Quantum Gravity and the Renormalization Group

Assignment 5 – Nov 17

Exercise 11: Ostrogradsky instabilities

Motivation: Higher-curvature invariants imply higher-order equations of motion. Here, we find out what this means for a toy model – a single higher-derivative harmonic oscillator.

We consider a single particle, which is a harmonic oscillator with a higher-derivative correction controlled by a parameter ϵ, namely

$$\mathcal{L} = -\frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 \,. \tag{11.I}$$

We will now study its solutions, count degrees of freedom, and look out for instabilities.

a) Show that the Euler-Lagrange equation for the Lagrangian in (11.I) is

$$\frac{\epsilon}{\omega^2}\ddddot{x} + \ddot{x} + \omega^2 x \,, \tag{11.II}$$

which has the solution

$$x(t) = C_+\cos(k_+ t) + S_+\sin(k_+ t) + C_-\cos(k_- t) + S_-\sin(k_- t)\,, \tag{11.III}$$

with the two frequencies

$$k_\pm = \omega\sqrt{\frac{1 \mp \sqrt{1-4\epsilon}}{2\epsilon}}\,, \tag{11.IV}$$

and where $C_\pm$ and $S_\pm$ are integration constants.

Bottom line: We need four pieces of initial data. Think before reading on: How many degrees of freedom usually require four pieces of initial data?

b) Ostrogradsky defined the canonical phase-space coordinates for second-order derivative Lagrangians as

$$\begin{aligned} X_1 &\equiv x\,, & P_1 &\equiv \frac{\partial L}{\partial \dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}}\,, \\ X_2 &\equiv \dot{x}\,, & P_2 &\equiv \frac{\partial L}{\partial \ddot{x}}\,. \end{aligned} \tag{11.V}$$

Show that in this case

$$P_1 = m\dot{x} + \frac{\epsilon m}{\omega^2}\dddot{x}, \qquad P_2 = -\frac{\epsilon m}{\omega^2}\ddot{x}\,. \tag{11.VI}$$

Check that this is equivalent to

$$\dddot{x} = \omega^2\frac{P_1 - mX_2}{\epsilon m}, \qquad \ddot{x} = -\frac{\omega^2 P_2}{\epsilon m}\,. \tag{11.VII}$$

c) The Hamiltonian is defined as

$$H \equiv \sum_{i=1}^{2} P_i \dot{X}_i - L\,. \tag{11.VIII}$$

Demonstrate that for our model, the Hamiltonian can be expressed as

$$H = P_1 X_2 - \frac{\omega^2}{2\epsilon m} P_2^2 - \frac{m}{2} X_2^2 + \frac{m\omega^2}{2} X_1^2\,. \tag{11.IX}$$

Think before moving on: What's problematic with this Hamiltonian?

d) Show that, evaluated on the solution in a), the Hamiltonian reads

$$H = \frac{m}{2}\sqrt{1-4\epsilon}\left[k_+^2(C_+^2+S_+^2) - k_-^2(C_-^2+S_-^2)\right]\,. \tag{11.X}$$

What's the difference between the two modes?

e) **[hard question]** The Hamiltonian in (11.X) is not bounded from above and below (why?). Yet, we could solve the differential equation in (11.II) to obtain the solution in (11.III) without instabilities in sight. Why is that? Will this property persist for more complicated models? For those who want to dive into this a bit more, there is a recent example in the literature.

a) For the Lagrangian in (11.I), we want to obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \mathcal{L}}{\partial \ddot{x}} = 0\,. \tag{11.1}$$

We first compute the derivatives:

$$\frac{\partial \mathcal{L}}{\partial x} = -m\omega^2 x\,, \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\,, \qquad \frac{\partial \mathcal{L}}{\partial \ddot{x}} = -\frac{\epsilon m}{\omega^2}\ddot{x}\,. \tag{11.2}$$

Hence, we obtain

$$\frac{\epsilon}{\omega^2}\ddddot{x} + \ddot{x} + \omega^2 x = 0\,. \tag{11.3}$$

Making the ansatz $x(t) = e^{ikt}$, one finds

$$\frac{\epsilon}{\omega^2}k^4 - k^2 + \omega^2 = 0\,. \tag{11.4}$$

This equation has the two pairs of solutions $k_\pm$ and $-k_\pm$, where

$$k_\pm = \omega\sqrt{\frac{1\mp\sqrt{1-4\epsilon}}{2\epsilon}}\,. \tag{11.5}$$

The general real solution is therefore

$$x(t) = C_+\cos(k_+ t) + S_+\sin(k_+ t) + C_-\cos(k_- t) + S_-\sin(k_- t)\,. \tag{11.6}$$

