Heidelberg University WS 2025/2026

Quantum Gravity and the Renormalization Group

Assignment 2 – Oct 27

Exercise 4: Alternative gravitational theories

Motivation: We will now investigate alternative theories for gravity. This has both theoretical and experimental motivations: on the theoretical side, we know that General Relativity breaks down, e.g., at the centre of black holes, and a quantum theory has to take over. On the experimental side, with recent observations of gravitational waves and black hole shadows, we will soon be able to test deviations from General Relativity. We will discuss these theories in some more detail in the lectures soon.

Let us study the simplest extensions of General Relativity by adding terms to the action with four derivatives of the metric, and investigate what impact these have on gravitational waves.

a) Find a basis of all independent terms with (in total) four derivatives that are constructed purely in terms of the metric, and that respect diffeomorphism invariance. *Hints:* This should involve curvatures and covariant derivatives only. How many derivatives of the metric does the Riemann tensor have, and what are its symmetries? Also, how many uncontracted indices can terms in the action have?

b) Derive the equations of motion from the generalised action that you have found.

c) **[hard question]** Let us consider one specific (and phenomenologically very important) model, the Starobinsky theory. Its action is that of General Relativity with an R^2-term added,

$$S^{\text{Staro}} = \int \mathrm{d}^4x \sqrt{-\det g} \left[\frac{R - 2\Lambda}{16\pi G_N} + \alpha R^2 \right] , \tag{4.1}$$

so it is a special case of the theory above. From the equations of motion that you derived in b), find out if solutions to Einstein's equations are still solutions to the Starobinsky theory. If yes, where is the "new physics" in this theory?

d) **[hard question]** Back to the full system of b): linearise the equations of motion in Minkowski space. Do gravitational waves propagate differently now? Can these differences be removed by a suitable gauge fixing?

a) The Riemann tensor has two derivatives of the metric. Terms in the action have no uncontracted indices. This means that we can either have two curvature tensors, or two derivatives acting on a single curvature tensor, with all indices contracted.

Let us start with two derivatives acting on a single curvature tensor. We cannot use the Riemann tensor, since it has four indices, and if we would contract all indices with derivatives, we would be at sixth derivative order. This means that the only combinations are

$$D^\mu D^\nu R_{\mu\nu} \, , \qquad D^2 R \, . \tag{4.2}$$

However, we can use the contracted differential Bianchi identity to simplify the first term. Starting from

$$D_{[\alpha} R_{\mu\nu]\rho\sigma} = 0\,, \tag{4.3}$$

(here, the angular brackets denote complete anti-symmetrisation over the indices within the brackets), and contracting the indices μ and ρ, we find

$$D_\alpha R_{\nu\sigma} - D_\nu R_{\alpha\sigma} + D^\rho R_{\nu\alpha\rho\sigma} = 0\,. \tag{4.4}$$

Now contracting α and σ, we find

$$D^\alpha R_{\nu\alpha} - D_\nu R + D^\rho R_{\nu\rho} = 0\,, \tag{4.5}$$

so that

$$D^\rho R_{\rho\sigma} = \frac{1}{2} D_\sigma R\,. \tag{4.6}$$

From this, we see that the only independent term with two derivatives and one curvature tensor is

$$D^2 R\,. \tag{4.7}$$

Next, we can form all contractions of two curvature tensors. From index considerations, it is clear that we can either contract two Riemann tensors, two Ricci tensors, or have the square of the Ricci scalar. The latter is a scalar, and is thus the first invariant:

$$R^2\,. \tag{4.8}$$

Two Ricci tensors can only be contracted in a single way, since it is symmetric:

$$R^{\mu\nu} R_{\mu\nu}\,. \tag{4.9}$$

Finally, let us discuss all contractions of two Riemann tensors,

$$R^{\mu\nu\rho\sigma} R_{\alpha\beta\gamma\delta}\,. \tag{4.10}$$

First, using symmetries we can always make it that μ and α are contracted with each other:

$$R^{\mu\nu\rho\sigma} R_{\mu\beta\gamma\delta}\,. \tag{4.11}$$

Next, we can contract ν either with β, or with γ – contracting it with δ instead is minus the contraction with γ due to the symmetry. In the first case, there is one way left to contract the remaining indices – the other is minus the same:

