Quantum Gravity and the Renormalization Group

Assignment 11 – Jan 19

Exercise 20: Heat kernel, part 2

Motivation: This is part 2 of the heat kernel. Are you exhausted yet? Good, it's going to become much worse. :)

At the end of the last sheet, you should have obtained the following recursion relation for the coefficients A_n:

$$\left(n - \frac{d}{2} + \frac{1}{2}\left(D^2\sigma(x,y)\right)\right) A_n(x,y) + \left(D^\mu\sigma(x,y)\right)\left(D_\mu A_n(x,y)\right) - D^2 A_{n-1}(x,y) = 0\,, \tag{20.I}$$

with

$$A_0(x,x) \equiv \overline{A_0} = 1\,, \qquad A_{-1}(x,y) = 0\,, \qquad n \geq 0\,. \tag{20.II}$$

From now on, we will drop the position arguments. The goal of this exercise is to compute the coincidence limit of the first heat kernel coefficient, $\overline{A_1}$.

a) Set $n = 1$ in (20.I) and take the coincidence limit to understand which ingredients you need to compute $\overline{A_1}$. Beware: the coincidence limit does *not* commute with covariant derivatives, i.e. $\overline{D^2 A_0} \neq D^2\overline{A_0}$!

b) One of the ingredients to compute $\overline{A_1}$ is $\overline{D^2 A_0}$. Derive an equation for the latter by acting with D^2 on (20.I) and taking the coincidence limit.

At this point you should get worried about the recursion, but maybe there is some hope after all. Instead of focussing on different derivatives of the A_n, let us switch our focus and try to compute coincidence limits of derivatives of the world function. Recall that

$$\frac{1}{2}(D^\mu\sigma(x,y))(D_\mu\sigma(x,y)) = \sigma(x,y)\,, \tag{20.III}$$

for *any* x, y, i.e. even away from the coincidence limit.

c) **[hard question]** Use (20.III) to compute $\overline{\sigma}$, $\overline{D_\mu\sigma}$, $\overline{D_\mu D_\nu\sigma}$, $\overline{D_\mu D_\nu D_\kappa\sigma}$ and $\overline{D_\mu D_\nu D_\kappa D_\lambda\sigma}$. *Hints:* take successive covariant derivatives of (20.III). Then take the coincidence limit of these equations and solve iteratively. You might have to commute covariant derivatives. Think about what you can pull out of the coincidence limit.

If you succeeded, you should feel relieved now if you look back at the equations for $\overline{A_1}$ and $\overline{D^2 A_0}$.

d) Use your results from c) to compute $\overline{D^2 A_0}$, and from there compute $\overline{A_1}$.

This illustrates the general procedure, and you can follow the same recipe to compute the A_n for larger n. It goes without saying that once again, this should not be done by hand.

a) Taking $n = 1$ and the coincidence limit, we find

$$\left(1 - \frac{d}{2} + \frac{1}{2}\left(\overline{D^2\sigma}\right)\right)\overline{A_1} + \left(\overline{D^\mu\sigma}\right)\left(\overline{D_\mu A_1}\right) - \overline{D^2 A_0} = 0\,. \tag{20.1}$$

To compute $\overline{A_1}$ from this, we thus need $\overline{D^2\sigma}$, $\overline{D^\mu\sigma}$, but also $\overline{D^2 A_0}$ and in particular $\overline{D_\mu A_1}$. The latter seems to break the recursion.

b) We act with D^2 on the recursion relation and find

$$\begin{aligned} \left(n - \frac{d}{2} + \frac{1}{2}\left(D^2\sigma(x,y)\right)\right) D^2 A_n(x,y) + \frac{1}{2}\left(D^2 D^2\sigma(x,y)\right) A_n(x,y) \\ + \left(D^\mu D^2\sigma(x,y)\right)\left(D_\mu A_n(x,y)\right) + \left(D^2 D^\mu\sigma(x,y)\right)\left(D_\mu A_n(x,y)\right) + \left(D^\mu\sigma(x,y)\right)\left(D^2 D_\mu A_n(x,y)\right) \\ +2\left(D^\nu D^\mu\sigma(x,y)\right)\left(D_\nu D_\mu A_n(x,y)\right) - D^2 D^2 A_{n-1}(x,y) = 0\,. \end{aligned} \tag{20.2}$$