The four integration constants $(C_\pm, S_\pm)$ are fixed by $(x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0)$. To fix one degree of freedom, one usually needs two initial conditions. Therefore, we find two independent dynamical degrees of freedom.

b) Using (11.2), we obtain

$$P_1 = m\dot{x} + \frac{\epsilon m}{\omega^2}\dddot{x}\,, \qquad P_2 = -\frac{\epsilon m}{\omega^2}\ddot{x}\,. \tag{11.7}$$

The inverse relations follow as

$$\dddot{x} = \frac{\omega^2}{\epsilon m}(P_1 - m\dot{x})\,, \qquad \ddot{x} = -\frac{\omega^2}{\epsilon m}P_2\,. \tag{11.8}$$

With $X_2 = \dot{x}$ these are

$$\dddot{x} = \frac{\omega^2}{\epsilon m}(P_1 - mX_2)\,, \qquad \ddot{x} = -\frac{\omega^2}{\epsilon m}P_2\,. \tag{11.9}$$

Since $\partial^2\mathcal{L}/\partial\ddot{x}^2 = -\epsilon m/\omega^2 \neq 0$, the Legendre transform is non-degenerate. This is necessary to obtain the Hamiltonian in a standard way.

c) The Hamiltonian is

$$H = \sum_i P_i\dot{X}_i - \mathcal{L} = P_1\dot{X}_1 + P_2\dot{X}_2 - \mathcal{L} = P_1X_2 + P_2\ddot{x} - \mathcal{L}\,. \tag{11.10}$$

Substitute $\ddot{x} = -\frac{\omega^2}{\epsilon m}P_2$:

$$H = P_1X_2 - \frac{\omega^2}{\epsilon m}P_2^2 - \mathcal{L}\,. \tag{11.11}$$

Now write $\mathcal{L}$ in phase-space variables:

$$\mathcal{L} = -\frac{\omega^2}{2\epsilon m}P_2^2 + \frac{m}{2}X_2^2 - \frac{m\omega^2}{2}X_1^2\,. \tag{11.12}$$

Hence

$$H = P_1X_2 - \frac{\omega^2}{2\epsilon m}P_2^2 - \frac{m}{2}X_2^2 + \frac{m\omega^2}{2}X_1^2\,. \tag{11.13}$$

Because H depends linearly on P_1, which can take any value between $-\infty$ and ∞, it is unbounded from above and below. This is Ostrogradsky's theorem in action.

d) Expressed in terms of Lagrangian coordinates, the Hamiltonian reads

$$H = \frac{m}{2}\left(\dot{x}^2 + \omega^2x^2\right) - \frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{\epsilon m}{\omega^2}\dot{x}\,\dddot{x}\,. \tag{11.14}$$

With

$$x = C_+\cos(k_+t) + S_+\sin(k_+t) + C_-\cos(k_-t) + S_-\sin(k_-t)\,, \tag{11.15}$$
$$\dot{x} = -k_+C_+\sin(k_+t) + k_+S_+\cos(k_+t) - k_-C_-\sin(k_-t) + k_-S_-\cos(k_-t)\,, \tag{11.16}$$
$$\ddot{x} = -k_+^2[C_+\cos(k_+t) + S_+\sin(k_+t)] - k_-^2[C_-\cos(k_-t) + S_-\sin(k_-t)]\,, \tag{11.17}$$
$$\dddot{x} = -k_+^3[-C_+\sin(k_+t) + S_+\cos(k_+t)] - k_-^3[-C_-\sin(k_-t) + S_-\cos(k_-t)]\,, \tag{11.18}$$

the Hamiltonian simplifies to

$$H = \frac{m}{2}\sqrt{1-4\epsilon}\left[k_+^2(C_+^2 + S_+^2) - k_-^2(C_-^2 + S_-^2)\right]. \tag{11.19}$$

The k_+-mode carries positive energy, while the k_--mode carries negative energy. The unboundedness here comes from the fact that the initial conditions can be chosen arbitrarily large in absolute value.

e) Both modes oscillate with real frequencies; classical trajectories are therefore dynamically stable. Nevertheless, H is unbounded because one oscillator contributes negatively. This tells us that unbounded energies are not sufficient to conclude that the theory is unstable. In this case, the two modes of energies with differing sign do not interact. Therefore, the instability cannot be triggered from the outset.

