- recall that the regulator term ΔS_k has to be <u>quadratic</u> in the field

  → will be a problem for theories with non-linear symmetries (non-Abelian, gravity)

Regulator

let us collect the necessary properties of R_k:

- $\lim\limits_{p^2/k^2 \to 0} R_k(p^2) > 0$ ← IR regularisation
- $\lim\limits_{k \to 0} R_k(p^2) = 0$ ← $\lim\limits_{k\to 0} \Gamma_k = \Gamma$
- $\lim\limits_{k \to \infty} R_k(p^2) = \infty$ ← $\lim\limits_{k\to\infty} \Gamma_k \simeq S$

the general shape is largely arbitrary - some popular choices:

Litim regulator: $$R_k(p^2) = (k^2-p^2)\,\theta(1-p^2/k^2)$$

Heaviside theta: $\theta(x)=\begin{cases}1, & x>0\\ 0, & x<0\end{cases}$

most popular because allows analytical evaluation of loop integrals in easy approximations

but causes significant problems for extended approximations because it is a distribution

check properties above!

Exponential regulator: $$R_k(p^2) = \frac{p^2}{e^{p^2/k^2}-1}$$

smooth ⇒ avoids above problems but in many cases, only numerical evaluation of loop integrals possible

→ do results depend on the choice of regulator?

yes, but:

- measurable quantities are independent

  (if no approximations have been made)

  → this e.g. includes the existence of fixed points and the critical exponents, but not fixed point values of couplings

  Knorr 2012.06499

- in approximations: residual regulator dependence

  → measure for how good the approximation is

FRG workflow:

- pick physics to investigate: field content + symmetries
- write down approximation for Γ_k
- choose regulator
- compute RHS of Wetterich equation to get β functions
- analyse β functions

including Wick rotation

- extract Γ and do physics with it

(• repeat with better approximation)

Symmetries

- Symmetries play central role in QFT

e.g.
- gauge symmetry of SM: $SU(3) \times SU(2) \times U(1)$
- diffeomorphism symmetry

gauge symmetry = actually redundancies in description

"B−L" baryon minus lepton number is conserved in any SM process

"global" symmetry, or actually physical symmetry

- in the following, we assume the absence of <u>anomalies</u>

<u>detour</u>: gauge anomalies → Tong's lecture notes on SM

- anomaly = a symmetry of the classical theory breaks upon quantisation

→ path integral measure breaks symmetry

- this is <u>fatal</u> for gauge symmetries - with a gauge anomaly, we cannot remove the ghostly states that are pure gauge

⇒ gauge anomaly = theory is dead

- example: QED with a single <u>Weyl</u> fermion has a gauge anomaly

offending term:

$$\sim \underset{\substack{\uparrow \\ \text{longitudinal mode of photon}}}{\lambda} F_{\mu\nu} F^{*\mu\nu}$$

⇒ we can discard a large set of theories just like that (the SM is anomaly-free)

→ we need to check what happens with symmetries in the FRG

upshot: interplay between regularisation and symmetry

similar in perturbation theory:

- dimensional regularisation preserves gauge symmetry (that's why we like it)
- UV cutoff breaks gauge symmetry, but this does not mean you cannot use it

→ it is just a bit more complicated

mantra:
physics cannot depend on unphysical stuff

plan: first discuss what happens from $S \to \Gamma$, then talk about Γ_k

→ starting point: symmetry with infinitesimal generator $\mathcal{G}$

this means $\mathcal{G}\phi$ is linear in ϕ

e.g. SU(N):

$$\mathcal{G}^a = -\mathcal{D}^{ab}_\mu \frac{\delta}{\delta A^b_\mu}$$

$$\uparrow$$

$$\partial_\mu \delta^{ab} - \underbrace{g}_{\text{SU(N) coupling}} \underbrace{f^{abc}}_{\text{structure constants}} A^c_\mu$$

⇒ if $\mathcal{G} Z = 0$, the theory possesses the symmetry

↳ what happens to Γ?

let us compute $\mathcal{G} Z[J] = 0$, assuming that the measure is invariant, $\mathcal{G}[\mathcal{D}\phi] = 0$:

$$0 = \int \mathcal{D}\phi \; \mathcal{G} \, e^{-S[\phi] + J\cdot\phi}$$

$$\overset{\mathcal{G}\text{ acts linearly}}{=} \int \mathcal{D}\phi \left\{ -\mathcal{G}S[\phi] + J\cdot\mathcal{G}\phi \right\} e^{-S[\phi]+J\cdot\phi}$$

divide by $Z[J]$:

$$0 = -\langle \mathcal{G}S \rangle + \langle J \cdot \mathcal{G}\phi \rangle$$

evaluate this at $J = J_{sup} = \frac{\delta \Gamma[\Phi]}{\delta \Phi}$:

