b) black holes:

- every vacuum solution of GR is also a vacuum solution to Quadratic gravity ↗ sheet 2

⇒ Kerr BHs are solutions

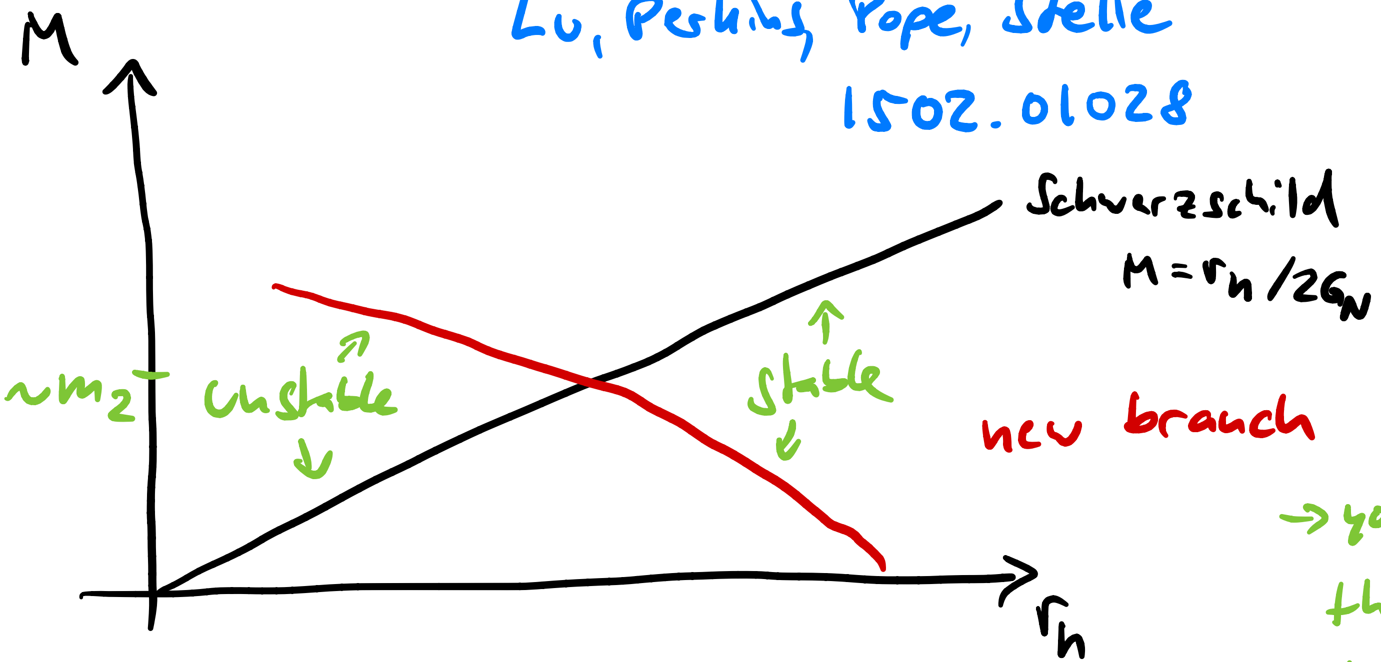
but there are others!

(only known numerically)

Lu, Perkins, Pope, Stelle 1502.01028

⇒ no uniqueness theorem as for GR in $d=4$

(in $d>4$ there can be other solutions like black strings)



→ you can also study the thermodynamics of these new solutions

⇝ light enough BHs might be described by this new branch

⇝ more branches exist, e.g. also wormholes and nonsingular holes

↪ lots of buzz in recent years to investigate solutions to classical gravity theories beyond GR

③ Renormalisibility

compute superficial degree of divergence of Quadratic Gravity:

at large momenta, the R^2-terms dominate

$\rightsquigarrow$ propagators $\propto p^{-4}$ recall $R \sim \partial\partial h \sim p^2 h$

vertices $\propto p^4$

in d dimensions, a diagram with P propagators, V vertices and L loops scales like

$$D = dL - 4P + 4V = (d-4)L + 4 = 4 \text{ in } d=4$$

$L = P - V + 1$

this means that divergences scale like p^4 at worst, independent of the loop order

→ we have p^4 terms in the action

→ we can absorb all divergences !