With interactions, the two modes couple and total energy can decrease without bound, producing runaway solutions. This is the Ostrogradsky instability. However, coupling the two particles does not inevitably lead to instabilities. There are actually integrable, interacting single-particle models with unbounded Hamiltonians which do not lead to instabilities; usually, because the dynamics are protected by some kind of symmetry. The referenced example is interacting, has an unbounded Hamiltonian and still does not lead to instabilities.

Exercise 12: Spin projectors

Motivation: A symmetric rank-two field $h_{\mu\nu}$ in flat spacetime carries ten components, but only two correspond to the physical graviton polarisations. The spin-projector formalism provides a clean algebraic decomposition of $h_{\mu\nu}$ into its irreducible components under the little group $SO(3)$. This exercise builds the spin projectors step by step and connects them to the propagating degrees of freedom in Einstein gravity.

We work in four-dimensional Minkowski spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$. Fourier modes with momentum p_μ can be classified in representations of the little group – projected on and normal to the momentum. Correspondingly, define the transverse and longitudinal projectors

$$\theta_\mu{}^\nu = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2}\,, \qquad \omega_\mu{}^\nu = \frac{p_\mu p^\nu}{p^2}\,. \tag{12.I}$$

a) Show that θ and ω are orthogonal and idempotent:

$$\theta_\mu{}^\rho \omega_\rho{}^\nu = 0\,, \qquad \theta_\mu{}^\rho \theta_\rho{}^\nu = \theta_\mu{}^\nu\,, \qquad \omega_\mu{}^\rho \omega_\rho{}^\nu = \omega_\mu{}^\nu\,. \tag{12.II}$$

Based on this result, argue that $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$ are a complete set projectors in the space of four-vectors.

b) Discuss which of the tensors in (12.I) project along the momentum, and which one onto the subspace normal to the momentum. What is the little group, i. e. the symmetry group of the normal subspace?

c) Decompose the four-vector A_μ into irreducible representations of the little group with the help of $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$. Which representations are those?

d) **Do this exercise before reading on:** A symmetric rank-two tensor $h_{\mu\nu} = h_{\nu\mu}$ like the graviton transforms as the tensor product of two four-vectors under Lorentz transformations. What are the transformation properties of the different objects one can construct from $h_{\mu\nu}$ and combinations of $\theta_\mu{}^\nu$ and $\omega_\mu{}^\nu$ under the little group?

e) To project onto the different representations, we need new projectors. Using θ and ω, show that requiring idempotency singles out the projectors

$$\begin{aligned}
\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{2}\left(\theta_\mu{}^\rho\theta_\nu{}^\sigma + \theta_\mu{}^\sigma\theta_\nu{}^\rho\right) - \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma}\,,\\
\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{2}\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right)\,,\\
\mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma}\,,\\
\mathcal{P}^{(0,w)}{}_{\mu\nu}{}^{\rho\sigma} &= \omega_{\mu\nu}\omega^{\rho\sigma}\,,
\end{aligned} \tag{12.III}$$

for the transverse-traceless, vector, and the two scalar modes (in that order). Show that these operators are an idempotent, orthogonal set of tensors, i.e.

$$\mathcal{P}^{(i)}{}_{\mu\nu}{}^{\alpha\beta}\,\mathcal{P}^{(j)}{}_{\alpha\beta}{}^{\rho\sigma} = \delta^{ij}\mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma}\,, \qquad \sum_i \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}(\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\sigma\delta_\nu{}^\rho) \equiv \mathbb{1}_{\mu\nu}{}^{\rho\sigma}\,, \tag{12.IV}$$

where $\mathbb{1}_{\mu\nu}{}^{\rho\sigma}$ denotes the identity in the space of symmetric Lorentz two-tensors.

f) Verify that the traces of the projectors, $\mathcal{P}^{(i)}{}_{\mu\nu}{}^{\mu\nu}$, are $5, 3, 1, 1$, respectively. Why is this a good cross check, i.e. what do these numbers mean? Which projector projects onto which representation of the little group?

