one can show that (with $\Lambda = 0$)

partial fraction decomposition

$$G_2(p^2) = \frac{4\lambda i}{p^4 - \frac{\lambda}{2} M_{Pl}^2 p^2} = \frac{8i}{M_{Pl}^2}\left[\frac{1}{-p^2} - \frac{1}{-p^2 + \frac{1}{2}\lambda M_{Pl}^2}\right]$$

for once, the i is important

massless spin 2 → standard graviton

massive spin 2 with negative residue

negative residue → opposite sign of kinetic term

→ (presumed) instability

→ "massive spin 2 ghost"

similarly, for the spin 0 part,

$$G_0(p^2) = \frac{4i}{M_{Pl}^2}\left[-\frac{1}{-p^2} + \frac{1}{-p^2 - \frac{\xi}{12}M_{Pl}^2}\right]$$

massless spin 0, already present in GR, but ghost ???

→ think about this!

massive spin 0, tachyon if $\xi < 0$

let us focus on the spin 2 sector

→ unitarity is broken by the ghost

detour: optical theorem in QFT

- unitarity in QM = probabilities are conserved

  ⇝ cannot "lose" or "create" information

- in QFT: define unitarity in terms of S-matrix
  = scattering matrix

  i.e. given an initial state $|a\rangle$ and a final state $|b\rangle$ in some Hilbert space, the S-matrix evolves $|a\rangle$ at $t = -\infty$ to $|b\rangle$ at $t = +\infty$,

$$|b\rangle = S|a\rangle$$

→ probability is conserved if

$$\langle a|a\rangle = \langle b|b\rangle = \langle a|S^\dagger S|a\rangle$$

$$\Rightarrow \quad \boxed{S^\dagger S = \mathbb{1}}$$

transfer matrix

we often write $S = \mathbb{1} + iT$

$$\Rightarrow \quad -i(T - T^\dagger) = T^\dagger T$$

sandwich with initial and final state:

$$-i\left[\langle b|T|a\rangle - \langle b|T^\dagger|a\rangle\right] = \langle b|T^\dagger T|a\rangle$$

insert a complete set of states, $\mathbb{1} = \sum_n |n\rangle\langle n|$

$$\Rightarrow \quad \langle b|T^\dagger T|a\rangle = \sum_n \langle b|T^\dagger|n\rangle\langle n|T|a\rangle$$

now pick an elastic process, $|b\rangle = |a\rangle$:

the optical theorem

$$2\,\mathrm{Im}\,\langle a|T|a\rangle = \sum_n |\langle a|T|a\rangle|^2 \geq 0$$

in particular, this has to hold for "$1\to 1$ scattering",

i.e. propagation

$\Rightarrow$ we need $\underline{\mathrm{Im}\left(i\,G(p^2)\right) \geq 0}$

for unitarity

to evaluate the optical theorem, we need to perform the Feynman trick $p^2 \to p^2 - i\varepsilon$ with $\varepsilon \searrow 0$

$$\to \operatorname{Im}\left(-i G_2(p^2)\right) \propto \lim_{\varepsilon \searrow 0} \operatorname{Im}\left[\frac{1}{p^2 - i\varepsilon} - \frac{1}{p^2 + m_2^2 - i\varepsilon}\right]$$

$$= \pi \left[\delta(p^2) - \delta(p^2 + m_2^2)\right] \not\geq 0$$

exercise: compute $\lim_{\varepsilon \searrow 0} \operatorname{Im} \frac{1}{p^2 - i\varepsilon}$

first note that $\frac{1}{p^2 - i\varepsilon} = \frac{1}{p^2 - i\varepsilon} \frac{p^2 + i\varepsilon}{p^2 + i\varepsilon} = \frac{p^2 + i\varepsilon}{p^4 + \varepsilon^2}$

$$\Rightarrow \operatorname{Im} \frac{1}{p^2 - i\varepsilon} = \frac{\varepsilon}{p^4 + \varepsilon^2}$$

note how this already behaves like $\delta(p^2)$ when $\varepsilon \to 0$:

