Quantum Gravity and the Renormalization Group

Assignment 10 – Jan 12

Exercise 19: Heat kernel, part 1

Motivation: The purpose of this exercise (split into two parts) is to generalise the integral over the loop momentum to curved spacetimes. In other words, we will define the supertrace on a curved manifold. This is necessary to compute beta functions in gravity.

In this exercise, we will define the supertrace with the help of a simple example. Consider a “free” (in quotation marks, since nothing is free in life and/or when gravity is present) scalar field ϕ in dimension d with the action

$$\Gamma_{\phi,k} = \int \mathrm{d}^d x \, \sqrt{g} \, \left[\frac{1}{2} (D_\mu \phi)(D^\mu \phi) \right] . \tag{19.I}$$

Our goal is to compute the contribution of the quantum fluctuations of this scalar field to the beta functions of Newton’s coupling and the cosmological constant, while neglecting metric quantum fluctuations. In a Feynman diagram picture, this would correspond to having all internal lines being scalar lines. By diffeomorphism invariance, we expect that whatever these quantum fluctuations generate, the RG flow should be parametrisable in the form

$$\frac{1}{2} \mathrm{STr} \left[\dots \right] = \int \mathrm{d}^d x \, \sqrt{g} \, \left[c_0 + c_1 \, R + c_2 \, R^2 + c_3 \, R_{\mu\nu} R^{\mu\nu} + \dots \right] . \tag{19.II}$$

Concretely, our task is to compute the numerical coefficients c_i. To ease the notation, from now on we strip away all delta functions and only work with the operators, and we will work in position space.

a) Follow similar steps as in Exercise 18 and formally compute the argument of the supertrace. You should get something like

$$\frac{1}{2} \mathrm{STr} \, W(\Delta) \tag{19.III}$$

for the right-hand side of the flow equation, for a suitable function W and where $\Delta = -D^2 = g^{\mu\nu} D_\mu D_\nu$.

We will now make a “small” detour – instead of computing the supertrace of a general function, let us first compute the supertrace of a very particular function:

$$\mathrm{STr} \, e^{-s\Delta} \equiv \mathrm{STr} \, H(s) \, . \tag{19.IV}$$

The matrix elements of $H(s)$ can be defined in a position basis,

$$H(x, y; s) = \langle y | H(s) | x \rangle \, . \tag{19.V}$$

If we were able to compute these matrix elements, the supertrace would simply correspond to the sum over the eigenvalues. This information will come in handy later.

b) Argue that the matrix elements $H(x,y;s)$ fulfill the *heat equation*

$$\partial_s H(x,y;s) = -\Delta_x H(x,y;s)\,, \tag{19.VI}$$

with initial condition

$$H(x,y;0) = \delta(x-y)\,. \tag{19.VII}$$

Above, the subscript x on Δ_x indicates that the operator acts with respect to the position x, and we will always take this convention, omitting the subscript. What is the meaning of the variable s?

We can now finally define the supertrace:

$$\boxed{\mathrm{STr}\, e^{-s\Delta} = \mathrm{tr}\int \mathrm{d}^d x\,\sqrt{g}\langle x|e^{-s\Delta}|x\rangle = \mathrm{tr}\int \mathrm{d}^d x\,\sqrt{g}\,H(x,x;s)\,.} \tag{19.VIII}$$

Here, tr indicates the trace over discrete indices (e.g. spacetime indices, or the $O(N)$ indices from Exercise 18). The expression $H(x,x;s)$ is called the *coincidence limit* of the heat kernel (since both points coincide). This is often indicated with an overbar, $H(x,x;s) \equiv \overline{H}(s)$.

c) Time for some sanity checks. Show that for a flat spacetime where $\Delta = -\partial^2$, the heat kernel elements read

$$H^{\mathrm{flat}}(x,y;s) = \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{(x-y)^2}{4s}}\,. \tag{19.IX}$$

Next, make sure that our definition of the supertrace actually reproduces the standard loop momentum integral in flat spacetime. That is, show that

$$\overline{H}^{\mathrm{flat}}(s) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{-s\,p^2} \tag{19.X}$$

by explicit computation. The right-hand side here should ring a bell (Exercise 18 again) and comes from (19.VIII) using that STr is essentially a loop momentum integral, just that the integrand is the exponential instead of the propagator times the derivative of the regulator.