Bonus: Derive the values of these traces purely from group theory.

g) Does the above set (12.III) form a basis in the space of rank-$(2,2)$ tensors that map symmetric rank-two tensors to symmetric rank-two tensors? If not, find the missing element(s). Are these new elements projectors? Compute all possible products of basis elements.

h) Take the graviton propagator in de Donder gauge from the lecture notes,

$$\mathcal{P}_{\mu\nu\rho\sigma} \simeq \frac{1}{p^2}\left[\mathbb{1}_{\mu\nu\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right]\,, \tag{12.V}$$

and decompose it into the basis that you have found. What do you learn from this?

a) We first verify that the projectors are orthogonal and idempotent:

$$\theta_\mu{}^\rho\theta_\rho{}^\nu = \left(\delta_\mu{}^\rho - \frac{p_\mu p^\rho}{p^2}\right)\left(\delta_\rho{}^\nu - \frac{p_\rho p^\nu}{p^2}\right) = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2} = \theta_\mu{}^\nu\,, \tag{12.1}$$

$$\omega_\mu{}^\rho\omega_\rho{}^\nu = \frac{p_\mu p^\rho p_\rho p^\nu}{(p^2)^2} = \frac{p_\mu p^\nu}{p^2} = \omega_\mu{}^\nu\,, \tag{12.2}$$

$$\theta_\mu{}^\rho\omega_\rho{}^\nu = \left(\delta_\mu{}^\rho - \frac{p_\mu p^\rho}{p^2}\right)\frac{p_\rho p^\nu}{p^2} = \frac{p_\mu p^\nu}{p^2} - \frac{p_\mu p^\nu}{p^2} = 0\,. \tag{12.3}$$

The identity on the space of four-vectors is $\delta_\mu{}^\nu$. A set of projectors is complete if it sums to the identity on the space considered. The projectors satisfy $\theta_\mu{}^\nu + \omega_\mu{}^\nu = \delta_\mu{}^\nu$. Thus, they form a complete set of projectors on four-vectors.

b) The little group is the subgroup of Lorentz transformations that leaves a reference momentum invariant:

$$\Lambda_\mu{}^\nu p_\nu = p_\mu\,. \tag{12.4}$$

For timelike $p_\mu = (m, 0, 0, 0)$, this group is $SO(3)$; for lightlike momenta $p_\mu = (E, 0, 0, E)$ it becomes $ISO(2)$. We use $SO(3)$ because its representations are labelled by integer spins and the massless case can be recovered in the limit $m \to 0$ in the end. The projector $\omega_\mu{}^\nu$ extracts the component of a vector along p_μ, while $\theta_\mu{}^\nu$ projects on the three-dimensional hypersurfaces orthogonal to p_μ. We can also see that from their traces, $\theta_\mu{}^\mu = 3$ and $\omega_\mu{}^\mu = 1$.

c) A four-vector decomposes uniquely as

$$A_\mu^\perp = \theta_\mu{}^\nu A_\nu \,, \qquad A_\mu^\parallel = \omega_\mu{}^\nu A_\nu \,. \tag{12.5}$$

From completeness, we get that $A_\mu = A_\mu^\perp + A_\mu^\parallel$. $A_\mu^\perp$ has three components transforming as a vector of $SO(3)$ (spin one), while $A_\mu^\parallel \propto p_\mu$ is invariant (spin zero).

d) As there are two open indices, we can construct the combinations

$$\omega_\mu{}^\rho \omega_\nu{}^\sigma h_{\rho\sigma} \,, \qquad (\omega_\mu{}^\rho \theta_\nu{}^\sigma + \omega_\nu{}^\rho \theta_\mu{}^\sigma) h_{\rho\sigma} \,, \qquad \theta_\mu{}^\rho \theta_\nu{}^\sigma h_{\rho\sigma} \,, \tag{12.6}$$

where we used that $h_{\mu\nu}$ is symmetric in its indices. We recall that $\omega_\mu{}^\nu$ projects onto a one-dimensional subspace, while $\theta_\mu{}^\nu$ projects onto a three-dimensional subspace. Therefore, $h^{pp} p_\mu p_\nu = \omega_\mu{}^\rho \omega_\nu{}^\sigma h_{\rho\sigma}$, which has one independent component, is a scalar (spin zero) under the little group, while $h^p_{(\mu} p_{\nu)}$, having three independent components, is a vector (spin one). We are left with $h_{\mu\nu}^\perp = \theta_\mu{}^\rho \theta_\nu{}^\sigma h_{\rho\sigma}$. This is a symmetric 3×3 matrix, which can be further divided into its scalar (spin zero) trace $h = h^\perp{}_\mu{}^\mu$ and the transverse traceless (spin two) part $h_{\mu\nu}^{\rm TT} = h_{\mu\nu}^\perp - h\theta_{\mu\nu}/3$. Note that here the second term must be proportional to $\theta_{\mu\nu}$ because otherwise we would leave the transverse subspace.

e) We now construct the explicit projectors that isolate the different spin sectors identified in d). They must satisfy three properties:

(a) They project onto the appropriate subspace (*idempotence*): $\mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} \mathcal{P}^{(i)}{}_{\rho\sigma}{}^{\kappa\lambda} = \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma}$.