Taking the coincidence limit and $n = 0$, we find

$$\begin{aligned} \left(-\frac{d}{2} + \frac{1}{2}\left(\overline{D^2\sigma}\right)\right)\overline{D^2 A_0} + \frac{1}{2}\left(\overline{D^2 D^2\sigma}\right)\overline{A_0} \\ + \left(\overline{D^\mu D^2\sigma}\right)\left(\overline{D_\mu A_0}\right) + \left(\overline{D^2 D^\mu\sigma}\right)\left(\overline{D_\mu A_0}\right) + \left(\overline{D^\mu\sigma}\right)\left(\overline{D^2 D_\mu A_0}\right) \\ +2\left(\overline{D^\nu D^\mu\sigma}\right)\left(\overline{D_\nu D_\mu A_0}\right) = 0\,. \end{aligned} \tag{20.3}$$

This means that in order to compute $\overline{D^2 A_0}$, we also need $\overline{D^2 D_\mu A_0}$, once again breaking the recursion. Is this all nonsense?

c) Maybe the world function saves the day. First of all, we already know that

$$\overline{\sigma} = 0\,. \tag{20.4}$$

Using this when taking the coincidence limit of the funny property of the world function (20.III), we can directly conclude that

$$\overline{D^\mu\sigma}\,\overline{D_\mu\sigma} = 0 \qquad \Rightarrow \qquad \overline{D_\mu\sigma} = 0\,. \tag{20.5}$$

Aha! This already kicks out some of the worrisome terms above. Let us now take a derivative of (20.III):

$$\left(D_\alpha D_\mu\sigma(x,y)\right)\left(D^\mu\sigma(x,y)\right) = D_\alpha\sigma(x,y)\,. \tag{20.6}$$

Taking the coincidence limit, we do not learn anything new, so another derivative it is:

$$\left(D_\beta D_\alpha D_\mu\sigma(x,y)\right)\left(D^\mu\sigma(x,y)\right) + \left(D_\alpha D_\mu\sigma(x,y)\right)\left(D_\beta D^\mu\sigma(x,y)\right) = D_\beta D_\alpha\sigma(x,y)\,. \tag{20.7}$$

In the coincidence limit, the first term on the left-hand side vanishes, and we find

$$\overline{D_\alpha D_\mu\sigma}\,\overline{D_\beta D^\mu\sigma} = \overline{D_\beta D_\alpha\sigma}\,. \tag{20.8}$$

This means that the coincidence limit of the second covariant derivative of the world function is idempotent. Which geometric quantities have this property? Exactly, only the metric, so we conclude

$$\overline{D_\beta D_\alpha\sigma} = g_{\beta\alpha}\,. \tag{20.9}$$

Convince yourself that this property makes sense in a flat manifold. As a special case, we have

$$\overline{D^2\sigma} = d\,. \tag{20.10}$$

Let us press on and take yet another derivative. We find

$$\begin{aligned}
&(D_\gamma D_\beta D_\alpha D_\mu \sigma(x,y))\,(D^\mu\sigma(x,y)) + (D_\beta D_\alpha D_\mu\sigma(x,y))\,(D_\gamma D^\mu\sigma(x,y))\\
&+(D_\gamma D_\alpha D_\mu\sigma(x,y))\,(D_\beta D^\mu\sigma(x,y)) + (D_\alpha D_\mu\sigma(x,y))\,(D_\gamma D_\beta D^\mu\sigma(x,y)) = D_\gamma D_\beta D_\alpha\sigma(x,y)\,.
\end{aligned} \tag{20.11}$$

Upon taking the coincidence limit and using our previous results, we get some simplifications, and arrive at

$$\overline{D_\beta D_\alpha D_\gamma\sigma} + \overline{D_\gamma D_\alpha D_\beta\sigma} + \overline{D_\gamma D_\beta D_\alpha\sigma} = \overline{D_\gamma D_\beta D_\alpha\sigma}\,, \tag{20.12}$$

or

$$\overline{D_\beta D_\alpha D_\gamma\sigma} + \overline{D_\gamma D_\alpha D_\beta\sigma} = 0\,. \tag{20.13}$$