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\,. \tag{4.12}$$

In the second case, we can also only find one contraction, namely

$$R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma}\,, \tag{4.13}$$

the other way is dependent:

$$R^{\mu\nu\rho\sigma} R_{\mu\sigma\nu\rho} = R^{\mu\nu\rho\sigma} R_{\nu\rho\mu\sigma} = -R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma}\,. \tag{4.14}$$

Is the second contraction really independent? Actually not – we can use the Bianchi identity

$$R_{[\mu\nu\rho]\sigma} = 0\,, \tag{4.15}$$

contract it with $R^{\mu\nu\rho\sigma}$, and find

$$R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma} = \frac{1}{2} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\,. \tag{4.16}$$

Summarising, we have four independent terms, one of which is a total derivative:

$$\boxed{D^2 R\,, \qquad R^2\,, \qquad R^{\mu\nu} R_{\mu\nu}\,, \qquad R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}\,.} \tag{4.17}$$

b) Let us make an ansatz for the total action that we consider:

$$S^{\mathrm{HD}} = \int \mathrm{d}^4x\, \sqrt{-\det g}\, \left[\frac{R-2\Lambda}{16\pi G_N} + \alpha\, R^2 + \beta\, R^{\mu\nu} R_{\mu\nu} + \gamma\, R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \delta\, D^2 R\right] . \tag{4.18}$$

Let us start with the positive side: we already computed the Einstein-Hilbert part of the equations of motion, and the last term is a total derivative so that it does not contribute to the equations of motion. So far, so good. Let us then compute the contributions of all the other pieces term by term. We start with the easiest, the R^2-term. For this, we already have all the ingredients:

$$\begin{aligned}
&\alpha \int \mathrm{d}^4x\, \sqrt{-\det \bar{g}}\, \left(1+\frac{1}{2}h\right) \left[\bar{R} + \bar{D}^\mu \bar{D}^\nu h_{\mu\nu} - \bar{D}^2 h\, - \bar{R}^{\mu\nu} h_{\mu\nu}\right]^2 \\
&\simeq \alpha \int \mathrm{d}^4x\, \sqrt{-\det \bar{g}}\, \left[\bar{R}^2 + \frac{1}{2} h\, \bar{R}^2 + 2\bar{R}\, \left(\bar{D}^\mu \bar{D}^\nu h_{\mu\nu} - \bar{D}^2 h\, - \bar{R}^{\mu\nu} h_{\mu\nu}\right)\right] \\
&\simeq \alpha \int \mathrm{d}^4x\, \sqrt{-\det \bar{g}}\, \left[\bar{R}^2 + \left(\frac{1}{2}\bar{R}^2\, \bar{g}^{\mu\nu} + 2\left(\bar{D}^\mu \bar{D}^\nu \bar{R}\right) - 2\left(\bar{D}^2 \bar{R}\right) \bar{g}^{\mu\nu} - 2\bar{R}\, \bar{R}^{\mu\nu}\right) h_{\mu\nu}\right] .
\end{aligned} \tag{4.19}$$

In the last step, we used partial integration so that all derivatives act on the curvature tensors. We can now read off the contribution to the equations of motion:

$$\alpha \left[\frac{1}{2}\bar{R}^2\, \bar{g}^{\mu\nu} + 2\left(\bar{D}^\mu \bar{D}^\nu \bar{R}\right) - 2\left(\bar{D}^2 \bar{R}\right) \bar{g}^{\mu\nu} - 2\bar{R}\, \bar{R}^{\mu\nu}\right] . \tag{4.20}$$

We continue with the $R^{\mu\nu}R_{\mu\nu}$-term. Here it is important to realise that the Ricci tensor is defined with lower indices, so that for the computation of perturbations of the Ricci tensor with upper indices, we cannot use the earlier formulas right away. Rather, we have to use