(b) They are *orthogonal*: $\mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} \mathcal{P}^{(j)}{}_{\rho\sigma}{}^{\kappa\lambda} = 0$ for $i \neq j$.

(c) Their sum yields the identity on the space of symmetric tensors:

$$\sum_i \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\sigma \delta_\nu{}^\rho) \equiv \mathbb{1}_{\mu\nu}{}^{\rho\sigma} \,. \tag{12.7}$$

We construct each projector from the two of the basic one- and three-dimensional projectors $\omega_\mu{}^\nu$ and $\theta_\mu{}^\nu$.

Spin 0: Let's start as simple as possible, namely with the spin zero part. Two scalar pieces exist: the longitudinal one (both indices along p_μ) and the transverse trace part.

As $\omega_{\mu\nu}\omega_{\rho\sigma} = \omega_{\rho\nu}\omega_{\mu\sigma} = \omega_{\sigma\nu}\omega_{\rho\mu}$, the most general combination of two ωs is

$$\mathcal{P}^{(0,w)}{}_{\mu\nu}{}^{\rho\sigma} = a_w \omega_{\mu\nu} \omega^{\rho\sigma} \,. \tag{12.8}$$

Idempotence then requires $a_w = 1$ such that

$$\mathcal{P}^{(0,w)}{}_{\mu\nu}{}^{\rho\sigma} = \omega_{\mu\nu} \omega^{\rho\sigma} \,. \tag{12.9}$$

The trace projector has to contract the indices of $h_{\mu\nu}$ with a transverse projector and take its trace. Thus, when contracted with two gravitons, it must satisfy

$$h^{\mu\nu} \mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = c(\theta^{\mu\nu} h_{\mu\nu})^2 \,, \tag{12.10}$$

for some constant c. Therefore, it can only have the shape

$$\mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\rho\sigma} = c\,\theta_{\mu\nu}\theta^{\rho\sigma}\,. \tag{12.11}$$

Idempotence yields

$$\mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\alpha\beta}\mathcal{P}^{(0,s)}{}_{\alpha\beta}{}^{\rho\sigma} = c^2(\theta_{\mu\nu}\theta^{\alpha\beta}\theta_{\alpha\beta}\theta^{\rho\sigma}) = 3c^2\,\theta_{\mu\nu}\theta^{\rho\sigma} \stackrel{!}{=} c\,\theta_{\mu\nu}\theta^{\rho\sigma} \Rightarrow c = \frac{1}{3}\,. \tag{12.12}$$

Hence,

$$\mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{3}\,\theta_{\mu\nu}\theta^{\rho\sigma}\,. \tag{12.13}$$

Spin 1: The spin one sector mixes one transverse and one longitudinal index. We construct the most general tensor $\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\rho\sigma}$ with one θ and one ω on each side:

$$\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = a_1\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right) + b_1\,\theta_{\mu\nu}\omega^{\rho\sigma} + c_1\,\omega_{\mu\nu}\theta^{\rho\sigma}\,. \tag{12.14}$$

The spin one projector must be orthogonal to both scalar projectors,

$$\mathcal{P}^{(1)}\cdot\mathcal{P}^{(0,s)} = \mathcal{P}^{(1)}\cdot\mathcal{P}^{(0,w)} = 0\,, \tag{12.15}$$

independently from which side they are contracted with it. This implies

$$\theta^{\mu\nu}\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = 0\,, \qquad \mathcal{P}^{(1)}{}_{\mu\nu}{}^{\rho\sigma}\omega_{\rho\sigma} = 0\,, \tag{12.16}$$

which eliminates the last two terms, that is $b_1 = c_1 = 0$. We fix a_1 by idempotence. Using $\theta^2 = \theta$, $\omega^2 = \omega$, and $\theta\cdot\omega = 0$, we obtain

$$\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\alpha\beta}\mathcal{P}^{(1)}{}_{\alpha\beta}{}^{\rho\sigma} = 2a_1^2\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right) \tag{12.17}$$

$$\stackrel{!}{=} a_1\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right) \Rightarrow a_1 = \frac{1}{2}\,. \tag{12.18}$$

