplify this by using

$$L = P - V + 1$$

which is a topological relation that holds for all loop diagrams proof: consider arbitrary diagram with P propagators, L loops and V vertices

→ successively cut propagators without cutting diagrams apart

↳ every cut removes 1 propagator and 1 loop

↳ repeat this until no loop is left

⇒ $P-L$ propagators are left

→ we still have V vertices that are connected by the $P-L$ propagators; there is at most a single connection between any two vertices (otherwise there is a loop); vertices cannot be connected directly and via another vertex (otherwise there is a loop) → vertices form a connected line

→ we need $V-1$ lines $= P-L$ propagators

plugging in this relation, we get

$$D = (d-2)L + 2$$

→ in $d=2$, only quadratic divergences = can be absorbed!

gravity is actually <u>topological</u> in $d=2$

in $d=2$, the Gauss-Bonnet term is exactly the Einstein term

→ in $d>2$, the degree of divergence increases with loop order, as claimed

Q: is perturbative QG still useful? <u>YES</u>

⇝ before we discuss possible solutions, we try to extract some physical insights from pert. QG

Perturbative QG as an EFT ← effective field theory

- order by order in perturbation theory, we can absorb the divergences of GR in finitely many counterterms

  ↳ as a recipe for a <u>fundamental</u> theory this fails

  ⇝ when curvature is "large", the theory is not predictive

↳ what happens at "small" curvature?

we have to clarify what this means since curvature has units

<u>observation</u>: n-loop terms come with G_N^n relative to GR

$$\hookrightarrow \; S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left[R + \alpha G_N R^2 + \beta G_N R_{\mu\nu} R^{\mu\nu} + \ldots \right]$$

remember that $[G_N] = -2$

$\hookrightarrow$ $1/G_N$ defines the square of a mass, the Planck mass M_{Pl}^2

$$M_{Pl} = \sqrt{\frac{\hbar c}{G_N}} \simeq 20 \mu g$$

$$\rightsquigarrow 10^{19}\,\text{GeV}$$

recall $m_{e^-} \simeq 511\,\text{keV}$

↳ the above expansion is in terms of curvature in Planck units

Q: when are higher order terms negligible?

for this, consider scattering of gravitons with energy $\sim E$

→ GR term contributes E^2

→ quadratic-in-curvature terms contribute E^4/M_{Pl}^2

relative factor : E^2/M_{Pl}^2

conclusion: if $\alpha, \beta \sim \mathcal{O}(1)$, the two contributions above are negligible if $E^2 \ll M_{Pl}^2$

(Conversely, if α and β are large, this reduces this bound)

$\rightsquigarrow$ EFT of gravity to determine the leading quantum gravity corrections

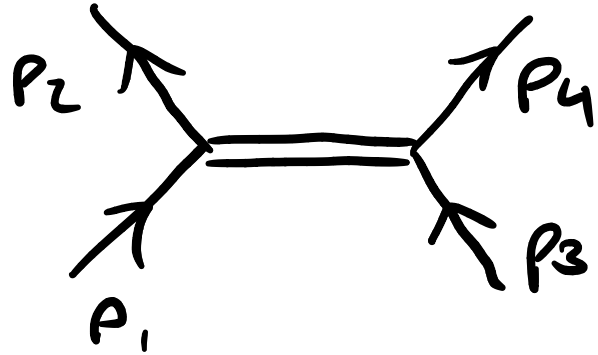
One-loop Newtonian potential

let us compute the gravitational attraction between two massive non-relativistic particles = "upgraded" Newtonian potential

a) tree level

this gives the classical Newtonian potential

consider the tree level diagram



~ scalar field that only interacts gravitationally

recall graviton propagator in de Donder gauge

once again we drop factors of "i"

$$P_{\mu\nu\rho\sigma} \propto \frac{1}{p^2}\left[\mathbb{1}_{\mu\nu\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right]$$

here: $p^2 = (p_1 - p_2)^2$

for the vertices, we assume a free massive scalar field action,

$$S_{mat} = \int d^4x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2\right]$$