How do we solve this equation? We have to commute derivatives. For this, we note that the world function is a bi-scalar (a scalar at both x and y). Thus,

$$\overline{D_\gamma D_\alpha D_\beta\sigma} = \overline{D_\gamma D_\beta D_\alpha\sigma} + \overline{D_\gamma\,[D_\alpha, D_\beta]\,\sigma} = \overline{D_\gamma D_\beta D_\alpha\sigma}\,. \tag{20.14}$$

With this, we can write

$$\overline{D_\beta D_\gamma D_\alpha\sigma} + \overline{D_\gamma D_\beta D_\alpha\sigma} = 0\,. \tag{20.15}$$

To get all derivatives into the same order, we have to also commute D_β and D_γ in the first term, but this time, we get a non-trivial commutator:

$$\begin{aligned}
\overline{D_\beta D_\gamma D_\alpha\sigma} = \overline{D_\gamma D_\beta D_\alpha\sigma} + \overline{[D_\beta, D_\gamma]\,D_\alpha\sigma} &= \overline{D_\gamma D_\beta D_\alpha\sigma} + \overline{R_{\beta\gamma\alpha}{}^{\kappa} D_\kappa\sigma}\\
&= \overline{D_\gamma D_\beta D_\alpha\sigma} + R_{\beta\gamma\alpha}{}^{\kappa}\,\overline{D_\kappa\sigma} = \overline{D_\gamma D_\beta D_\alpha\sigma}\,.
\end{aligned} \tag{20.16}$$

Going to the second line, we pulled the Riemann tensor out of the coincidence limit. This works since the Riemann tensor is just a tensor, not a bi-tensor. Coming back to the above, we thus conclude

$$\overline{D_\gamma D_\beta D_\alpha\sigma} = 0\,. \tag{20.17}$$

Boooo, boring. What about four derivatives? Let's see:

$$\begin{aligned}
&(D_\delta D_\gamma D_\beta D_\alpha D_\mu\sigma(x,y))\,(D^\mu\sigma(x,y)) + (D_\gamma D_\beta D_\alpha D_\mu\sigma(x,y))\,(D_\delta D^\mu\sigma(x,y))\\
&+(D_\delta D_\beta D_\alpha D_\mu\sigma(x,y))\,(D_\gamma D^\mu\sigma(x,y)) + (D_\beta D_\alpha D_\mu\sigma(x,y))\,(D_\delta D_\gamma D^\mu\sigma(x,y))\\
&+(D_\delta D_\gamma D_\alpha D_\mu\sigma(x,y))\,(D_\beta D^\mu\sigma(x,y)) + (D_\gamma D_\alpha D_\mu\sigma(x,y))\,(D_\delta D_\beta D^\mu\sigma(x,y))\\
&+(D_\delta D_\alpha D_\mu\sigma(x,y))\,(D_\gamma D_\beta D^\mu\sigma(x,y)) + (D_\alpha D_\mu\sigma(x,y))\,(D_\delta D_\gamma D_\beta D^\mu\sigma(x,y)) = D_\delta D_\gamma D_\beta D_\alpha\sigma(x,y)\,.
\end{aligned} \tag{20.18}$$

Taking the coincidence limit, many terms drop out, and we get

$$\overline{D_\gamma D_\beta D_\alpha D_\delta\sigma} + \overline{D_\delta D_\beta D_\alpha D_\gamma\sigma} + \overline{D_\delta D_\gamma D_\alpha D_\beta\sigma} + \overline{D_\delta D_\gamma D_\beta D_\alpha\sigma} = \overline{D_\delta D_\gamma D_\beta D_\alpha\sigma}\,, \tag{20.19}$$

that is,

$$\overline{D_\gamma D_\beta D_\alpha D_\delta\sigma} + \overline{D_\delta D_\beta D_\alpha D_\gamma\sigma} + \overline{D_\delta D_\gamma D_\alpha D_\beta\sigma} = 0\,. \tag{20.20}$$

Once again, we have to sort covariant derivatives. In a first step, we have

$$\overline{D_\gamma D_\beta D_\delta D_\alpha\sigma} + \overline{D_\delta D_\beta D_\gamma D_\alpha\sigma} + \overline{D_\delta D_\gamma D_\beta D_\alpha\sigma} = 0\,. \tag{20.21}$$