$$\begin{aligned}
R^{\mu\nu} &= g^{\mu\alpha}\, R_{\alpha\beta}\, g^{\beta\nu} \\
&\simeq \left(\bar{g}^{\mu\alpha} - h^{\mu\alpha}\right) \left[\bar{R}_{\alpha\beta} + \frac{1}{2}\bar{D}^\gamma \left(\bar{D}_\alpha h_{\gamma\beta} + \bar{D}_\beta h_{\gamma\alpha} - \bar{D}_\gamma h_{\alpha\beta}\right) - \frac{1}{2}\bar{D}_\alpha \bar{D}_\beta h\right] \left(\bar{g}^{\beta\nu} - h^{\beta\nu}\right) \\
&\simeq \bar{R}^{\mu\nu} - h^{\mu\alpha}\bar{R}_\alpha{}^\nu - \bar{R}^{\mu\alpha} h_\alpha{}^\nu + \frac{1}{2}\bar{D}^\gamma \left(\bar{D}^\mu h_\gamma{}^\nu + \bar{D}^\nu h_\gamma{}^\mu - \bar{D}_\gamma h^{\mu\nu}\right) - \frac{1}{2}\bar{D}^\mu \bar{D}^\nu h\, .
\end{aligned} \tag{4.21}$$

As you can see, there are two extra terms! Combining everything, we can compute

$$\begin{aligned}
&\beta \int \mathrm{d}^4x\, \sqrt{-\det \bar{g}}\, \left(1+\frac{1}{2}h\right) \times \\
&\qquad \left[\bar{R}^{\mu\nu} - h^{\mu\alpha}\bar{R}_\alpha{}^\nu - \bar{R}^{\mu\alpha} h_\alpha{}^\nu + \frac{1}{2}\bar{D}^\gamma \left(\bar{D}^\mu h_\gamma{}^\nu + \bar{D}^\nu h_\gamma{}^\mu - \bar{D}_\gamma h^{\mu\nu}\right) - \frac{1}{2}\bar{D}^\mu \bar{D}^\nu h\right] \times \\
&\qquad \left[\bar{R}_{\mu\nu} + \frac{1}{2}\bar{D}^\beta \left(\bar{D}_\mu h_{\beta\nu} + \bar{D}_\nu h_{\beta\mu} - \bar{D}_\beta h_{\mu\nu}\right) - \frac{1}{2}\bar{D}_\mu \bar{D}_\nu h\right] \\
&\simeq \beta \int \mathrm{d}^4x\, \sqrt{-\det \bar{g}}\, \Bigg[\bar{R}^{\mu\nu}\bar{R}_{\mu\nu} + \frac{1}{2}h\, \bar{R}^{\mu\nu}\bar{R}_{\mu\nu} + \bar{R}^{\mu\nu}\left(\frac{1}{2}\bar{D}^\beta \left(\bar{D}_\mu h_{\beta\nu} + \bar{D}_\nu h_{\beta\mu} - \bar{D}_\beta h_{\mu\nu}\right) - \frac{1}{2}\bar{D}_\mu \bar{D}_\nu h\right) \\
&\qquad + \left(-h^{\mu\alpha}\bar{R}_\alpha{}^\nu - \bar{R}^{\mu\alpha} h_\alpha{}^\nu + \frac{1}{2}\bar{D}^\gamma \left(\bar{D}^\mu h_\gamma{}^\nu + \bar{D}^\nu h_\gamma{}^\mu - \bar{D}_\gamma h^{\mu\nu}\right) - \frac{1}{2}\bar{D}^\mu \bar{D}^\nu h\right) \bar{R}_{\mu\nu}\Bigg] \\
&\simeq \beta \int \mathrm{d}^4x\, \sqrt{-\det \bar{g}}\, \Bigg[\bar{R}^{\mu\nu}\bar{R}_{\mu\nu} + \Bigg(\frac{1}{2}\bar{R}^{\alpha\beta}\bar{R}_{\alpha\beta}\bar{g}^{\mu\nu} - 2\bar{R}^{\mu\alpha}\bar{R}_\alpha{}^\nu - \left(\bar{D}^\alpha \bar{D}^\beta \bar{R}_{\alpha\beta}\right) \bar{g}^{\mu\nu} \\
&\qquad + \left(\bar{D}_\alpha \bar{D}^\mu \bar{R}^{\alpha\nu}\right) + \left(\bar{D}_\alpha \bar{D}^\nu \bar{R}^{\alpha\mu}\right) - \left(\bar{D}^2 \bar{R}^{\mu\nu}\right)\Bigg) h_{\mu\nu}\Bigg] .
\end{aligned} \tag{4.22}$$