we have to expand this to linear order in $h_{\mu\nu}$ about $\eta_{\mu\nu}$

and take functional derivatives

recall $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G_N}\, h_{\mu\nu}$

$$\hookrightarrow \quad V^{\mu\nu}(p_1, p_2) \propto \sqrt{8\pi G_N}\left[p_1^{(\mu} p_2^{\nu)} - \frac{1}{2}\eta^{\mu\nu}(p_1 \cdot p_2 + m^2)\right]$$

symmetrisation of indices

assuming two different mass scalar fields

$$\rightsquigarrow \quad V^{\mu\nu}(p_1, p_2)\, P_{\mu\nu\rho\sigma}(p_1 - p_2)\, V^{\rho\sigma}(p_3, p_4) = -\frac{16\pi G_N m_1^2 m_2^2}{(p_1 - p_2)^2}$$

recall that $p^2 = m^2$ for external lines

$$\approx -\frac{16\pi G_N m_1^2 m_2^2}{(\vec{p}_1 - \vec{p}_2)^2}$$

nonrelativistic limit

e.g. $p_{1,2\,\mu} = \left(\sqrt{m_\phi^2 + \vec{p}_{1,2}^{\,2}}, \vec{p}_{1,2}\right)$

$\hookrightarrow (p_1 - p_2)^2 \approx (\vec{p}_1 - \vec{p}_2)^2 + \mathcal{O}\left(\frac{|\vec{p}|^4}{m_\phi^2}\right)$

Scattering potential $\propto$ Fourier transform in $\vec{q} = \vec{p}_1 - \vec{p}_2$ "momentum transfer"

up to normalisation factors $= \frac{1}{4 m_1 m_2}$

$$\hookrightarrow V(r) = \int \frac{d^3\vec{q}}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, \frac{-16\pi G_N m_1^2 m_2^2}{\vec{q}^{\,2}}\, \frac{1}{4 m_1 m_2} = -\frac{G_N m_1 m_2}{r}$$

sanity check satisfied

b) one-loop

this computation has a long history, and it took many attempts (= wrong papers) to get the final result

Donoghue gr-qc/9310024, Bjerrum-Bohr/Donoghue/Holstein hep-th/0211072

$\rightsquigarrow$ redo above but with one loop:

only here: restored c and $\hbar$

$$V(r) = -\frac{G_N m_1 m_2}{r}\left[1 + 3\,\frac{G_N(m_1+m_2)}{r\,c^2} + \frac{41}{10\pi}\,\frac{G_N \hbar}{r^2 c^3} + \dots\right]$$

"GR!"

this is interesting: both classical and quantum

correction from one-loop diagram

$\hookrightarrow$ fact: loop expansion $\neq$ $\hbar$-expansion

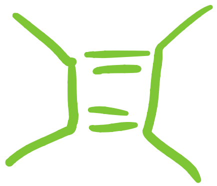
Holstein / Donoghue hep-th/0405239

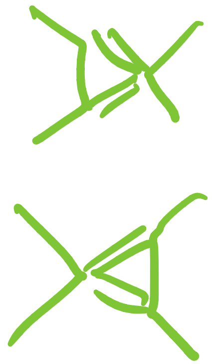
$\to$ one can calculate Post-Minkowskian (PM) corrections from loop diagrams

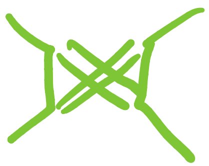
$\to$ has opened up a new interesting line of research in recent years: using QFT methods to improve GW calculations which are needed to interpret LIGO data

let us discuss the quantum term

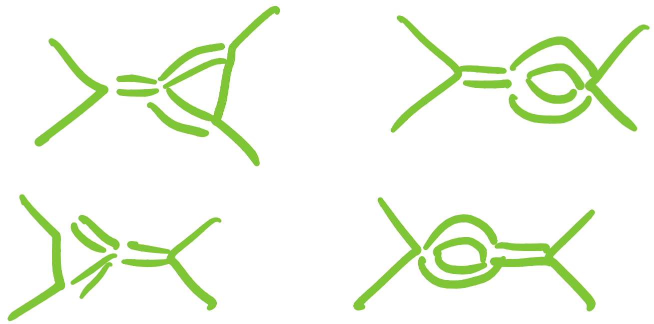
diagrams:

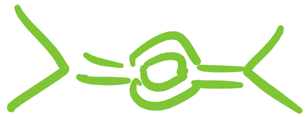
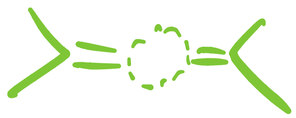
box 

triangle 

crossed box 

double seagull 

vertex correction diagrams 

vacuum polarisation  

grav. Faddeev-Popov ghost

diagrams all have divergences → does the counterterm enter the potential (including unknown prefactors)?