It is time to get serious now. How does the heat kernel $H(x,y;s)$ look like in an arbitrary, curved spacetime? Let us take some inspiration from the flat heat kernel (19.IX). The first thing we have to "upgrade" is the coordinate difference in the exponential, $(x-y)^2$. What would a suitable generalisation look like? If you guessed the geodesic distance, you are correct! For reasons, we define the *world function* $\sigma(x,y)$ to be *half* of the square of the geodesic distance. The world function has a funny property, namely

$$\frac{1}{2}(D_\mu\sigma(x,y))(D^\mu\sigma(x,y)) = \sigma(x,y)\,. \tag{19.XI}$$

Recall that all derivatives act on the x-coordinate. Convince yourself that this relation makes sense in flat spacetime.

So far so good, but this might not be the only upgrade needed. Let us thus make the ansatz

$$H(x,y;s) = \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}}\,\Omega(x,y;s)\,, \tag{19.XII}$$

where Ω parameterises the information on the curvature of the manifold. In particular, we have the boundary condition that $\Omega = 1$ whenever the manifold is flat.

d) What is the coincidence limit of the world function σ, that is, what is $\overline{\sigma} = \sigma(x,x)$?

e) What are the mass dimensions of s and Ω? Use this to argue with which powers of s curvatures and covariant derivatives in Ω have to come. From this, can you make the boundary condition $\Omega = 1$ when the spacetime is flat more precise?

We will now use the heat equation to compute Ω.

f) Insert the ansatz for the heat kernel into the heat equation to derive a differential equation for Ω.

g) Make a power series ansatz for Ω in powers of s, i.e. assume that

$$\Omega(x,y;s) \sim \sum_{n\geq 0} s^n A_n(x,y)\,, \qquad s \to 0\,. \tag{19.XIII}$$

Using this ansatz, derive a recursion relation for the coefficients A_n (this should still be a differential equation). What does the boundary condition from e) translate to in terms of the A_n?

This completes the setup of the computation. In part 2 of this exercise, we will solve the recursion relation to some low order.

a) Following the same steps as previously, we find

$$W(\Delta) = \frac{k\partial_k R_k(\Delta)}{\Delta + R_k(\Delta)}\,. \tag{19.1}$$

b) For this, we take the definition $H(s) = e^{-s\Delta}$. Taking the s-derivative, we find

$$\partial_s H(s) = \partial_s e^{-s\Delta} = -\Delta\, e^{-s\Delta} = -\Delta\, H(s)\,. \tag{19.2}$$

The initial condition is $H(0) = e^{-0\Delta} = 1$, or in position basis,

$$H(x,y;0) = \langle y|x\rangle = \delta(x-y)\,. \tag{19.3}$$

Looking at the heat equation, the meaning of s is that of the time variable.

c) To show that (19.IX) solves the flat heat equation, we compare the s-derivative with the Laplacian acting on it. On the one hand,

$$\partial_s H^{\text{flat}}(x,y;s) = \left[-\frac{d}{2s} + \frac{(x-y)^2}{4s^2}\right] H^{\text{flat}}(x,y;s)\,. \tag{19.4}$$