Thus, the spin one projector is

$$\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right)\,. \tag{12.19}$$

Spin 2: The spin two part corresponds to the transverse traceless component of $h_{\mu\nu}$. As this component is transverse in both indices, we begin with the most general ansatz built purely from θs:

$$\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\rho\sigma} = a\,\left(\theta_\mu{}^\rho\theta_\nu{}^\sigma + \theta_\mu{}^\sigma\theta_\nu{}^\rho\right) + b\,\theta_{\mu\nu}\theta^{\rho\sigma}\,. \tag{12.20}$$

Because the projector acts on symmetric tensors, here we symmetrised in $(\mu\nu)$ and $(\rho\sigma)$. That the sector is traceless means that contracting any pair of indices with the metric should give zero:

$$\eta^{\mu\nu}\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\rho\sigma} = (2a + 3b)\,\theta^{\rho\sigma} \stackrel{!}{=} 0 \Rightarrow b = -\frac{2}{3}a\,. \tag{12.21}$$

Idempotence fixes the overall normalisation. Contracting once with itself and using $\theta^2 = \theta$, one finds $a = \frac{1}{2}$. Thus

$$\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\theta_\mu{}^\rho\theta_\nu{}^\sigma + \theta_\mu{}^\sigma\theta_\nu{}^\rho\right) - \frac{1}{3}\,\theta_{\mu\nu}\theta^{\rho\sigma}\,. \tag{12.22}$$

Orthogonality. On the way, we already showed idempotence for all projectors, and ensured orthogonality between $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(0,s)}$ as well as $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(0,w)}$. Besides, $\mathcal{P}^{(0,s)}$ and $\mathcal{P}^{(0,w)}$ as well as $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(0,w)}$ are orthogonal because they are entirely made up of a single type of orthogonal projectors. Similarly, $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(1)}$ are orthogonal because $\mathcal{P}^{(2)}$ depends only on θs while $\mathcal{P}^{(1)}$ has ωs in each term. Finally, $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(0,s)}$ are orthogonal because

$$\begin{aligned}
\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\alpha\beta}\mathcal{P}^{(0,s)}{}_{\alpha\beta}{}^{\rho\sigma} &= \left[\frac{1}{2}(\theta_\mu{}^\alpha\theta_\nu{}^\beta + \theta_\mu{}^\beta\theta_\nu{}^\alpha) - \frac{1}{3}\theta_{\mu\nu}\theta^{\alpha\beta}\right] \tfrac{1}{3}\theta_{\alpha\beta}\theta^{\rho\sigma} \\
&= \tfrac{1}{6}\big(\theta_\mu{}^\alpha\theta_\nu{}^\beta + \theta_\mu{}^\beta\theta_\nu{}^\alpha\big)\theta_{\kappa\lambda}\theta^{\rho\sigma} - \tfrac{1}{9}\theta_{\mu\nu}\theta^{\alpha\beta}\theta_{\alpha\beta}\theta^{\rho\sigma} \\
&= \tfrac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma} - \tfrac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma} \\
&= 0.
\end{aligned} \tag{12.23}$$

Thus, we have shown the first equation in (12.IV).

Decomposition of identity. We now show explicitly that the four projectors sum to the identity. Start by inserting the explicit definitions:

$$\begin{aligned}
\sum_i \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} &= \left[\frac{1}{2}\left(\theta_\mu{}^\rho\theta_\nu{}^\sigma + \theta_\mu{}^\sigma\theta_\nu{}^\rho\right) - \frac{1}{3}\,\theta_{\mu\nu}\theta^{\rho\sigma}\right] + \left[\frac{1}{2}\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right)\right] \\
&\quad + \left[\frac{1}{3}\theta_{\mu\nu}\theta^{\rho\sigma}\right] + \left[\omega_{\mu\nu}\omega^{\rho\sigma}\right] \\
&= \frac{1}{2}\left(\theta_\mu{}^\rho\theta_\nu{}^\sigma + \theta_\mu{}^\sigma\theta_\nu{}^\rho\right) + \frac{1}{2}\left(\theta_\mu{}^\rho\omega_\nu{}^\sigma + \theta_\mu{}^\sigma\omega_\nu{}^\rho + \theta_\nu{}^\rho\omega_\mu{}^\sigma + \theta_\nu{}^\sigma\omega_\mu{}^\rho\right) + \omega_{\mu\nu}\omega^{\rho\sigma}\,.
\end{aligned} \tag{12.24}$$