We thus find the contribution to the equations of motion to be

$$\beta\left[\frac{1}{2}\bar{R}^{\alpha\beta}\bar{R}_{\alpha\beta}\bar{g}^{\mu\nu}-2\bar{R}^{\mu\alpha}\bar{R}_{\alpha}{}^{\nu}-\left(\bar{D}^{\alpha}\bar{D}^{\beta}\bar{R}_{\alpha\beta}\right)\bar{g}^{\mu\nu}+\left(\bar{D}_{\alpha}\bar{D}^{\mu}\bar{R}^{\alpha\nu}\right)+\left(\bar{D}_{\alpha}\bar{D}^{\nu}\bar{R}^{\alpha\mu}\right)-\left(\bar{D}^{2}\bar{R}^{\mu\nu}\right)\right]. \tag{4.23}$$

This can actually be simplified via the differential Bianchi identity. We commute the covariant derivatives to form divergences (which also creates Riemann tensors):

$$\begin{aligned}\bar{D}_{\alpha}\bar{D}^{\mu}\bar{R}^{\alpha\nu}&=\left[\bar{D}_{\alpha},\bar{D}^{\mu}\right]\bar{R}^{\alpha\nu}+\bar{D}^{\mu}\bar{D}_{\alpha}\bar{R}^{\alpha\nu}\\&=\bar{R}^{\alpha}{}_{\beta\alpha}{}^{\mu}\bar{R}^{\beta\nu}+\bar{R}^{\nu}{}_{\beta\alpha}{}^{\mu}\bar{R}^{\alpha\beta}+\bar{D}^{\mu}\bar{D}_{\alpha}\bar{R}^{\alpha\nu}\\&=\bar{R}^{\mu\alpha}\bar{R}_{\alpha}{}^{\nu}-\bar{R}_{\alpha\beta}\bar{R}^{\mu\alpha\nu\beta}+\bar{D}^{\mu}\bar{D}_{\alpha}\bar{R}^{\alpha\nu}.\end{aligned} \tag{4.24}$$

As a small check, verify that via this and the Bianchi identity, the contribution can be written as

$$\beta\left[\frac{1}{2}\bar{R}^{\alpha\beta}\bar{R}_{\alpha\beta}\bar{g}^{\mu\nu}-2\bar{R}_{\alpha\beta}\bar{R}^{\mu\alpha\nu\beta}-\left(\bar{D}^{2}\bar{R}^{\mu\nu}\right)-\frac{1}{2}\left(\bar{D}^{2}\bar{R}\right)\bar{g}^{\mu\nu}+\left(\bar{D}^{\mu}\bar{D}^{\nu}\bar{R}\right)\right]. \tag{4.25}$$

Finally, let us discuss the Riemann-squared-term. For this, note that we can write

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}=R^{\rho\sigma\mu\nu}R_{\mu\nu\rho\sigma}=g^{\alpha\mu}g^{\beta\rho}R_{\alpha\sigma\beta}{}^{\nu}R_{\mu\nu\rho}{}^{\sigma}, \tag{4.26}$$

which brings the Riemann tensor into its defining index form. Repeating the computation that we did in exercise 2 for the Ricci tensor, we find (check this!)

$$\begin{aligned}R_{\mu\nu\rho}{}^{\sigma}-\bar{R}_{\mu\nu\rho}{}^{\sigma}&\simeq\bar{D}_{\nu}\left(\Gamma^{\sigma}{}_{\mu\rho}-\bar{\Gamma}^{\sigma}{}_{\mu\rho}\right)-\bar{D}_{\mu}\left(\Gamma^{\sigma}{}_{\nu\rho}-\bar{\Gamma}^{\sigma}{}_{\nu\rho}\right)\\&\simeq\frac{1}{2}\bar{D}_{\nu}\left(\bar{D}_{\mu}h^{\sigma}{}_{\rho}+\bar{D}_{\rho}h^{\sigma}{}_{\mu}-\bar{D}^{\sigma}h_{\mu\rho}\right)-\frac{1}{2}\bar{D}_{\mu}\left(\bar{D}_{\nu}h^{\sigma}{}_{\rho}+\bar{D}_{\rho}h^{\sigma}{}_{\nu}-\bar{D}^{\sigma}h_{\nu\rho}\right).\end{aligned} \tag{4.27}$$

