Quantum Gravity and the Renormalization Group

Assignment 5 – Nov 17

Exercise 11: Ostrogradsky instabilities

Motivation: Higher-curvature invariants imply higher-order equations of motion. Here, we find out what this means for a toy model – a single higher-derivative harmonic oscillator.

We consider a single particle, which is a harmonic oscillator with a higher-derivative correction controlled by a parameter ϵ, namely

$$\mathcal{L} = -\frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2\,. \tag{11.I}$$

We will now study its solutions, count degrees of freedom, and look out for instabilities.

a) Show that the Euler-Lagrange equation for the Lagrangian in (11.I) is

$$\frac{\epsilon}{\omega^2}\ddddot{x} + \ddot{x} + \omega^2 x\,, \tag{11.II}$$

which has the solution

$$x(t) = C_+\cos(k_+t) + S_+\sin(k_+t) + C_-\cos(k_-t) + S_-\sin(k_-t)\,, \tag{11.III}$$

with the two frequencies

$$k_\pm = \omega\sqrt{\frac{1\mp\sqrt{1-4\epsilon}}{2\epsilon}}\,, \tag{11.IV}$$

and where $C_\pm$ and $S_\pm$ are integration constants.

Bottom line: We need four pieces of initial data. Think before reading on: How many degrees of freedom usually require four pieces of initial data?

b) Ostrogradsky defined the canonical phase-space coordinates for second-order derivative Lagrangians as

$$\begin{aligned} X_1 &\equiv x\,, & P_1 &\equiv \frac{\partial L}{\partial \dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}}\,, \\ X_2 &\equiv \dot{x}\,, & P_2 &\equiv \frac{\partial L}{\partial \ddot{x}}\,. \end{aligned} \tag{11.V}$$

Show that in this case

$$P_1 = m\dot{x} + \frac{\epsilon m}{\omega^2}\dddot{x}, \qquad P_2 = -\frac{\epsilon m}{\omega^2}\ddot{x}\,. \tag{11.VI}$$

Check that this is equivalent to

$$\dddot{x} = \omega^2\frac{P_1 - mX_2}{\epsilon m}, \qquad \ddot{x} = -\frac{\omega^2 P_2}{\epsilon m}\,. \tag{11.VII}$$

c) The Hamiltonian is defined as

$$H \equiv \sum_{i=1}^{2} P_i \dot{X}_i - L \,. \tag{11.VIII}$$

Demonstrate that for our model, the Hamiltonian can be expressed as

$$H = P_1 X_2 - \frac{\omega^2}{2\epsilon m} P_2^2 - \frac{m}{2} X_2^2 + \frac{m\omega^2}{2} X_1^2 \,. \tag{11.IX}$$

Think before moving on: What's problematic with this Hamiltonian?

d) Show that, evaluated on the solution in a), the Hamiltonian reads

$$H = \frac{m}{2}\sqrt{1-4\epsilon}\left[k_+^2(C_+^2 + S_+^2) - k_-^2(C_-^2 + S_-^2)\right] \,. \tag{11.X}$$

What's the difference between the two modes?

e) **[hard question]** The Hamiltonian in (11.X) is not bounded from above and below (why?). Yet, we could solve the differential equation in (11.II) to obtain the solution in (11.III) without instabilities in sight. Why is that? Will this property persist for more complicated models? For those who want to dive into this a bit more, there is a recent example in the literature.

Exercise 12: Spin projectors

Motivation: A symmetric rank-two field $h_{\mu\nu}$ in flat spacetime carries ten components, but only two correspond to the physical graviton polarisations. The spin-projector formalism provides a clean algebraic decomposition of $h_{\mu\nu}$ into its irreducible components under the little group $SO(3)$. This exercise builds the spin projectors step by step and connects them to the propagating degrees of freedom in Einstein gravity.

