one-loop vs. two-loop

and a bit of c)

generalisation of QCD

consider $SU(N_c)$ Yang-Mills theory coupled to N_f fermions in the fundamental representation

no need to worry about the details, it is enough here to understand that there is a theory with this β function

$$\rightarrow \quad \beta_g = -\underbrace{\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{g^3}{16\pi^2}}_{\text{one loop term}} + \mathcal{O}(g^5)$$

g: $SU(N_c)$ gauge coupling

$\Rightarrow$ if $N_f < \frac{11}{2}N_c$, $\beta_g < 0$, i.e. antiscreening

$\Rightarrow$ asymptotic freedom

$\Rightarrow$ for $N_f > \frac{11}{2}N_c$, asymptotic freedom is lost

↳ can we get Asymptotic Safety instead?

→ add two loop term to investigate this case

↳ $\beta_g = \left(B + C \frac{g^2}{16\pi^2} \right) \frac{g^3}{16\pi^2}$, $B > 0$

$(N_f > \frac{11}{2} N_c)$

one can show: $C > 0$ if only fermions are present

⇒ no AS fixed point ☹

but: one can get $C < 0$ if one adds scalar fields

with a Yukawa coupling $(y \phi \bar{\psi} \psi)$

we will analyse this in the Veneziano limit:

we take $N_f, N_c \to \infty$ while

keeping $\varepsilon = \frac{N_f}{N_c} - \frac{11}{2}$ finite and arbitrarily small

pert. in ε

→ this gives perturbative control over the AS fixed point!

↳ we can neglect higher order loops

for this limit, we have to consider rescaled couplings:

$$\hat{\alpha}_y = \frac{N_c}{16\pi^2} y^2 , \quad \hat{\alpha}_g = \frac{N_c}{16\pi^2} g^2$$

Yukawa term

important: N_c

not important: $y \to g^2$, $g \to g^2$, $16\pi^2$

$$\Rightarrow \quad \beta_{\hat{\alpha}_g} = \left[\frac{4}{3}\varepsilon + \left(25 + \frac{26}{3}\varepsilon\right)\hat{\alpha}_g - 2\left(\frac{11}{2}+\varepsilon\right)^2 \hat{\alpha}_y \right] \hat{\alpha}_g^2$$

MVP

this minus allows for cancellations of one-loop and two-loop → a)

$$\beta_{\hat{\alpha}_y} = \left[(13 + 2\varepsilon)\hat{\alpha}_y - 6\hat{\alpha}_g \right] \hat{\alpha}_y$$

screening vs. antiscreening → c)

→ "LiSa" model

Litim, Sannino

JHEP 1412 178 (2014)

review: Eichhorn 1810.07615

Exercise : compute the AS fixed point and critical exponents to leading order in ε

$\hat{\alpha}_{g,y*} \neq 0$

FP :

- from $\beta_{\hat{\alpha}_y} = 0$, we get

$$\hat{\alpha}_{y*} = \frac{6}{13} \hat{\alpha}_{g*}$$

why can we neglect $\varepsilon \cdot \hat{\alpha}_{y*}$ and $\varepsilon \cdot \hat{\alpha}_{g*}$?

- from $\beta_{\hat{\alpha}_g} = 0$, we get

$$\frac{4}{3}\varepsilon + 25 \hat{\alpha}_{g*} - \frac{121}{2} \hat{\alpha}_{y*} = 0$$

$\rightarrow$ two equations for two unknowns

$\rightarrow$ solution: $\quad \hat{\alpha}_{g*} = \frac{26}{57}\varepsilon \quad , \quad \hat{\alpha}_{y*} = \frac{4}{19}\varepsilon$

$\rightarrow$ this is why we can neglect $\varepsilon \cdot \hat{\alpha}_{g/y*}$ – these products are higher order in ε