Next, we commute the second and third derivative in the first two terms:

$$\overline{D_\gamma D_\delta D_\beta D_\alpha \sigma} + \overline{D_\gamma \left[D_\beta, D_\delta\right] D_\alpha \sigma} + \overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} + \overline{D_\delta \left[D_\beta, D_\gamma\right] D_\alpha \sigma} + \overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} = 0\,. \tag{20.22}$$

Both commutators give a single Riemann tensor. Note that the derivative to the left acts on both the Riemann tensor and the world function! We thus have

$$\begin{aligned} 0 &= \overline{D_\gamma D_\delta D_\beta D_\alpha \sigma} + \overline{D_\gamma R_{\beta\delta\alpha}{}^{\kappa} D_\kappa \sigma} + \overline{D_\delta R_{\beta\gamma\alpha}{}^{\kappa} D_\kappa \sigma} + 2\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} \\ &= \overline{D_\gamma D_\delta D_\beta D_\alpha \sigma} + \left(D_\gamma R_{\beta\delta\alpha}{}^{\kappa}\right) \overline{D_\kappa \sigma} + R_{\beta\delta\alpha}{}^{\kappa} \overline{D_\gamma D_\kappa \sigma} \\ &\quad + \left(D_\delta R_{\beta\gamma\alpha}{}^{\kappa}\right) \overline{D_\kappa \sigma} + R_{\beta\gamma\alpha}{}^{\kappa} \overline{D_\delta D_\kappa \sigma} + 2\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} \\ &= \overline{D_\gamma D_\delta D_\beta D_\alpha \sigma} + R_{\beta\delta\alpha\gamma} + R_{\beta\gamma\alpha\delta} + 2\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma}\,. \end{aligned} \tag{20.23}$$

Here, we once again used the coincidence limits of one and two derivatives acting on the world function. One final commutator is left:

$$\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} + \overline{\left[D_\gamma, D_\delta\right] D_\beta D_\alpha \sigma} + R_{\beta\delta\alpha\gamma} + R_{\beta\gamma\alpha\delta} + 2\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} = 0 \tag{20.24}$$

Expanding this gives

$$\begin{aligned} 0 &= \overline{R_{\gamma\delta\beta}{}^{\kappa} D_\kappa D_\alpha \sigma} + \overline{R_{\gamma\delta\alpha}{}^{\kappa} D_\beta D_\kappa \sigma} + R_{\beta\delta\alpha\gamma} + R_{\beta\gamma\alpha\delta} + 3\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} \\ &= R_{\gamma\delta\beta\alpha} + R_{\gamma\delta\alpha\beta} + R_{\beta\delta\alpha\gamma} + R_{\beta\gamma\alpha\delta} + 3\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} \\ &= R_{\beta\delta\alpha\gamma} + R_{\beta\gamma\alpha\delta} + 3\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma}\,. \end{aligned} \tag{20.25}$$

Finally a non-trivial result! We read off

$$\overline{D_\delta D_\gamma D_\beta D_\alpha \sigma} = -\frac{1}{3}\left(R_{\beta\delta\alpha\gamma} + R_{\beta\gamma\alpha\delta}\right)\,. \tag{20.26}$$

As a special case, we need

$$\overline{D^2 D^2 \sigma} = -\frac{2}{3} R\,. \tag{20.27}$$

d) We can finally go back to the heat kernel coefficients. First, we compute $\overline{D^2 A_0}$ from (20.3). Inserting all the above information, we find that most terms cancel, and we get

$$\begin{aligned} 0 &= \frac{1}{2}\left(\overline{D^2 D^2 \sigma}\right) \overline{A_0} + 2\left(\overline{D^\mu D^\nu \sigma}\right)\left(\overline{D_\mu D_\nu A_0}\right) \\ &= \frac{1}{2}\left(-\frac{2}{3}R\right) + 2\overline{D^2 A_0}\,, \end{aligned} \tag{20.28}$$

or

$$\overline{D^2 A_0} = \frac{1}{6} R\,. \tag{20.29}$$

All "dangerous" terms in the recursion, that is those that could break the bootstrap, drop out due to the coincidence limits of the world function. Finally, throwing everything into (20.1), we find

$$\overline{A_1} = \overline{D^2 A_0} = \frac{1}{6} R\,. \tag{20.30}$$

Going back to our (semi-)starting point, we thus find

$$\mathrm{STr}\, e^{-s\Delta} \sim \left(\frac{1}{4\pi s}\right)^{d/2} \mathrm{tr} \int \mathrm{d}^d x\, \sqrt{g}\left[1 + \frac{s}{6} R\right]\,, \qquad s \to 0\,. \tag{20.31}$$

Note how the heat kernel coefficients that we computed are independent of the dimension d. This is actually true also for all higher orders in the small-s expansion (also called early time expansion). The only dependence on the dimension comes from the prefactor!