We work in four-dimensional Minkowski spacetime with metric $\eta_{\mu\nu} = \text{diag}(-,+,+,+)$. Fourier modes with momentum p_μ can be classified in representations of the little group – projected on and normal to the momentum. Correspondingly, define the transverse and longitudinal projectors

$$\theta_\mu{}^\nu = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2} \,, \qquad \omega_\mu{}^\nu = \frac{p_\mu p^\nu}{p^2} \,. \tag{12.I}$$

a) Show that θ and ω are orthogonal and idempotent:

$$\theta_\mu{}^\rho \omega_\rho{}^\nu = 0 \,, \qquad \theta_\mu{}^\rho \theta_\rho{}^\nu = \theta_\mu{}^\nu \,, \qquad \omega_\mu{}^\rho \omega_\rho{}^\nu = \omega_\mu{}^\nu \,. \tag{12.II}$$

Based on this result, argue that $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$ are a complete set projectors in the space of four-vectors.

b) Discuss which of the tensors in (12.I) project along the momentum, and which one onto the subspace normal to the momentum. What is the little group, i. e. the symmetry group of the normal subspace?

c) Decompose the four-vector A_μ into irreducible representations of the little group with the help of $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$. Which representations are those?

d) **Do this exercise before reading on:** A symmetric rank-two tensor $h_{\mu\nu} = h_{\nu\mu}$ like the graviton transforms as the tensor product of two four-vectors under Lorentz transformations. What are the transformation properties of the different objects one can construct from $h_{\mu\nu}$ and combinations of ${\theta_\mu}^\nu$ and ${\omega_\mu}^\nu$ under the little group?

e) To project onto the different representations, we need new projectors. Using θ and ω, show that requiring idempotency singles out the projectors

$$\begin{aligned}
{\mathcal{P}^{(2)}}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{2}\left({\theta_\mu}^\rho {\theta_\nu}^\sigma + {\theta_\mu}^\sigma {\theta_\nu}^\rho\right) - \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma}\,, \\
{\mathcal{P}^{(1)}}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{2}\left({\theta_\mu}^\rho {\omega_\nu}^\sigma + {\theta_\mu}^\sigma {\omega_\nu}^\rho + {\theta_\nu}^\rho {\omega_\mu}^\sigma + {\theta_\nu}^\sigma {\omega_\mu}^\rho\right)\,, \\
{\mathcal{P}^{(0,s)}}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma}\,, \\
{\mathcal{P}^{(0,w)}}_{\mu\nu}{}^{\rho\sigma} &= \omega_{\mu\nu}\omega^{\rho\sigma}\,,
\end{aligned} \tag{12.III}$$

for the transverse-traceless, vector, and the two scalar modes (in that order). Show that these operators are an idempotent, orthogonal set of tensors, i.e.

$${\mathcal{P}^{(i)}}_{\mu\nu}{}^{\alpha\beta}\,{\mathcal{P}^{(j)}}_{\alpha\beta}{}^{\rho\sigma} = \delta^{ij}{\mathcal{P}^{(i)}}_{\mu\nu}{}^{\rho\sigma}\,, \qquad \sum_i {\mathcal{P}^{(i)}}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}({\delta_\mu}^\rho {\delta_\nu}^\sigma + {\delta_\mu}^\sigma {\delta_\nu}^\rho) \equiv \mathbb{1}_{\mu\nu}{}^{\rho\sigma}\,, \tag{12.IV}$$

where $\mathbb{1}_{\mu\nu}{}^{\rho\sigma}$ denotes the identity in the space of symmetric Lorentz two-tensors.

f) Verify that the traces of the projectors, ${\mathcal{P}^{(i)}}_{\mu\nu}{}^{\mu\nu}$, are $5, 3, 1, 1$, respectively. Why is this a good cross check, i.e. what do these numbers mean? Which projector projects onto which representation of the little group?

Bonus: Derive the values of these traces purely from group theory.

g) Does the above set (12.III) form a basis in the space of rank-$(2,2)$ tensors that map symmetric rank-two tensors to symmetric rank-two tensors? If not, find the missing element(s). Are these new elements projectors? Compute all possible products of basis elements.

h) Take the graviton propagator in de Donder gauge from the lecture notes,

$$\mathcal{P}_{\mu\nu\rho\sigma} \simeq \frac{1}{p^2}\left[\mathbb{1}_{\mu\nu\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right]\,, \tag{12.V}$$

and decompose it into the basis that you have found. What do you learn from this?