<u>crit. exp.</u>: stability matrix $M = \left.\begin{pmatrix} \frac{\partial \beta_{\hat{\alpha}_g}}{\partial \hat{\alpha}_g} & \frac{\partial \beta_{\hat{\alpha}_g}}{\partial \hat{\alpha}_y} \\ \frac{\partial \beta_{\hat{\alpha}_y}}{\partial \hat{\alpha}_g} & \frac{\partial \beta_{\hat{\alpha}_y}}{\partial \hat{\alpha}_y} \end{pmatrix}\right|_*$

leading order in ε

- $\left.\frac{\partial \beta_{\hat{\alpha}_g}}{\partial \hat{\alpha}_g}\right|_* = \left.\frac{\partial [\dots]}{\partial \hat{\alpha}_g} \hat{\alpha}_g^2\right|_* \simeq 25\,\hat{\alpha}_{g*}^2$

why no contribution from $[\,]\left.\frac{\partial \hat{\alpha}_g^2}{\partial \hat{\alpha}_g}\right|_*$?

- $$\left.\frac{\partial \beta_{\hat{\alpha}_g}}{\partial \hat{\alpha}_y}\right|_* = \left.\frac{\partial [\ldots]}{\partial \hat{\alpha}_y}\,\hat{\alpha}_g^2\right|_* \simeq -\frac{121}{2}\,\hat{\alpha}_{g*}^2$$

evaluation at FP $\Rightarrow [\ldots]=0$
$\Rightarrow$ no contribution

- $$\left.\frac{\partial \beta_{\hat{\alpha}_y}}{\partial \hat{\alpha}_g}\right|_* = \left.\frac{\partial [\ldots]}{\partial \hat{\alpha}_g}\,\hat{\alpha}_y\right|_* \simeq -6\,\hat{\alpha}_{y*} = -\frac{36}{13}\,\hat{\alpha}_{g*}$$

- $$\left.\frac{\partial \beta_{\hat{\alpha}_y}}{\partial \hat{\alpha}_y}\right|_* = \left.\frac{\partial [\ldots]}{\partial \hat{\alpha}_y}\,\hat{\alpha}_y\right|_* \simeq 13\,\hat{\alpha}_{y*} = 6\,\hat{\alpha}_{g*}$$

$$\Rightarrow \quad M \simeq \begin{pmatrix} 25\,\hat{\alpha}_{g*}^2 & -\frac{121}{2}\,\hat{\alpha}_{g*}^2 \\ -\frac{36}{13}\,\hat{\alpha}_{g*} & 6\,\hat{\alpha}_{g*} \end{pmatrix}$$

For critical exponents, we need (minus) the eigenvalues of M:

- $\det M = (-\theta_1)\cdot(-\theta_2) = \theta_1 \cdot \theta_2$

$$= -\frac{228}{13}\hat{\alpha}_{g*}^3 = \mathcal{O}(\varepsilon^3)$$

- $\operatorname{tr} M = -\theta_1 - \theta_2$

$$= \underbrace{6\hat{\alpha}_{g*}}_{\mathcal{O}(\varepsilon)} + \underbrace{25\hat{\alpha}_{g*}^2}_{\mathcal{O}(\varepsilon^2)}$$

$\Rightarrow$ one θ is $\mathcal{O}(\varepsilon)$, the other is $\mathcal{O}(\varepsilon^2)$

$\Rightarrow$ $\theta_1 \simeq \frac{104}{171}\varepsilon^2$, $\theta_2 \simeq -\frac{52}{19}\varepsilon$ $\Rightarrow$ $\hat{\alpha}_y$ can be predicted in terms of $\hat{\alpha}_g$!

canonical vs. quantum scaling

consider again β function in $SU(N_c)$ Yang-Mills theory

in $d=4$: $$\beta_g = -\frac{11}{3} N_c \frac{g^3}{16\pi^2} + \dots$$

different ϵ than above

__now__ consider $d>4$, in particular let $d = 4 + \epsilon$

$\rightarrow$ the gauge coupling acquires a mass dimension!