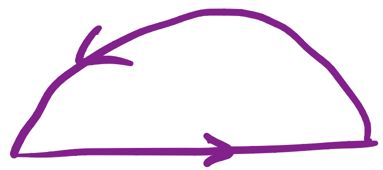
if $p^2 \neq 0$, $\varepsilon \to 0$ gives 0

if $p^2 = 0$, we have $1/\varepsilon \to \infty$

second, consider a test function $\phi(p^2)$ that falls off sufficiently fast* in the complex plane, and consider

$$\int_{-\infty}^{\infty} dz \, \phi(z) \frac{\varepsilon}{z^2 + \varepsilon^2}$$

contour integral close above real line



such that integral over arc is zero

pole at $z = i\varepsilon$

$$= 2\pi i \, \phi(i\varepsilon) \frac{\varepsilon}{i\varepsilon + i\varepsilon}$$

$$= \pi \phi(i\varepsilon)$$

$\Rightarrow$ in the limit $\varepsilon \to 0$, this gives $\pi\,\phi(0)$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} \frac{\varepsilon}{p^4+\varepsilon^2} = \pi\,\delta(p^2)$$

$\Rightarrow$ the massive spin 2 ghost violates the optical theorem at tree level

$\rightarrow$ <u>Quadratic Gravity violates</u> <u>standard unitarity</u> !

in recent years, people have tried to get around this, e.g. via different prescriptions of $i\varepsilon$

to date, no final consensus has been achieved whether this issue can be fixed

an overview can e.g. be found in
Buoninfante 2501.04097

the full story is complicated

↳ we will try something else

Asymptotic Safety

recall : EFT of GR is fine for low energies, but is not fundamental due to having ∞ many free parameters

= loss of predictivity

→ can we find a principle that fixes (almost) all of them?

in the mathematical sense, i.e. all but finitely many

→ can we do so without Ostrogradsky-type problems?

we will spend most of our time on the first question

Scale symmetry

- our starting point: we want to hold on to QFT

→ we know how couplings depend on the scale

↳ or: can compute it (in principle)

→ for now, the notion of scale will remain general; along the way, we will introduce the so-called Functional RG

FRG scale determines approximately in which range of momenta quantum fluctuations have been integrated out thus, suppose we have a set of scale-dependent

couplings $\vec{G}(k)$ → k is the scale

⇝ β functions $\vec{\beta}_{\vec{G}} = k\,\partial_k \vec{G}(k)$ encode this

scale dependence

in principle we can do this for the

∞ many couplings in gravity

↳ what we need to fix the predictivity issue

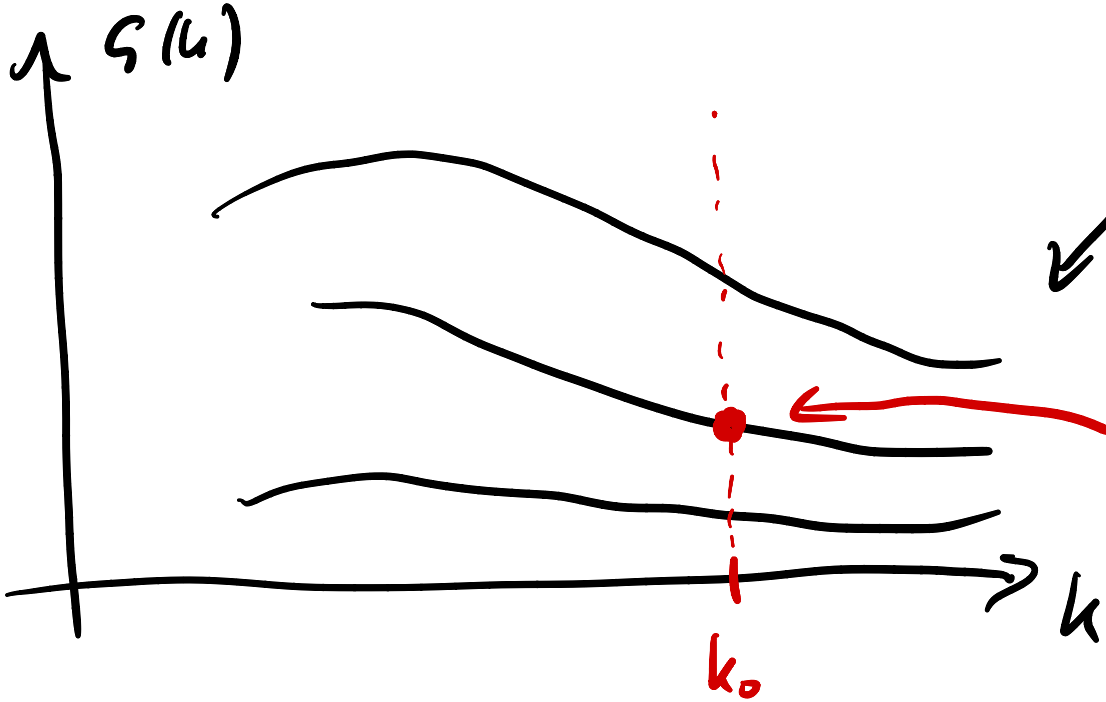
is a principle that prescribes initial conditions