On the other hand,

$$\begin{aligned}
\Delta_x H^{\text{flat}}(x,y;s) &= -\eta^{\mu\nu}\partial_\mu\partial_\nu H^{\text{flat}}(x,y;s) \\
&= -\eta^{\mu\nu}\partial_\mu \left[\left(-\frac{x_\nu - y_\nu}{2s}\right) H^{\text{flat}}(x,y;s)\right] \\
&= -\eta^{\mu\nu}\left[-\frac{\eta_{\mu\nu}}{2s} + \frac{x_\nu - y_\nu}{2s}\frac{x_\mu - y_\mu}{2s}\right] H^{\text{flat}}(x,y;s) \\
&= \left[\frac{d}{2s} - \frac{(x-y)^2}{4s^2}\right] H^{\text{flat}}(x,y;s)\,.
\end{aligned} \tag{19.5}$$

By comparison, we see that $H^{\text{flat}}(x,y;s)$ indeed solves the heat equation. Also the initial condition is fulfilled: whenever $x \neq y$, the exponential forces the whole expression to zero as $s \to 0$. However, if $x = y$, the exponential is the identity, and in the limit $s \to 0$ we get infinity.

For the second part, we first have that the coincidence limit of the flat heat kernel is

$$\overline{H}(s) = \left(\frac{1}{4\pi s}\right)^{d/2} . \tag{19.6}$$

On the other hand, using the previous definition, the supertrace of the exponential is

$$\begin{aligned} \operatorname{STr} e^{-s\,p^2} &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{-s\,p^2} \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}\Omega \int_0^\infty \mathrm{d}p\, p^{d-1}\, e^{-s\,p^2} \\ &= \frac{1}{(2\pi)^d} \left[\frac{2\pi^{d/2}}{\Gamma(d/2)}\right] \left[\frac{\Gamma(d/2)}{2s^{d/2}}\right] \\ &= \left(\frac{1}{4\pi s}\right)^{d/2} . \end{aligned} \tag{19.7}$$

Here, we used d-dimensional spherical coordinates, and the integral over $\mathrm{d}\Omega$ represents the integral over the $(d-1)$ angles.

d) The world function is proportional to the square of the geodesic distance, so its coincidence limit vanishes,

$$\overline{\sigma} = 0 . \tag{19.8}$$

e) The heat kernel $H(s) = e^{-s\Delta}$ has the product $s\Delta$ in the exponential, which to make sense should be dimensionless. This implies that s has mass dimension -2. For the mass dimension of Ω, we compare our ansatz for the curved heat kernel with that of the flat heat kernel, we find that Ω must be dimensionless. This is also clear from the boundary condition $\Omega = 1$ when the spacetime is flat.

Since Ω parameterises curvature information, we can span it in powers of curvatures and covariant derivatives. Because Ω is dimensionless, all curvatures are accompanied by a factor of s, and all covariant derivatives are accompanied by a factor of $\sqrt{s}$. Since derivatives always come in even powers, we will only have integer powers of s in Ω.

Following this logic, sending the curvature to zero can be implemented by sending s to zero. This implies that the precise boundary condition reads

$$\overline{\Omega}(0) = 1 . \tag{19.9}$$

We will see soon why the coincidence limit is necessary here.

f) Let us first compute the s-derivative of our ansatz. We have

$$\partial_s H(x,y;s) = \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}} \left[\left(-\frac{d}{2s} + \frac{\sigma(x,y)}{2s^2}\right) \Omega(x,y;s) + \partial_s \Omega(x,y;s)\right] . \tag{19.10}$$