check: $$S \propto \int d^d x \, F_{\mu\nu} F^{\mu\nu}$$ suppressing colour indices

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iG [A_\mu, A_\nu]$

- kinetic term $\sim (\partial A)^2$

$$\Rightarrow 2\underbrace{[\partial]}_{=1} + 2[A] + \underbrace{[d^dx]}_{=-d} \overset{!}{=} 0$$

$$\Rightarrow [A] = \frac{d-2}{2}$$

- field strength tensor must have consistent mass dimension

$$\Rightarrow [\partial A] = [G] + [A^2]$$

$$\Rightarrow \quad [G] = [\partial] - [A] = \frac{4-d}{2} = -\frac{\epsilon}{2} < 0$$

$d = 4 + \epsilon$

$$\Rightarrow d_G = -\frac{\epsilon}{2}$$

↳ in $d = 4 + \epsilon$, β_g acquires a linear term:

$$\beta_g = \frac{\epsilon}{2} g - \frac{11}{3} N_c \frac{g^3}{16\pi^2} + \ldots$$

canonical scaling, screening

quantum fluctuation, anti-screening

$\Rightarrow$ for $\epsilon \ll 1$, there is a perturbative interacting fixed point at $g_*^2 = \frac{16\pi^2}{11 N_c} \frac{3}{2} \epsilon$

↳ for $g < g_*$, the one-loop term is sub-leading this mechanism is relevant for quantum gravity, because in $d=4$, $[G_N]=-2$, so that

$$\beta_g = 2g + \mathcal{O}(g^2)$$

$\Rightarrow$ we need antiscreening

recall earlier example

$$\beta_g = 2g - 2g^2/g_*$$

<u>note</u>: in general dimension, $[G_N] = 2-d$ check this!

$\hookrightarrow$ in $d=2+\epsilon$, $\beta_g = \epsilon g - \frac{2}{3}(19 + 6N_v - \frac{1}{2}N_f - N_s)g^2 + \dots$

vectors # fermions # scalars

$\Rightarrow$ QG is asymptotically safe in $2+\epsilon$ dimensions!

(for some matter content)

this however only gives some motivation, a lot can go wrong between $d=2$ and $d=4$:

$\rightarrow$ $g_* \propto \epsilon$, but $\epsilon \to 2$

$\rightarrow$ in $d=2$, $C_{\mu\nu\rho\sigma} = 0$, $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$

in $d=3$, $C_{\mu\nu\rho\sigma} = 0$

$\hookrightarrow$ unclear how higher-order curvature terms are "turned on" as $\epsilon \to 2$

Functional Renormalisation Group

or: how to compute non-perturbative β functions (finally!)

we follow this logic:

- evaluating the full path integral would be nice, but it is too complicated = integrals are hard [citation needed]
- instead of computing the full path integral in one step, we will integrate in small steps, and then derive a differential equation for how these "partial path integrals" change under a small step = differential equations are "easy"
- using an action functional is convenient because we can compute eqs. of motion, propagators, ...

↳ goal: compute effective action $\Gamma[\Phi]$ — we will make it precise later

→ analogue of classical action $S[\phi]$ but includes <u>all</u> quantum effects

→ with quantum fluctuations, EoMs of a "classical" field configuration do not have any physical meaning — path integral sums over all of them

$$\Rightarrow \quad \frac{\delta \Gamma}{\delta \Phi} = 0$$ are the EoMs for the expectation value $\Phi = \langle \phi \rangle$

for simplicity, we will now discuss everything for a scalar field ϕ, and discuss the generalisation to grav later

Effective action

starting point: path integral

$$= \int d^dx \, J(x)\phi(x)$$

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi] + iJ\cdot\phi}$$

related to $\frac{\delta Z}{\delta J}$

note: $\langle \phi \rangle_J := \dfrac{\int \mathcal{D}\phi \, \phi \, e^{iS[\phi]+iJ\cdot\phi}}{\int \mathcal{D}\phi \, e^{iS[\phi]+iJ\phi}}$ normalised