for the couplings, i.e. a principle that determines their values at some scale k_0

this could e.g. be ∞ many relations

like $G_2(k_0) = 7G_1(k_0)^2$, $G_3(k_0) = 27$, ...

as long as only finitely many free parameters are left to be fixed experimentally



curves are fixed by β function

principle needed that says this is the "correct" value idea: we demand a renormalisation group fixed point,

i.e. we demand scale symmetry

important: this is not defined via $\vec{\beta}_{\vec{G}} = 0$

because some couplings G_i have a non-zero

mass dimension like Newton's constant

$\rightarrow$ recall $[G_N] = -2$

$\hookrightarrow$ these are not pure numbers, but they define a scale

↳ we introduce dimensionless couplings $\vec{g}$ via

$$g_i = G_i\, k^{-d_{G_i}}$$

d_{G_i} : mass dimension of G_i

→ $\boxed{\text{scale symmetry is achieved if } \vec{\beta}_{\vec{g}} = 0}$

fixed point condition

how does this restore predictivity?

→ $\vec{\beta}_{\vec{g}} = 0$ provides the same number of equations as couplings that are studied

β functions are in general coupled systems of polynomials (in pert. theory) or non-polynomial expressions (e.g. algebraic or exponential)

$\rightsquigarrow$ from experience, these systems usually have only few real zeros

$\hookrightarrow$ finitely many choices

Example: a simple approximation of the β function of the dimensionless Newton coupling $g = G_N k^2$ is

$(d=4)$

$$\beta_g = 2g - 2g^2/g_* \qquad \text{with } g_* > 0$$

this term comes from the mass dimension:

$$\beta_g = k\partial_k(Gk^2) = 2k^2 G + k^2(k\partial_k G)$$
$$= 2g + \# g^2$$

fixed point condition $\beta_g = 0$ gives

either $g = 0$ or $g = g_*$

$\Rightarrow$ a scale-symmetric theory is expected to predict the values of all (dimensionless) couplings

however: nature is not scale-invariant!