Exercise 21: Heat kernel, part 3, or the inverse Laplace transform

Motivation: This is part 3 of the heat kernel – I lied that there would be only two parts. Remember where we started? Good, we have to actually come back and compute the original supertrace.

The starting point of the heat kernel exercises was that we originally wanted to compute

$$\mathrm{STr}\, W(\Delta)\,, \tag{21.I}$$

for some general function W. We spent a lot of time to compute the supertrace for an exponential, but in general we will not deal with only exponential functions, so we still need a recipe to connect the two.
For this, suppose we could write something like

$$W(\Delta) = \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-s\Delta}\,, \tag{21.II}$$

for some new function $\tilde{W}$. Wouldn't this be great? We could simply use this equation and use all previous results:

$$\mathrm{STr}\, W(\Delta) = \mathrm{STr} \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-s\Delta} = \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, \underbrace{\mathrm{STr}\, e^{-s\Delta}}_{\text{we did this!}}\,. \tag{21.III}$$

The only thing left to do would be to actually compute $\tilde{W}$ and perform the integrals over s, and we would be done. Also, we assumed that we can exchange the integral with the supertrace, but shhhhhhh.
Let us give some substance to this idea. The integral transform (21.II) is called the *inverse Laplace transform*. You can think of it like this: the original function W is the Laplace transform of some (a priori unknown) function $\tilde{W}$. Of course, there are some conditions on its existence, but let's simply assume for the moment that it exists. In the two previous exercises, we computed the supertrace of the exponential in an expansion in powers of s, so that

$$\mathrm{STr}\, W(\Delta) \sim \int_0^\infty \mathrm{d}s\, \tilde{W}(s) \left(\frac{1}{4\pi s}\right)^{d/2} \sum_{n\geq 0} s^n \int \mathrm{d}^d x\, \sqrt{g}\, \overline{A_n}\,. \tag{21.IV}$$

We thus have to deal with integrals over $\tilde{W}$ multiplying either negative (small n) or positive (large n) powers of s. Do we now really have to compute $\tilde{W}$? Actually, no.

a) Negative powers: show that for $n > 0$,

$$\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1}\, W(z)\,. \tag{21.V}$$

This means that one can map these integrals over $\tilde{W}$ to integrals over the original function W! The integral over z has the interpretation of the integral over the loop momentum.

b) Non-negative powers: show that for $n \geq 0$,

$$\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^n = (-1)^n\, W^{(n)}(0)\,, \tag{21.VI}$$

that is, these integrals can be mapped to derivatives of the original function at vanishing argument.

c) **[hard question]** Use your combined knowledge to compute

$$\mathrm{STr}\, W(\Delta) \tag{21.VII}$$

up to linear order in curvature, in arbitrary dimension, for W taken from Exercise 19. To evaluate the integrals, use the Litim regulator. What exactly did you just compute?

a) There are at least two different ways to prove this equation. The easier one is "backwards", i.e. we start with the right-hand side, insert the inverse Laplace transform, commute the integrals, and perform the integral over z:

$$\begin{aligned}\frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\, W(z) &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-sz} \\ &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}z\, z^{n-1}\, \tilde{W}(s)\, e^{-sz} \\ &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^{-n}\, \Gamma(n) \\ &= \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^{-n}\,.\end{aligned} \tag{21.1}$$

For the integral over z to converge, we need $n > 0$, in agreement with the requirement. If you do not like this way since you need to know the result in advance, we can also prove it in the other direction. For this, we use the representation of s^{-n} in terms of its inverse Laplace transform,

$$s^{-n} = \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\, e^{-sz}\,. \tag{21.2}$$

Inserting this on the left-hand side and once again commuting integrals, we have

$$\begin{aligned}\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^{-n} &= \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\,\frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\, e^{-sz} \\ &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-sz} \\ &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\, W(z)\,.\end{aligned} \tag{21.3}$$

In the last step, we used the definition of the inverse Laplace transform.