⇒ Quadratic Gravity is perturbatively renormalisable !

so this is it ? can we go home and call it a day ?

↳ Is perturbative renormalisability enough to ensure that a theory is fundamental, i.e. valid at all scales?

⇝ NO.

⇝ renormalisability only ensures that a finite number of counterterms is needed to absorb all divergences

⇝ no implication by itself on how couplings change under the renormalisation group (RG) flow

↳ can still have divergences in the finitely many retained couplings → Landau pole

We will introduce the RG flow more properly later;
for now the idea is the following:

- we can probe the theory at different scales

  e.g.: <u>experimentally</u> by performing scattering at different energies

  <u>theoretically</u> by introducing a cutoff and shifting it around

  these are related, more on this later (Pandora's box ahead)

$\rightsquigarrow$ values of couplings change

- we call the "probing scale" $\underset{\text{units of energy = momentum}}{\underline{\mu}}$ for now

and we introduce the β-function

$$\beta_g = \underbrace{\mu \partial_\mu}_{= \partial_{\ln \mu}} g(\mu)$$

for the coupling $g(\mu)$

<u>Example</u>: ϕ^4 theory in $d=4$ in Minkowski space

$\rightsquigarrow$ action $\quad S = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\lambda}{4!}\phi^4 \right]$

$\hookrightarrow$ is renormalisable $\nearrow$ sheet 3

but "trivial", i.e. it has a <u>Landau pole</u>

to see this, investigate β-function of λ at one loop:

$$\beta_\lambda = \frac{3}{16\pi^2}\lambda^2 + \mathcal{O}(\lambda^3)$$

higher loop contributions

recall what this means – it is a differential equation

$$\mu \partial_\mu \lambda(\mu) = \frac{3}{16\pi^2}\lambda(\mu)^2$$

the solution reads

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 + \frac{3}{16\pi^2}\lambda(\mu_0)\ln\frac{\mu_0}{\mu}}$$

μ_0 = reference scale

$\rightsquigarrow$ integration constant

for fixed μ_0 and $\lambda(\mu_0) \neq 0$, we encounter a pole at $\mu = \mu_0 e^{-\frac{16\pi^2}{3\lambda(\mu_0)}}$

$\Rightarrow$ divergence (= Landau pole) at finite scale

unless $\lambda(\mu_0) = 0$

$\hookrightarrow$ $\lambda(\mu) = 0$, so the theory is non-interacting (trivial)

$\Rightarrow$ $\boxed{\text{perturbative renormalisability is } \underline{\text{not enough}}}$

<u>note</u>: the above argument relies on the one-loop β-function even though λ becomes large close to the Landau pole, so that the perturbative series breaks down

<u>however</u>, there is non-perturbative evidence from lattice studies that strongly suggests that ϕ^4-theory is indeed trivial

back to Quadratic Gravity: is it UV-complete?

= does it avoid Landau poles?

to investigate this, we reparametrise the action and use the fact that we can pick any two out of the three invariants at $\mathcal{O}(R^4)$ plus topological term

$$\hookrightarrow \quad S = \int d^4x \sqrt{-g} \left[\frac{R - 2\Lambda}{16\pi G_N} - \frac{1}{2\lambda} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \frac{1}{\xi} R^2 \right]$$

C is the Weyl tensor = completely traceless version of the Riemann tensor

in $d=4$, $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho} R_{\sigma]\nu} + g_{\nu[\rho} R_{\sigma]\mu} + \frac{1}{3} g_{\mu[\rho} g_{\sigma]\nu} R$

→ this system has 4 couplings (you can also add the Gauss-Bonnet term with an extra coupling)

we are interested in the high-energy limit of Quadratic Gravity, so we focus on λ, ξ

Buccio, Donoghue, Menezes, Percacci
2403.02397

$$\beta_\lambda = -\left(\frac{1}{4\pi}\right)^2 \frac{1617\lambda - 20\xi}{90}\lambda$$

$$\beta_\xi = -\left(\frac{1}{4\pi}\right)^2 \frac{\xi^2 - 36\lambda\xi - 2520\lambda^2}{36}$$