↳ there are distinct scales in nature,

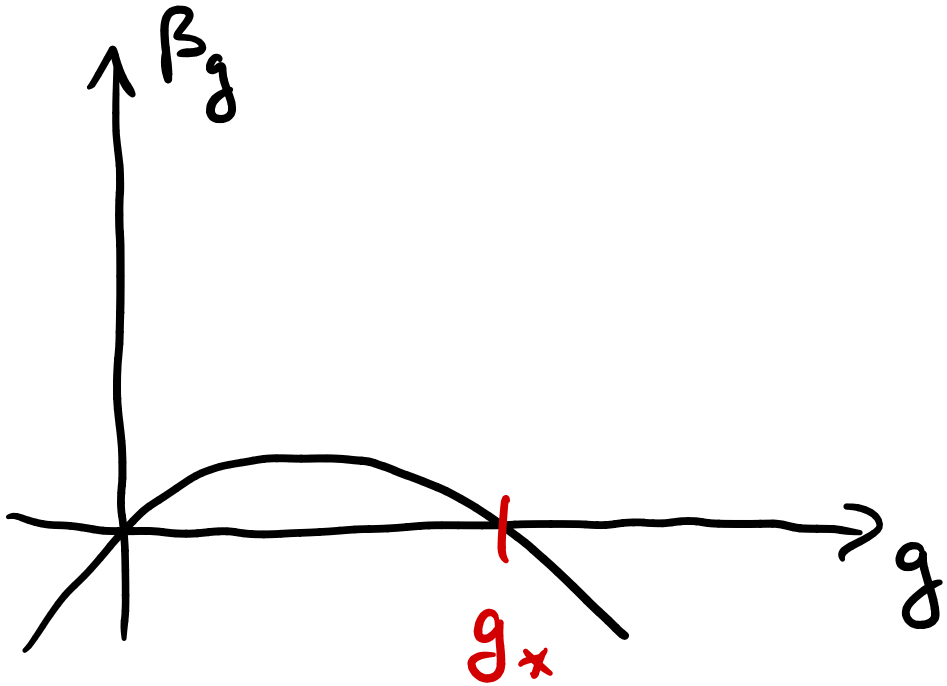
e.g. masses of elementary particles

consequence: scale symmetry can (at best) be realised asymptotically – in our case at very large k (= microscopically) – but there must be a scale k_{tr} at which a transition away from scale invariance takes place

→ how can this happen?

→ does this destroy predictivity again? (at least for $k < k_{tr}$)

consider again the above example, $\beta_g = 2g - 2g^2/g_*$



- if $\beta_g < 0$, then $k\partial_k g < 0$

  this means that the coupling grows towards lower k

  = the coupling is antiscreened

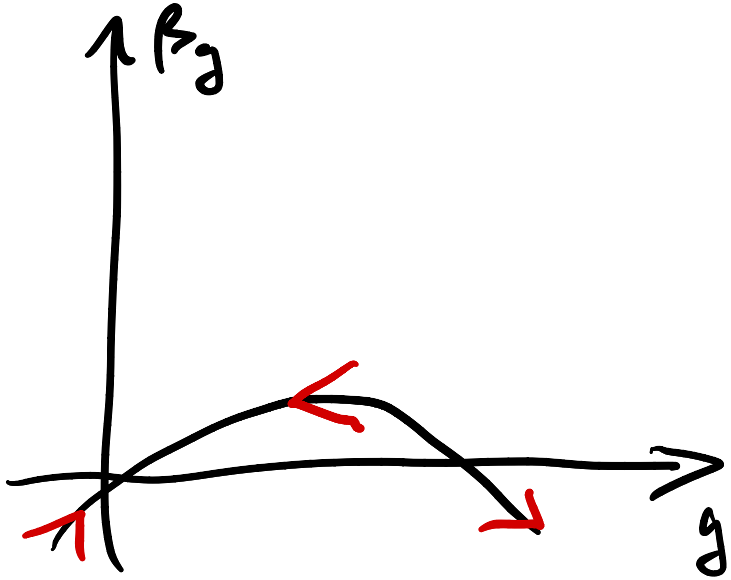
- if $\beta_g > 0$, then $k\partial_k g > 0$

  this means that the coupling decreases towards lower k

  = screening

different convention from earlier

arrows towards smaller k



$\Rightarrow$ the fixed point at $g = g_*$ is unstable in the sense that a tiny perturbation drives the coupling away from scale symmetry

we call such a coupling <u>relevant</u>, because the perturbation away from scale symmetry generically grows large at low energies, i.e., it is relevant for the dynamics of the theory