Now we replace $\theta = \delta - \omega$ everywhere. This gives

$$\begin{aligned}
\sum_i \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{2}\Big[\left(\delta_\mu{}^\rho - \omega_\mu{}^\rho\right)\left(\delta_\nu{}^\sigma - \omega_\nu{}^\sigma\right) + \left(\delta_\mu{}^\sigma - \omega_\mu{}^\sigma\right)\left(\delta_\nu{}^\rho - \omega_\nu{}^\rho\right)\Big] \\
&\quad + \frac{1}{2}\Big[\left(\delta_\mu{}^\rho - \omega_\mu{}^\rho\right)\omega_\nu{}^\sigma + \left(\delta_\mu{}^\sigma - \omega_\mu{}^\sigma\right)\omega_\nu{}^\rho + \left(\delta_\nu{}^\rho - \omega_\nu{}^\rho\right)\omega_\mu{}^\sigma + \left(\delta_\nu{}^\sigma - \omega_\nu{}^\sigma\right)\omega_\mu{}^\rho\Big] \\
&\quad + \omega_{\mu\nu}\omega^{\rho\sigma} \\
&= \mathbb{1}_{\mu\nu}{}^{\rho\sigma} - \frac{1}{2}\left[\delta_\mu{}^\rho\omega_\nu{}^\sigma + \delta_\mu{}^\sigma\omega_\nu{}^\rho + \delta_\nu{}^\rho\omega_\mu{}^\sigma + \delta_\nu{}^\sigma\omega_\mu{}^\rho\right] + \frac{1}{2}\left[\omega_\mu{}^\rho\omega_\nu{}^\sigma + \omega_\mu{}^\sigma\omega_\nu{}^\rho\right] \\
&\quad + \frac{1}{2}\left[\delta_\mu{}^\rho\omega_\nu{}^\sigma + \delta_\mu{}^\sigma\omega_\nu{}^\rho + \delta_\nu{}^\rho\omega_\mu{}^\sigma + \delta_\nu{}^\sigma\omega_\mu{}^\rho\right] - \left[\omega_\mu{}^\rho\omega_\nu{}^\sigma + \omega_\mu{}^\sigma\omega_\nu{}^\rho\right] \\
&\quad + \omega_{\mu\nu}\omega^{\rho\sigma} \\
&= \mathbb{1}_{\mu\nu}{}^{\rho\sigma} - \frac{1}{2}\left[\omega_\mu{}^\rho\omega_\nu{}^\sigma + \omega_\mu{}^\sigma\omega_\nu{}^\rho\right] + \omega_{\mu\nu}\omega^{\rho\sigma}\,.
\end{aligned} \tag{12.25}$$

Finally, we insert the definition of ω to see that all the ω-terms cancel, and we indeed get the identity:

$$\sum_i \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\rho\sigma} = \mathbb{1}_{\mu\nu}{}^{\rho\sigma} - \frac{1}{2}\left[\frac{p_\mu p^\rho}{p^2}\frac{p_\mu p^\sigma}{p^2} + \frac{p_\mu p^\sigma}{p^2}\frac{p_\nu p^\rho}{p^2}\right] + \frac{p_\mu p_\nu}{p^2}\frac{p^\rho p^\sigma}{p^2} = \mathbb{1}_{\mu\nu}{}^{\rho\sigma}\,. \tag{12.26}$$

Thus, we just proved the second equation in (12.IV).

f) We now compute the traces of the projectors and interpret the result in terms of the spin representations of the little group. The trace of a projector equals the dimension of the subspace it projects onto, because for any projector $\mathcal{P}$,

$$\mathrm{Tr}(\mathcal{P}^2) = \mathrm{Tr}(\mathcal{P})\,, \tag{12.27}$$

so the trace simply counts the number of independent components in the image of $\mathcal{P}$. For our rank-four tensors acting on the space of symmetric tensors,

$$\mathrm{Tr}\,\mathcal{P}^{(i)} = \mathcal{P}^{(i)}{}_{\mu\nu}{}^{\mu\nu}\,. \tag{12.28}$$

We compute this explicitly for each case.