On the other hand, acting with the Laplacian on it gives

$$
\begin{aligned}
\Delta\, H(x,y;s) &= -g^{\mu\nu} D_\mu D_\nu H(x,y;s) \\
&= -g^{\mu\nu} D_\mu \left[\left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}} \left(-\frac{1}{2s} (D_\nu \sigma(x,y))\, \Omega(x,y;s) + (D_\nu \Omega(x,y;s)) \right) \right] \\
&= -g^{\mu\nu} \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}} \times \\
&\quad \left[-\frac{1}{2s} (D_\mu \sigma(x,y)) \left(-\frac{1}{2s} (D_\nu \sigma(x,y))\, \Omega(x,y;s) + (D_\nu \Omega(x,y;s)) \right) + \right. \\
&\quad \left. \left(-\frac{1}{2s} (D_\mu D_\nu \sigma(x,y))\, \Omega(x,y;s) - \frac{1}{2s} (D_\nu \sigma(x,y))\, (D_\mu \Omega(x,y;s)) + (D_\mu D_\nu \Omega(x,y;s)) \right) \right] \\
&= \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}} \left[-\frac{1}{4s^2} (D^\mu \sigma(x,y))(D_\mu \sigma(x,y)) \Omega(x,y;s) + \frac{1}{s} (D^\mu \sigma(x,y))(D_\mu \Omega(x,y;s)) \right. \\
&\quad \left. - \frac{1}{2s} (D^2 \sigma(x,y))\, \Omega(x,y;s) - (D^2 \Omega(x,y;s)) \right] \\
&= \left(\frac{1}{4\pi s}\right)^{d/2} e^{-\frac{\sigma(x,y)}{2s}} \left[-\frac{1}{2s^2} \sigma(x,y) \Omega(x,y;s) + \frac{1}{s} (D^\mu \sigma(x,y))(D_\mu \Omega(x,y;s)) \right. \\
&\quad \left. - \frac{1}{2s} (D^2 \sigma(x,y))\, \Omega(x,y;s) - (D^2 \Omega(x,y;s)) \right] .
\end{aligned} \tag{19.11}
$$

In the last step, we used the funny property of the world function, (19.XI), to rewrite the first term. If we now equate both sides, exactly this term cancels, and we find

$$
\left[\left(-\frac{d}{2s} + \partial_s - D^2 + \frac{1}{2s} (D^2 \sigma(x,y)) \right) \Omega(x,y;s) + \frac{1}{s} (D_\mu \sigma(x,y))(D^\mu \Omega(x,y;s)) \right] = 0 . \tag{19.12}
$$

g) We now insert the ansatz in powers of s and collect equal powers. For this, we first note that

$$
\partial_s \Omega(x,y;s) = \sum_{n \geq 0} n\, s^{n-1}\, A_n(x,y) . \tag{19.13}
$$

Covariant derivatives simply act on the A_n and do not change the structure in s. With this at hand, we have

$$
\sum_{n \geq 0} s^n \left[\left(\frac{n}{s} - \frac{d}{2s} + \frac{1}{2s} (D^2 \sigma(x,y)) \right) A_n(x,y) + \frac{1}{s} (D_\mu \sigma(x,y))(D^\mu A_n(x,y)) - D^2 A_n(x,y) \right] = 0 . \tag{19.14}
$$

Only the last term has no extra $1/s$, and we thus have to shift the index on it. Enforcing that the equation is true for all powers of s individually, we then find

$$
\left(n - \frac{d}{2} + \frac{1}{2} \left(D^2 \sigma(x,y) \right) \right) A_n(x,y) + (D^\mu \sigma(x,y))\, (D_\mu A_n(x,y)) - D^2 A_{n-1}(x,y) = 0 . \tag{19.15}
$$

This equation is true for $n \geq 0$, and we have to impose $A_{-1}(x,y) = 0$. The original boundary condition translates into $A_0 = 1$.

Extra material 5: Heat kernel beyond free scalar fields

The case of a scalar field that we are treating is of course only the simplest case. The whole heat kernel technology can be applied more generally though. The first (and straightforward) generalisation is that you can add an *endomorphism* $\mathfrak{E}$ to the Laplacian. That is, one can compute

$$\mathrm{STr}\, e^{-s(\Delta+\mathfrak{E})} \tag{19.16}$$

with the same techniques. What is $\mathfrak{E}$? This should be a multiplicative operator, and not a differential operator. The simplest case would be a mass term, $\mathfrak{E} = m^2$, but more commonly it refers to a curvature term, e.g. $\mathfrak{E} = R$. Everything goes through in the same way (convince yourself of that!), the dependence of the heat kernel is stored entirely in Ω, and the recursion relation for the A_n simply gets an extra term,