At the level of an action, this gives

$$\begin{aligned}&\gamma\int\mathrm{d}^4x\,\sqrt{-\det\bar{g}}\,\left(1+\frac{1}{2}h\right)\left[\bar{g}^{\alpha\mu}-h^{\alpha\mu}\right]\left[\bar{g}^{\beta\rho}-h^{\beta\rho}\right]\times\\&\qquad\left[\bar{R}_{\alpha\sigma\beta}{}^{\nu}+\frac{1}{2}\bar{D}_{\sigma}\left(\bar{D}_{\alpha}h^{\nu}{}_{\beta}+\bar{D}_{\beta}h^{\nu}{}_{\alpha}-\bar{D}^{\nu}h_{\alpha\beta}\right)-\frac{1}{2}\bar{D}_{\alpha}\left(\bar{D}_{\sigma}h^{\nu}{}_{\beta}+\bar{D}_{\beta}h^{\nu}{}_{\sigma}-\bar{D}^{\nu}h_{\sigma\beta}\right)\right]\times\\&\qquad\left[\bar{R}_{\mu\nu\rho}{}^{\sigma}+\frac{1}{2}\bar{D}_{\nu}\left(\bar{D}_{\mu}h^{\sigma}{}_{\rho}+\bar{D}_{\rho}h^{\sigma}{}_{\mu}-\bar{D}^{\sigma}h_{\mu\rho}\right)-\frac{1}{2}\bar{D}_{\mu}\left(\bar{D}_{\nu}h^{\sigma}{}_{\rho}+\bar{D}_{\rho}h^{\sigma}{}_{\nu}-\bar{D}^{\sigma}h_{\nu\rho}\right)\right]\\&\simeq\gamma\int\mathrm{d}^4x\,\sqrt{-\det\bar{g}}\left[\bar{R}^{\mu\nu\rho\sigma}\bar{R}_{\mu\nu\rho\sigma}+\frac{1}{2}h\,\bar{R}^{\mu\nu\rho\sigma}\bar{R}_{\mu\nu\rho\sigma}-2h^{\alpha\beta}\bar{R}_{\alpha}{}^{\nu\rho\sigma}\bar{R}_{\beta\nu\rho\sigma}-4\bar{R}^{\mu\rho\nu\sigma}\bar{D}_{\sigma}\bar{D}_{\rho}h_{\mu\nu}\right]\\&\simeq\gamma\int\mathrm{d}^4x\,\sqrt{-\det\bar{g}}\left[\bar{R}^{\mu\nu\rho\sigma}\bar{R}_{\mu\nu\rho\sigma}+\left(\frac{1}{2}\bar{R}^{\alpha\beta\gamma\delta}\bar{R}_{\alpha\beta\gamma\delta}\bar{g}^{\mu\nu}-2\bar{R}^{\mu\alpha\beta\gamma}\bar{R}^{\nu}{}_{\alpha\beta\gamma}-4\left(\bar{D}_{\rho}\bar{D}_{\sigma}\bar{R}^{\mu\rho\nu\sigma}\right)\right)h_{\mu\nu}\right].\end{aligned} \tag{4.28}$$

Look at the second step – a lot of simplifications appear due to the symmetries of the Riemann tensor (check this explicitly)! We read off the contribution to the equations of motion as

$$\gamma\left[\frac{1}{2}\bar{R}^{\alpha\beta\gamma\delta}\bar{R}_{\alpha\beta\gamma\delta}\bar{g}^{\mu\nu}-2\bar{R}^{\mu\alpha\beta\gamma}\bar{R}^{\nu}{}_{\alpha\beta\gamma}-4\left(\bar{D}_{\rho}\bar{D}_{\sigma}\bar{R}^{\mu\rho\nu\sigma}\right)\right]. \tag{4.29}$$

Once again, we can use the Bianchi identity – actually we can use it a second time after sorting covariant derivatives. Check the intermediate steps! We arrive at

$$\gamma\left[\frac{1}{2}\bar{R}^{\alpha\beta\gamma\delta}\bar{R}_{\alpha\beta\gamma\delta}\bar{g}^{\mu\nu}-2\bar{R}^{\mu\alpha\beta\gamma}\bar{R}^{\nu}{}_{\alpha\beta\gamma}-4\bar{R}_{\alpha\beta}\bar{R}^{\mu\alpha\nu\beta}+4\bar{R}^{\mu\alpha}\bar{R}_{\alpha}{}^{\nu}-4\left(\bar{D}^{2}\bar{R}^{\mu\nu}\right)+2\left(\bar{D}^{\mu}\bar{D}^{\nu}\bar{R}\right)\right]. \tag{4.30}$$