β-functions of Quadratic Gravity have been computed earlier, e.g. Stelle PRD 16 (1977) 953 but recently there has been some discussion about this for <u>asymptotic freedom</u> (= no Landau pole, couplings go to zero as $\mu \to \infty$)

we need $\beta < 0$

yellow region

$\Rightarrow$ for

$\lambda > 0$ and

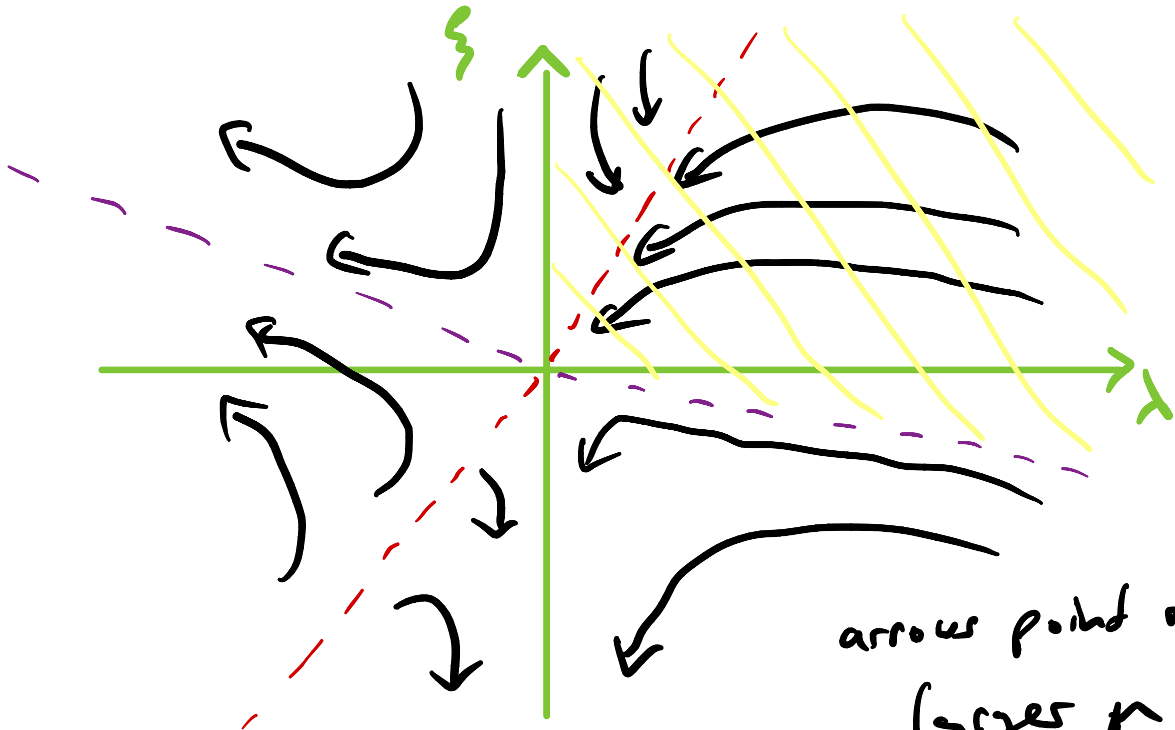
$$\xi > \frac{569 - \sqrt{386761}}{15} \lambda$$

both couplings go to zero as $\mu \to \infty$

$\rightarrow$ like QCD!

analysing this for two couplings is more complicated than our ϕ^4-example

one way is to plot the integral curves (= solutions) in a 2D plot to get an idea



arrows point towards larger μ

intermediate conclusion:

Quadratic gravity is

- phenomenologically viable with very weak constraints
- perturbatively renormalisable
- free of Landau poles (at least at high energies)

→ why is it not <u>the</u> theory of QG?

= why are you still here?