From our heuristic definition, we might want to define $\Gamma[\Phi]$ as the integrand evaluated at $\Phi = \langle\phi\rangle$ and with $S \to \Gamma$, so

$$Z[J] \overset{?}{=} e^{i\Gamma[\Phi] + iJ\cdot\Phi}$$

more precisely, $\Gamma[\Phi]$ and $Z[J]$ are related by a Legendre transform (almost) :

$$\Gamma[\Phi] = \sup_J \Big[-J \cdot \Phi \underbrace{- i \ln Z[J]}_{\text{often called } W[J] \text{ = Schwinger functional}} \Big]$$

!

choose J such that $[\ldots]$ is maximised for each Φ

$\Rightarrow$ $J_{\sup} = J_{\sup}[\Phi]$

$\rightsquigarrow$ Φ is the conjugate variable of J

this is the same "conjugate" as going from Lagrangian to Hamiltonian check: for a conjugate variable, we need

use path integral above

$$\Phi \overset{!}{=} \frac{\delta W[J_{sup}]}{\delta J_{sup}} = -i \frac{1}{Z[J_{sup}]} \frac{\delta Z[J_{sup}]}{\delta J_{sup}} = \langle \phi \rangle_{J_{sup}} \checkmark$$

J_{sup}: J that maximises [...] above for given Φ

EoM: compute

recall $J_{sup} = J_{sup}[\Phi]$!

$$\frac{\delta \Gamma}{\delta \Phi} = \frac{\delta}{\delta \Phi}\left[W[J_{sup}] - J_{sup} \cdot \Phi \right]$$

$$= -J_{sup} + \frac{\delta J_{sup}}{\delta \Phi} \underbrace{\left[\frac{\delta W}{\delta J_{sup}} - \Phi \right]}_{=0\,!}$$

$$\Rightarrow \quad \frac{\delta \Gamma}{\delta \Phi} = - J_{sup}$$

EoM for Φ in presence of source J_{sup}

$\Rightarrow$ for many interesting scenarios $J=0$, in analogy to classical EoM $\frac{\delta S}{\delta \Phi} = 0$

the analogy with the classical action suggests that we can parametrise Γ as

all possible field monomials, including derivatives

$$\Gamma[\Phi] = \int d^d x \sum_n G_n \, \mathcal{O}_n(\Phi)$$

physical values of couplings that include all quantum effects

$\Rightarrow$ we want to compute G_n !

example: recall the general scalar field theory with $\mathbb{Z}_2$ symmetry considered in the section on the stability matrix

$$\rightsquigarrow \quad \Gamma[\Phi] = \int d^dx \Big[\frac{1}{2} (\partial_\mu \Phi)(\partial^\mu \Phi) + \sum_{n\geq 1} G_{2n} \Phi^{2n}$$