b) For this case, we pull out some tricks:

$$\begin{aligned}\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^n &= \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, s^n\, e^{-sz}\bigg|_{z=0} \\ &= \left[(-1)^n\partial_z^n\int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-sz}\right]\bigg|_{z=0} \\ &= \left[(-1)^n\partial_z^n\, W(z)\right]\big|_{z=0} \\ &= (-1)^n\, W^{(n)}(0)\,.\end{aligned} \tag{21.4}$$

Convince yourself that this trick indeed only works for non-negative n.

c) Let us put everything together. From (19.1), we had

$$W(z) = \frac{k\partial_k R_k(z)}{z + R_k(z)} \,. \tag{21.5}$$

We just showed that we do not need to compute its inverse Laplace transform in order to evaluate the supertrace. Next, we use the inverse Laplace transform and the formula for the heat kernel to write

$$\begin{aligned}
\mathrm{STr}\, W(\Delta) &= \mathrm{STr} \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, e^{-s\Delta} \\
&= \int_0^\infty \mathrm{d}s\, \tilde{W}(s)\, \mathrm{STr}\, e^{-s\Delta} \\
&\sim \int_0^\infty \mathrm{d}s\, \tilde{W}(s) \left(\frac{1}{4\pi s}\right)^{d/2} \int \mathrm{d}^d x \sqrt{g} \left[1 + \frac{s}{6} R\right] .
\end{aligned} \tag{21.6}$$

Next, assuming that $d > 2$, we can use the formulas from a) and write

$$\mathrm{STr}\, W(\Delta) \sim \left(\frac{1}{4\pi}\right)^{d/2} \int \mathrm{d}^d x \sqrt{g} \left[\frac{1}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1}\, W(z) + \frac{R}{6} \frac{1}{\Gamma\left(\frac{d}{2}-1\right)} \int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-2}\, W(z) \right] . \tag{21.7}$$

The ultimate step is now to evaluate the threshold integrals. Recall from Exercise 18 that for the scale derivative of the Litim regulator, we can drop the delta function, so that effectively

$$k\partial_k R_k(z) \simeq 2k^2 \theta\left(1 - \frac{z}{k^2}\right) . \tag{21.8}$$

Inserting this, we have

$$\begin{aligned}
\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1}\, W(z) &= \int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1} \frac{2k^2\theta\left(1-\frac{z}{k^2}\right)}{z + (k^2 - z)\theta\left(1 - \frac{z}{k^2}\right)} \\
&= \int_0^{k^2} \mathrm{d}z\, z^{\frac{d}{2}-1} \frac{2k^2}{k^2} = 2 \int_0^{k^2} \mathrm{d}z\, z^{\frac{d}{2}-1} \\
&= 2k^d \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-1} = \frac{4k^d}{d} \,.
\end{aligned} \tag{21.9}$$

Here, we rescaled z to make it dimensionless, $z = k^2 x$. In the same way, we can compute the second threshold integral:

$$\begin{aligned}
\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-2}\, W(z) &= \int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-2} \frac{2k^2\theta\left(1-\frac{z}{k^2}\right)}{z + (k^2 - z)\theta\left(1 - \frac{z}{k^2}\right)} \\
&= \int_0^{k^2} \mathrm{d}z\, z^{\frac{d}{2}-2} \frac{2k^2}{k^2} = 2 \int_0^{k^2} \mathrm{d}z\, z^{\frac{d}{2}-2} \\
&= 2k^{d-2} \int_0^1 \mathrm{d}x\, x^{\frac{d}{2}-2} = \frac{4k^{d-2}}{d-2} \,.
\end{aligned} \tag{21.10}$$

Finally, finally, finally, we have

$$\boxed{\mathrm{STr}\, W(\Delta) \sim \left(\frac{1}{4\pi}\right)^{d/2} \int \mathrm{d}^d x \sqrt{g} \left[\frac{4k^d}{d\,\Gamma\left(\frac{d}{2}\right)} + \frac{R}{6} \frac{4k^{d-2}}{(d-2)\Gamma\left(\frac{d}{2}-1\right)} \right]} \,. \tag{21.11}$$

You can see how the powers of k appear in the way required by mass dimension. So what did we compute? This is the contribution of the quantum fluctuations of a free scalar field to the beta functions of the cosmological constant and Newton's constant.