You might now think that this is finally it. Ha, gotcha! There is one final simplification, which is specific to four dimensions, and comes from the idea of "oversymmetrisation". This idea states that if one antisymmetrises over a number of indices which is larger than the spacetime dimension, the resulting expression is zero (convince yourself that this is the case). Now consider

$$0 = \left(\delta_{\mu}{}^{[\nu} R_{\kappa\lambda}{}^{\alpha\beta} R_{\tau\omega}{}^{\gamma\delta]}\right) \delta_{\alpha}{}^{\kappa} \delta_{\beta}{}^{\lambda} \delta_{\gamma}{}^{\tau} \delta_{\delta}{}^{\omega} \tag{4.31}$$

where the antisymmetrisation is indicated by the angular brackets. We thus completely antisymmetrise over the upper indices $\{\nu, \alpha, \beta, \gamma, \delta\}$. With a lot of patience, one finds the relation

$$R_{\mu\alpha\beta\gamma} R_{\nu}{}^{\alpha\beta\gamma} = \frac{1}{4} g_{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + 2R^{\alpha\beta} R_{\mu\alpha\nu\beta} + 2R_{\mu\alpha} R^{\alpha}{}_{\nu} - g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} - R\, R_{\mu\nu} + \frac{1}{4} g_{\mu\nu} R^2 \,. \tag{4.32}$$

If you want to understand this better, decompose the Riemann tensor into the Weyl tensor, Ricci tensor and Ricci scalar. The Weyl tensor C is the completely tracefree version of the Riemann tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left[R_{\mu\rho} g_{\nu\sigma} - R_{\mu\sigma} g_{\nu\rho} + R_{\nu\sigma} g_{\mu\rho} - R_{\nu\rho} g_{\mu\sigma}\right] + \frac{1}{6}\left[g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right] R \,. \tag{4.33}$$

With this final simplification, we arrive at

$$\gamma \left[-8\bar{R}_{\alpha\beta} \bar{R}^{\mu\alpha\nu\beta} + 2\bar{g}^{\mu\nu} \bar{R}^{\alpha\beta} \bar{R}_{\alpha\beta} + 2\bar{R}^{\mu\nu} \bar{R} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R}^2 - 4\left(\bar{D}^2 \bar{R}^{\mu\nu}\right) + 2\left(\bar{D}^{\mu} \bar{D}^{\nu} \bar{R}\right)\right] . \tag{4.34}$$

Combining all the contributions, including the Einstein-Hilbert term, we have

$$-\frac{1}{16\pi G_N}\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu}\right) - 2(\beta + 4\gamma) R_{\alpha\beta} R^{\mu\alpha\nu\beta} + \frac{\beta + 4\gamma}{2} R^{\alpha\beta} R_{\alpha\beta} g^{\mu\nu} - 2(\alpha - \gamma) R^{\mu\nu} R$$
$$+ \frac{\alpha - \gamma}{2} R^2 g^{\mu\nu} - (\beta + 4\gamma) D^2 R^{\mu\nu} + (2\alpha + \beta + 2\gamma) D^{\mu} D^{\nu} R - \frac{4\alpha + \beta}{2} (D^2 R) g^{\mu\nu} = 0 \,. \tag{4.35}$$

Note the following curiosity: if we shift the couplings α, β, γ in the following way,

$$\alpha \mapsto \alpha + c \,, \quad \beta \mapsto \beta - 4c \,, \quad \gamma \mapsto \gamma + c \,, \tag{4.36}$$

the equations of motion do not change. We will come back to the reason for this later.

c) Let us set $\beta = \gamma = 0$ to get the equations of motion for the Starobinsky model:

$$-\frac{1}{16\pi G_N}\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu}\right) - 2\alpha R^{\mu\nu} R + \frac{\alpha}{2} R^2 g^{\mu\nu} + 2\alpha D^{\mu} D^{\nu} R - 2\alpha (D^2 R) g^{\mu\nu} = 0 \tag{4.37}$$

We can rewrite this in a suggestive way:

$$-\frac{1}{16\pi G_N}\left((R^{\mu\nu} - \Lambda g^{\mu\nu}) - \frac{1}{2}(R - 4\Lambda) g^{\mu\nu}\right) - 2\alpha\, R\left(R^{\mu\nu} - \frac{1}{4} R g^{\mu\nu}\right) + 2\alpha\left(D^{\mu} D^{\nu} - g^{\mu\nu} D^2\right) R = 0 \,. \tag{4.38}$$