↳ relevant couplings allow us to get away from exact scale symmetry

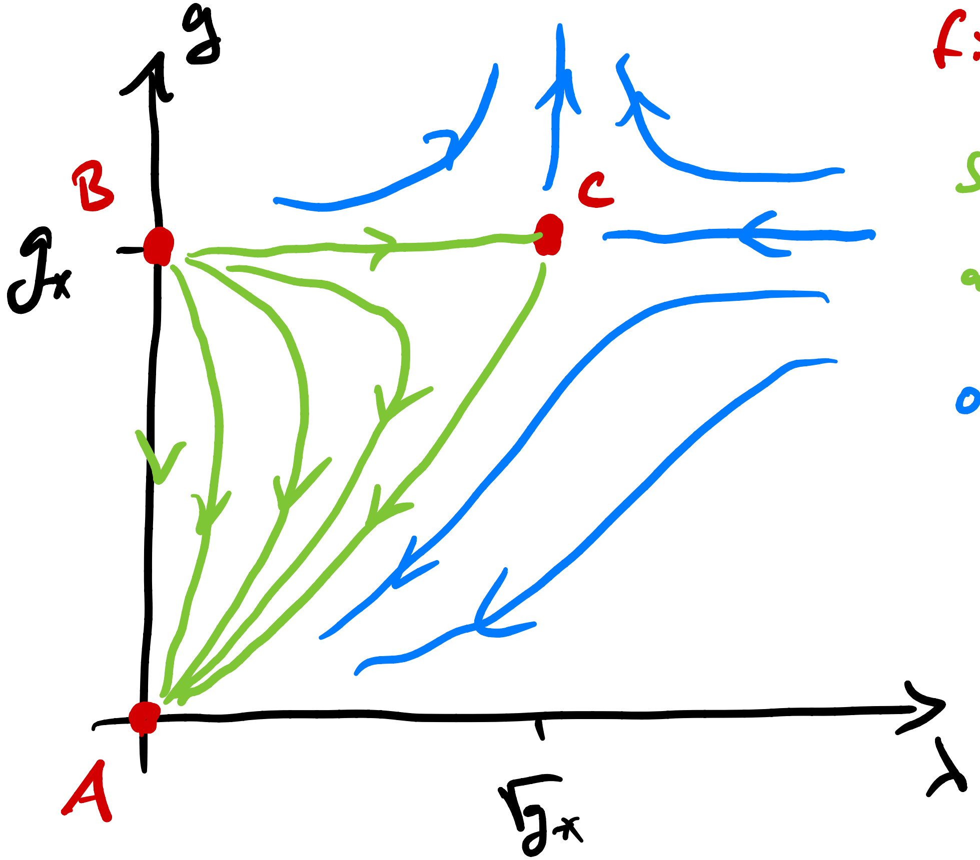
↳ we can reconcile scale symmetry at high energies/short distances with the existence of scales at low energies/large distances to answer the question about predictivity, we have to consider a second coupling λ

$$\rightarrow \begin{cases} \beta_g = 2g - 2\,g^2/g_* \\ \beta_\lambda = -g\lambda + \lambda^3 \end{cases}$$

fixed points: $g = 0$ $\Downarrow$ $\lambda = 0$ A

$g = g_*$ → $\lambda = 0$ B, $\lambda = \pm\sqrt{g_*}$ C

for simplicity, suppose in the following that we need $\lambda \geq 0$ → could be a gauge coupling phase diagram:



fixed points

solutions starting and ending at fixed points

other solutions

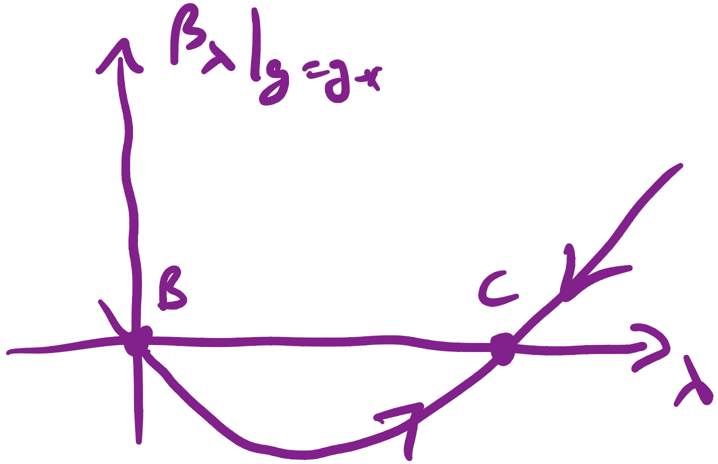
conditions apply, we will come back to this shortly

B has two relevant directions

$\Rightarrow$ two free parameters* $(g(k_{IR}), \lambda(k_{IR}))$

C has <u>one relevant</u> direction (g) and one irrelevant direction

we can see this by considering β_λ at $g = g_*$:



$\Rightarrow$ $\lambda(k_{IR})$ is predicted

A has two irrelevant directions

Summary: B is an example of how a fixed point can have predictive power even if the RG flow moves away from it