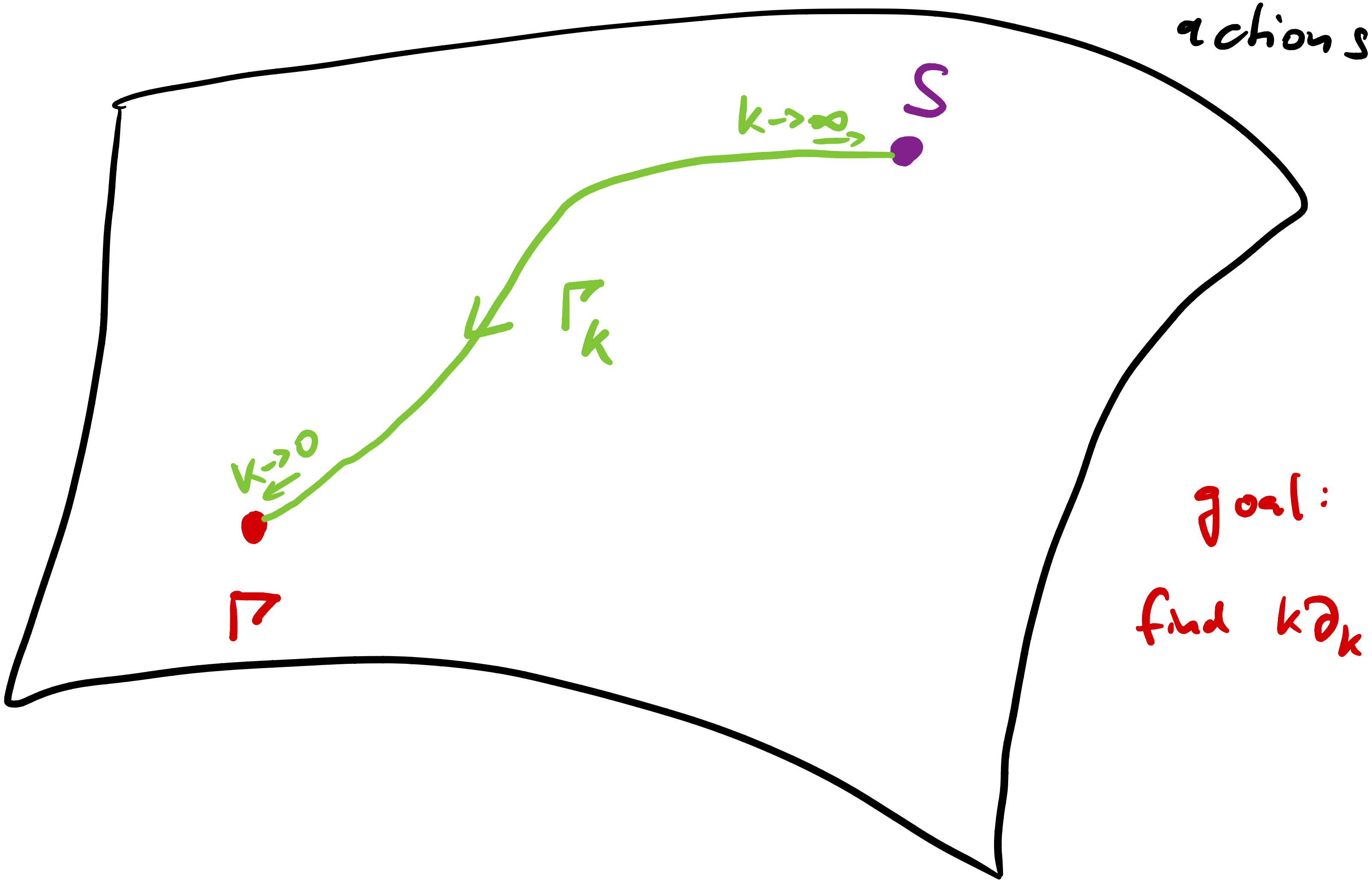
$$+ \sum_{n\geq 1} H_{2n} \Phi^{2n} (\partial_\mu \Phi)(\partial^\mu \Phi)$$

$$+ \dots \Big]$$

$\Rightarrow$ want to compute the G_{2n}, H_{2n}, ...

reminder

plan: instead of computing G_n directly, we compute $G_{n,k}$ which are the corresponding couplings for the "partial path integrals" at scale k (to be made precise below) in a sketch:



goal:

find $k\partial_k \Gamma_k$

Γ_k: effective action where we have performed the "partial path integral" for modes $\phi(p)$ with $p^2 > k^2$

$\rightsquigarrow$ high frequency modes

problem: in Minkowski signature, "high energy" does not necessarily mean p^2 large, because

$$p^2 = -E^2 + \vec{p}^{\,2}$$

$\Rightarrow$ any covariant cutoff on p^2 does not restrict the wavenumber nor the frequency

$\Rightarrow$ a cutoff on E and $\vec{p}$ separately breaks Lorentz invariance

Baldazzi, Percacci, Skrinjar 1811.03369

"solution": we consider the Euclidean generating functional

(and hope that this doesn't break anything...)