$$0 = -\langle \mathcal{G} S \rangle + \langle J_{\sup} \cdot \mathcal{G} \phi \rangle$$

$$= -\langle \mathcal{G} S \rangle + \langle \Gamma^{(1)}[\Phi] \cdot \mathcal{G} \phi \rangle$$

independent of ϕ

$\Rightarrow$ can be pulled out of $\langle \dots \rangle$

$$= -\langle \mathcal{G} S \rangle + \underbrace{\Gamma^{(1)}[\Phi] \cdot \underbrace{\langle \mathcal{G} \phi \rangle}_{= \mathcal{G} \langle \phi \rangle = \mathcal{G} \Phi}}_{= \mathcal{G} \Gamma[\Phi]}$$

$$\Rightarrow \quad 0 = -\langle \mathcal{G} S \rangle + \mathcal{G} \Gamma$$

↳ if the measure and the microscopic action S are invariant under a symmetry, then so is Γ ! ☺

of course, things are more complicated:

Complication #1: gauge symmetry → needs gauge-fixing procedure   we will talk about this soon

now assume $\mathcal{G}S = 0$

a similar derivation shows the Ward identity

$$\mathcal{W} = \mathcal{G}\Gamma - \langle \mathcal{G}(S_{gf} + S_{FP}) \rangle = 0$$

gauge fixing action   Faddeev-Popov ghost action fun fact: this forbids a mass term for the gluon

$m_A^2 A_\mu^a A^{a\mu}$ for Γ

$\Gamma \to \Gamma_k$

↘

Complication #2: Symmetry breaking by regulator

- best case scenario: regulator preserves symmetry
  - → can be achieved e.g. for O(N), chiral symmetry
- impossible for non-linear symmetries
  - → includes non-Abelian + gravity ☹

why? regulator term ΔS_k must be quadratic in field

⇒ non-linear symmetry does not

preserve this   this is BAD   you don't know it yet but trust me, it's bad

⇒ there is no gauge-invariant regulator

↳ modified Ward identity:

$$W_k = \mathcal{G}(\Gamma_k + \Delta S_k)[\Phi] - \langle \mathcal{G}(S_{gf} + S_{FP} + \Delta S_k)(\phi) \rangle = 0$$

note:
- $k \to 0$: $W_k \to W$
- one can show that $k\partial_k W_k \propto W_k$

  ⇒ RG flow preserves W_k but truncations break it

fun fact part 2: at $k>0$, we need a gluon mass $m_{A,k}^2 \simeq g^2 k^2$

this is because the regulator does its job
and provides k-dependent masses

morale: don't break symmetries (if you can)

for diffeomorphism invariance, we need to gauge-fix,

but there is more

why can't it ever be simple?

→ here be dragons

Regularisation in gravity

- recall: ΔS_k is quadratic in field
- try to construct a regulator in gravity:

$$\Delta S_k \overset{?}{=} \frac{1}{2}\int d^dx \sqrt{g}\; g_{\mu\nu}\, R_k(\_)^{\mu\nu\rho\sigma}\, g_{\rho\sigma}$$

problems:

- det g contains g $\Rightarrow$ not quadratic

  but $\sqrt{\det g}$ needed for coordinate invariance
- argument of regulator should be

$$\sim \Delta = -g^{\mu\nu} D_\mu D_\nu$$

not linear ↗ ↖ $D\bar{g} = 0$!

- indices of regulator need to be contracted – with what ?

in lack of a better way to do this – there is a high price to pay

"solution": background field method ← first lecture !

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

→ path integral over h

→ keep $\bar{g}$ general → background independence

→ we can write a regulator for h !

$$\hookrightarrow \quad \Delta S_k = \frac{1}{2}\int d^dx\, \underline{\sqrt{\bar{g}}}\; h_{\mu\nu}\, \underline{R_k(\bar{\Delta})^{\mu\nu\rho\sigma}}\, h_{\rho\sigma}$$

constructed from $\bar{g}$ alone, no h

the price : ~~your soul~~ diffeomorphism symmetry is broken (badly)

why? ΔS_k is <u>not</u> a functional of $\bar{g}+h=g$

without regulator, but with $g=\bar{g}+h$: two implementations of diff. :

quantum diff : $\bar{g}\to\bar{g}$, $h\to h+\mathcal{L}(\bar{g}+h)$

$\to$ all of the transformation is carried by h

background diff: $\bar{g} \to \bar{g} + \mathcal{L}_\xi \bar{g}$, $h \to h + \mathcal{L}_\xi h$

-> each field keeps "its" transformation

actually we even have a new symmetry: "split symmetry"

$$\bar{g} \to \bar{g} + X \,, \quad h \to h - X$$

with regulator (same for gauge-fixing but less bad):

-> background diff is kept

-> quantum diff is broken

-> split symmetry is broken

BIG headache, problem for future-you