We can now insert the solution to Einstein's equations,

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \,, \qquad R = 4\Lambda \,, \tag{4.39}$$

and confirm that this is indeed a solution. Now where is the new physics? For this, let us take the trace of the equations of motion:

$$R - 4\Lambda - 96\pi G_N \alpha (D^2 R) = 0 \,. \tag{4.40}$$

The new physics lies in the observation that $R = 4\Lambda$ is a solution to this equation, but it is not the only one.

d) Linearising the equations of motion, we can directly throw out all the terms quadratic in curvature – only the GR-part and the part with covariant derivatives survives. For the same reason, we do not have to linearise the covariant derivatives. As before, we also have to set $\Lambda = 0$. We furthermore lower the indices and multiply with (-1) for convenience, so that we have to linearise

$$\frac{1}{16\pi G_N}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) + (\beta+4\gamma)D^2 R_{\mu\nu} - (2\alpha+\beta+2\gamma)D_\mu D_\nu R + \frac{4\alpha+\beta}{2}(D^2R)g_{\mu\nu} \simeq 0\,. \quad (4.41)$$

We can now insert the formulas that we have derived before, (3.1) and (3.2):

$$\begin{aligned}
&\frac{1}{32\pi G_N}\left[\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\left(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h\right)\right] \\
&+(\beta+4\gamma)\partial^2\frac{1}{2}\left[\partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\right] \\
&-(2\alpha+\beta+2\gamma)\partial_\mu\partial_\nu\left[\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h\right] + \frac{4\alpha+\beta}{2}\eta_{\mu\nu}\left[\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h\right] = 0\,.
\end{aligned} \quad (4.42)$$

To investigate gravitational waves, let us now impose a gauge condition of the form

$$\partial^\mu h_{\mu\nu} = c\,\partial_\nu h\,, \quad (4.43)$$

and for the moment forget about h. This is justified since the spin two part (i.e., the actual wave) is carried by the transverse-traceless part. With this, we have

$$\left[-\frac{1}{32\pi G_N}\partial^2 - \frac{\beta+4\gamma}{2}\partial^4\right] h_{\mu\nu} \simeq 0\,. \quad (4.44)$$

This is not a standard wave equation anymore, and the extra terms cannot be removed by a gauge-fixing. The change is simply the physical effect of the Ricci and Riemann terms in the action.

Extra material 2: Beyond four derivatives and generalised wave equations

It is interesting to think about further extensions of the gravitational action. One way – called the derivative expansion – proceeds by adding terms according to the number of derivatives acting on the metric, as we did. This gets complicated **very** quickly. To get some handle on this, one can use group theory – the relevant buzz word to look for is Young tableau. The standard reference in this context is Fulling, King, Wybourne, Cummins, Class. Quant. Grav. 9 1151. Even if we discard boundary terms (like the term D^2R above), at sixth order there are 10 independent terms (which reduces to eight in four dimensions)!

Starting at sixth order in derivatives, however, not all terms contribute to the wave equation when the equations of motion are linearised. For example, the term R^3 does not contribute to it, while RD^2R does. Convince yourself that the only terms that contribute to the wave equation in flat space are in fact terms with at most two curvatures, but an arbitrary number of derivatives.

Exercise 5: Evidence for Quantum Gravity?

In the lecture, we discussed an argument for the quantisation of gravity that relies on what the gravitational field would be of a superposition of a massive object at different positions.

In 1981, Page and Geilker wrote a paper about an experiment they conducted (Phys. Rev. Lett. 47 (1981) 979-982). Read the paper and critically comment on whether their experiment realises the above argument.

As we discussed during the tutorial, there are several issues with this paper. Probably the main issue is the insufficient discussion of decoherence. Some more discussion can be found in the two comments on the paper, together with their replies (B. Hawkins, Phys. Rev. Lett. 48 (1982) 520 + Don N. Page, Phys. Rev. Lett. 48 (1982) 521; L. E. Ballentine, Phys. Rev. Lett. 48 (1982) 522 + Don N. Page, Phys. Rev. Lett. 48 (1982) 523). Sabine Hossenfelder also discussed this paper long ago on her blog, but in my opinion it is a rather superficial discussion.