④ Stability

$\rightsquigarrow$ potential problem tied to the fact that Quadratic Gravity is a fourth-order theory

↳ generic potential issue with many of such systems:

Ostrogradsky instability

we will review this in the original setting of classical mechanics

↳ consider a Lagrange function $L(x, \dot{x}, \ddot{x})$

generalisation of standard mechanics where $L = L(x, \dot{x})$

→ equations of motion have higher-order time derivatives

→ need <u>more</u> initial conditions to specify unique solution

= extra degree of freedom

⇝ this leads to a Hamiltonian H that

is unbounded from below

↑

generally expected to lead to

instabilities / runaway, but needs

to be discussed

Euler-Lagrange equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0$$

$\rightarrow$ 4 initial conditions needed

= number of canonical variables

(generalised positions and momenta)

Ostrogradsky chose:

$$x_1 = x \qquad p_1 = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}$$

$$x_2 = \dot{x} \qquad p_2 = \frac{\partial L}{\partial \ddot{x}}$$

$$\Rightarrow \quad H = p_1 \dot{x}_1 + p_2 \dot{x}_2 - L$$

$$= p_1 \dot{x} + p_2 \ddot{x} - L$$

linear in p_1

$\Rightarrow$ unbounded from below

detailed example: sheet 5

Quadratic Gravity has the same problem   at least naively

→ R^2, $R_{\mu\nu}R^{\mu\nu}$ have 4 derivatives

→ due to local Lorentz invariance, these cannot only be spatial derivatives

→ unstable (?)

not so long ago the discussion would have stopped here, but:

① in gravity, the Hamiltonian is a sum of constraints

⇒ for solutions, $H=0$!

↳ not actually obvious that Ostrogradsky's argument holds

why $H=0$ in gravity:

- H is the generator of time translations (= time evolution)
- but gravity has reparametrisation invariance (time redefinition is a diffeomorphism)

→ "time evolution" is subtle in gravity

→ even more so in quantum gravity, long discussions about the "problem of time" in Loop Quantum Gravity

② recent numerical studies of classical Quadratic Gravity show that stable evolution can exist in at least parts of phase space

the fact that H is unbounded from below does not imply instability

Deffayet, Held, Mukohyama, Vikman 2305.09631

Held, Lim 2306.04725

Figueras, Held, Kovács 2407.08775

...

→ maybe classically the theory is not as unstable as expected – what about quantum?

for this, we will look at the propagator

here, we will finally see the extra massive modes that lead to the extra Yukawa terms in the Newtonian potential

it turns out that we can write

$$P_{\mu\nu\rho\sigma} = G_2(p^2)\,\mathcal{P}^{(2)}_{\mu\nu\rho\sigma} + G_0(p^2)\,\mathcal{P}^{(0)}_{\mu\nu\rho\sigma} + \text{gauge stuff}$$

$\mathcal{P}^{(0)/(2)}$: spin projectors

→ project on spin 0 or 2

→ they are specific combinations of the 5 basis elements discussed earlier quick detour: spin projectors $\rightsquigarrow$ representation theory of $so(3,1)$

more on sheet 5

① vectors can be decomposed into transverse and longitudinal parts:

$$V_\mu \longrightarrow \underbrace{V_\mu^T}_{\text{spin } 1} + \underbrace{\partial_\mu V^L}_{\text{spin } 0}$$

→ projectors in Fourier space:

$$T_\mu{}^\nu = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2}, \quad L_\mu{}^\nu = \frac{p_\mu p^\nu}{p^2}$$

check: $T\cdot V = V^T$, $L\cdot V = \partial v^L$, $T^2 = T$, $L^2 = L$, $T\cdot L = L\cdot T = 0$

② gravitons can be decomposed into

- spin 2 – transverse traceless

$$P^{(2)}{}_{\mu\nu}{}^{\rho\sigma} = T_{(\mu}{}^{\rho} T_{\nu)}{}^{\sigma} - \frac{1}{3} T_{\mu\nu} T^{\rho\sigma}$$

- spin 1 – transverse

$$P^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = 2\, T_{(\mu}{}^{(\rho} L_{\nu)}{}^{\sigma)}$$

- two spin 0

$$P^{(0,tr)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{3} T_{\mu\nu} T^{\rho\sigma}$$

$$P^{(0,L)}{}_{\mu\nu}{}^{\rho\sigma} = L_{\mu\nu} L^{\rho\sigma}$$

spin 1 and one spin 0 = gauge degrees of freedom

this is a pain

↓

+ mixing between these!