$$\left(n - \frac{d}{2} + \frac{1}{2}\left(D^2\sigma(x,y)\right)\right) A_n(x,y) + \left(D^\mu\sigma(x,y)\right)\left(D_\mu A_n(x,y)\right) - D^2 A_{n-1}(x,y) + \mathfrak{E}(x)\, A_{n-1}(x,y) = 0\,. \tag{19.17}$$

The second extension is more profound, but important. Say you want to compute the heat kernel for the kinetic operator of a photon. In harmonic gauge, the kinetic term reads

$$\frac{1}{2} A^\mu \left[\Delta\, \delta_\mu{}^\nu + R_\mu{}^\nu\right] A_\nu\,. \tag{19.18}$$

We see the immediate complication: the operator has indices (and an endomorphism, but we already discussed this). More generally, let's say we have a general field Φ^A, where A denotes any collection of indices, be it spacetime, gauge, or $O(N)$ indices. A is then called a de Witt superindex. By definition, the field always has a single upper superindex. This means that e.g. for the metric, $\Phi^A \mapsto g_{\mu\nu}$, so A corresponds to *two lower spacetime indices*.
A general kinetic term will then contain a differential operator that has two such indices,

$$\frac{1}{2} \Phi_A\, \Delta^A{}_B\, \Phi^B\,. \tag{19.19}$$

We will assume that the operator $\Delta^A{}_B$ is of second order and *minimal*, that means that the operator is basically a fancy Laplacian. You can of course add an endomorphism, which now also has two superindices, $\mathfrak{E}^A{}_B$. For example for gravity, you could have $\mathfrak{E}^A{}_B \mapsto R_{\mu\nu}{}^{\rho\sigma}$.
How does the heat kernel technique change? Actually, not that much. We would now compute

$$\mathrm{STr}\, e^{-s\Delta^A{}_B}\,, \tag{19.20}$$

and correspondingly, also H (and Ω, and the A_n) receive two superindices. As a matter of fact, all these objects are bi-tensors: the first superindex refers to the first position (x above), whereas the second superindex refers to the second position (y above). This means for example that

$$\Omega^A(x,y;s)_B \tag{19.21}$$

is an object that transforms like the field itself at x, and like the dual field at y. Notice the fancy index notation to indicate this (but you can just stick to normal notation). The tr in the definition of the supertrace then simply corresponds to the trace over the superindices, that is

$$\mathrm{tr} H^A{}_B = H^A{}_B \delta^B{}_A\,. \tag{19.22}$$

How are such indices contracted? This needs the identity operator for superindices, which by definition is given by

$$\delta^A{}_B\,\delta(x-y) = \frac{\delta\Phi^A(x)}{\delta\Phi^B(y)}\,. \tag{19.23}$$

Note that the world function will remain indexfree.
Often, the operator $\Delta^A{}_B$ contains a linear derivative term – think of the kinetic operator of a charged field. This can be absorbed by using gauge-covariant derivatives. For example, if the kinetic operator would read

$$\Delta = -\partial^2 - 2a^\mu\partial_\mu + b\,, \tag{19.24}$$

one can introduce a covariant derivative $\nabla_\mu = \partial_\mu + a_\mu$, so that

$$\Delta = -\nabla^2 + (\partial^\mu a_\mu) + a^\mu a_\mu + b = -\nabla^2 + (\nabla^\mu a_\mu) + b\,. \tag{19.25}$$