<u>no</u>: a term $1/r^3$ comes from Fourier-transforming

$$\int \frac{d^3\vec{q}}{(2\pi)^3} \ln \frac{\vec{q}^{\,2}}{\mu^2}\, e^{i\vec{q}\cdot\vec{r}}$$

this is distinct from local counterterms $\sim R^2, R_{\mu\nu}R^{\mu\nu}$

these produce polynomials in q^2

$\Rightarrow$ we can disentangle the IR (=long-range) effect from the UV (=short-distance) effect

correction to $V(r)$, $r \to \infty$

local counterterms ("$r \to 0$")

$\hookrightarrow$ we <u>can</u> combine quantum effects with gravity and obtain unambiguous predictions within EFT

the problem is that the EFT breaks down at scales $r \simeq \ell_{Pl}$ and loses predictivity $\Rightarrow$ we <u>still</u> need a UV completion

QG crossroads

- perturbative QFT only works as an EFT
- experimental clues out of sight   recall $M_{Pl} \sim 10^{19}\,\mathrm{GeV}$

  $\to$ many approaches to QG, each with different assumptions/starting points/...

what could we do differently?

① pert. QFT is fine, but the gravitational starting point is wrong

$\rightsquigarrow$ at one loop, we get R^2 and $R_{\mu\nu}R^{\mu\nu}$, what if we add them from the start?

Quadratic gravity

② pert. QFT is wrong, but QFT is fine

$\rightsquigarrow$ we need a non-pert. treatment

Asymptotic Safety

Loop Quantum Gravity

③ QFT breaks down

String theory

Causal set theory

Quadratic gravity

if you can't beat them, join them

instead of quantising GR perturbatively, let us consider

$$\boxed{S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,R + \int d^4x\sqrt{-g}\left[\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right]}$$

as the starting point

we will discuss:

1. is adding these terms compatible with observations?
2. what is their effect on black holes + cosmology?
3. is this theory renormalisable?
4. why is this not considered to be *the* QG?

① Compatible with observations?

- in experimentally accessible situations, curvature is small

↪ $R/M_{Pl}^2 \ll 1$

even for black holes

e.g. Schwarzschild BH of mass M:

Kretschmann scalar $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

$$K_{SBH} = \frac{48\, G_N^2 M^2}{r^6}$$

↪ at horizon:

$(r = r_H = 2 G_N M)$

recall $G_N = \frac{1}{M_{Pl}^2}$

$$\left.\frac{K_{SBH}}{M_{Pl}^4}\right|_{r=r_H} = \frac{48}{M_{Pl}^4} \frac{G_N^2 M^2}{64\, G_N^6 M^6}$$

$$= \frac{3}{4} \frac{M_{Pl}^4}{M^4}$$

tiny

- strongest bounds on α, β come from constraints on modifications of Newton's law

$$\rightsquigarrow \phi(r) = -\frac{G_N m}{r}\left[1 + \frac{1}{3}e^{-m_0 r} - \frac{4}{3}e^{-m_2 r}\right]$$

$$m_0 = \sqrt{32\pi G_N(3\alpha-\beta)}$$
$$m_2 = \sqrt{16\pi G_N \beta}$$

$$\hookrightarrow \quad |\alpha|, |\beta| < 10^{60}$$

short range terms
→ weak constraints

note: we can see the appearance of two massive modes from the Yukawa terms $\sim e^{-mr}$

→ will play an important role later

② Phenomenology

there are (at least) two settings where modifications of GR could be expected to play a major role

a) early universe cosmology:

- R^2-term gives a minimal model for inflation known as Starobinsky model

- generally good agreement with measurements of cosmic microwave background for $\beta = 0$, $\alpha \sim 10^5$!

  → but: recently under pressure

- attractive because there is only a single parameter (α)