This operator is now again of the form Laplacian plus endomorphism, with $\mathfrak{E} = (\nabla^\mu a_\mu) + b$. All covariant derivatives in the computation of heat kernel coefficients now correspond to the gauge-covariant ∇.
There is one final complication, but this will only appear at a later point in the heat kernel computation. We will nevertheless discuss it now. At some point in the computation, one has to compute commutators of the covariant derivative acting on bi-tensors like A_n. For example, we will need to compute

$$[\nabla_\mu, \nabla_\nu]\; A_n^A(x,y)_B\,. \tag{19.26}$$

Once again, our convention is that covariant derivatives act at the position x. Now it is extremely crucial that A_n is a bi-tensor, i.e. at x it transforms like the field Φ^A, and not like an object with two superindices! What is the commutator of (gauge-)covariant derivatives? If you guessed something like curvature or field strength, you are correct. We define the generalised field strength/curvature tensor corresponding to the above commutator as $\mathcal{F}$, so that

$$[\nabla_\mu, \nabla_\nu]\; A_n^A(x,y)_B = \mathcal{F}_{\mu\nu}{}^A{}_C\, A_n^C(x,y)_B\,. \tag{19.27}$$

This generalised field strength depends on the precise meaning of the superindices. For example, suppose that we are looking at the heat kernel coefficient of a photon. Then $A=\mu$, and we have (taking $\nabla = D$ for simplicity)

$$[D_\mu, D_\nu]\; A_n^\alpha(x,y)_\beta = R_{\mu\nu}{}^\alpha{}_\lambda\, A_n^\lambda(x,y)_\beta\,, \tag{19.28}$$

and we read off

$$\mathcal{F}_{\mu\nu}{}^A{}_C \mapsto R_{\mu\nu}{}^\alpha{}_\lambda\,, \qquad (A\mapsto\alpha\,, C\mapsto\lambda)\,. \tag{19.29}$$

Notice how it is really crucial that the commutator only “sees” the upper index of A_n, and not the lower one that “sits at y”.
Instead, if we considered the heat kernel of the graviton, we would have

$$[D_\mu, D_\nu]\; A_n^{\alpha\beta}(x,y)_{\gamma\delta} = R_{\mu\nu}{}^\alpha{}_\lambda\, A_n^{\lambda\beta}(x,y)_{\gamma\delta} + R_{\mu\nu}{}^\beta{}_\lambda\, A_n^{\alpha\lambda}(x,y)_{\gamma\delta}\,, \tag{19.30}$$

and so

$$\mathcal{F}_{\mu\nu}{}^A{}_C \mapsto \frac{1}{2}R_{\mu\nu}{}^\alpha{}_\kappa\delta^\beta{}_\lambda + \frac{1}{2}R_{\mu\nu}{}^\alpha{}_\lambda\delta^\beta{}_\kappa + \frac{1}{2}R_{\mu\nu}{}^\beta{}_\kappa\delta^\alpha{}_\lambda + \frac{1}{2}R_{\mu\nu}{}^\beta{}_\lambda\delta^\alpha{}_\kappa\,, \qquad (A\mapsto\alpha\beta\,, C\mapsto\kappa\lambda)\,. \tag{19.31}$$

Last but not least, consider a pure $U(1)$ gauge-covariant derivative in flat spacetime, that is $\nabla = \partial + a$, and assume that we are computing the heat kernel for a charged scalar. In that case,

$$[\nabla_\mu, \nabla_\nu] \, A_n(x,y) = [(\partial_\mu a_\nu) - (\partial_\nu a_\mu)] \, A_n(x,y) = F_{\mu\nu} \, A_n(x,y) \,, \tag{19.32}$$

so that the generalised field strength is just the standard Abelian field strength, ${\mathcal{F}_{\mu\nu}}^A{}_C \mapsto F_{\mu\nu}$. Makes sense, eh?

Some final words of caution: sometimes the generalised field strength is defined with a minus sign in (19.27), so it is always important to check conventions. Also, in some works, it is called Ω instead of $\mathcal{F}$ – don't confuse it with the factor in the heat kernel itself!