Spin 2:

$$\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\mu\nu} = \frac{1}{2}(\theta_\mu{}^\mu\theta_\nu{}^\nu + \theta_{\mu\nu}\theta^{\nu\mu}) - \frac{1}{3}\theta_{\mu\nu}\theta^{\mu\nu}\,. \tag{12.29}$$

Using $\theta_\mu{}^\mu = 3$ and $\theta_{\mu\nu}\theta^{\mu\nu} = \theta_\mu{}^\mu = 3$, we find

$$\mathcal{P}^{(2)}{}_{\mu\nu}{}^{\mu\nu} = 5\,. \tag{12.30}$$

This corresponds to the five independent components of a symmetric, traceless, transverse tensor, i. e. the spin two representation.

Spin 1:

$$\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\mu\nu} = \frac{1}{2}(\theta_\mu{}^\mu\omega_\nu{}^\nu + \theta_{\mu\nu}\omega^{\nu\mu} + \theta_\nu{}^\mu\omega_\mu{}^\nu + \theta_\nu{}^\nu\omega_\mu{}^\mu)\,. \tag{12.31}$$

Since $\theta_{\mu\nu}\omega^{\mu\nu} = 0$, $\theta_\mu{}^\mu = 3$, and $\omega_\mu{}^\mu = 1$, the trace reduces to

$$\mathcal{P}^{(1)}{}_{\mu\nu}{}^{\mu\nu} = 3\,. \tag{12.32}$$

This corresponds to the three independent components of a vector (spin one) representation of $SO(3)$.

Transverse scalar sector:

$$\mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\mu\nu} = \frac{1}{3}\theta_{\mu\nu}\theta^{\mu\nu} = 1\,. \tag{12.33}$$

Hence, this represents one spin zero degree of freedom (the trace in the transverse subspace).

Longitudinal scalar sector:

$$\mathcal{P}^{(0,w)}{}_{\mu\nu}{}^{\mu\nu} = \omega_{\mu\nu}\omega^{\mu\nu} = 1\,. \tag{12.34}$$

This corresponds to the single longitudinal scalar component (proportional to $p_\mu p_\nu$).

Consistency check. The total number of components is

$$5 + 3 + 1 + 1 = 10\,, \tag{12.35}$$

which matches the number of independent components of a symmetric 4×4 tensor $h_{\mu\nu}$.

Bonus: The four numbers $5, 3, 1, 1$ correspond to the dimensions of the irreducible spin representations of the little group $SO(3)$ that appear when decomposing a symmetric rank-two tensor $h_{\mu\nu}$ under Lorentz transformations. We now derive them purely from the group-theoretic viewpoint, without referring to the explicit projectors.

A Lorentz vector transforms under the $(\frac{1}{2}, \frac{1}{2})$ representation of the covering group $SL(2,\mathbb{C}) \simeq SO(1,3)$. In other words, a single index A_μ carries spin $(\frac{1}{2}, \frac{1}{2})$. A general rank-two tensor $T_{\mu\nu}$ therefore transforms under the tensor product of two vector representations:

$$(\tfrac{1}{2}, \tfrac{1}{2}) \otimes (\tfrac{1}{2}, \tfrac{1}{2}) = (1,1) \oplus (1,0) \oplus (0,1) \oplus (0,0)\,. \tag{12.36}$$

Here (j_L, j_R) labels the irreducible representations of $SU(2)_L \times SU(2)_R$, the two spinor factors of $SL(2,\mathbb{C})$. The antisymmetric part of $T_{\mu\nu}$ corresponds to $(1,0)\oplus(0,1)$ (since it has six components, like an antisymmetric two-tensor), while the symmetric part corresponds to $(1,1)\oplus(0,0)$. Thus, for $h_{\mu\nu}=h_{\nu\mu}$, we are concerned only with $(1,1)\oplus(0,0)$.

The irreducible representations of the little group $SO(3)$ are labeled by the spin quantum number $j=0,1,2,\ldots$ with dimensions $2j+1$. When restricting a Lorentz representation (j_L,j_R) to spatial rotations, the relevant $SO(3)$ spin is obtained by adding the two $SU(2)$ spins:

$$j=|j_L-j_R|,|j_L-j_R|+1,\ldots,j_L+j_R\,. \tag{12.37}$$

For $(1,1)$ we have $j_L=j_R=1$, so the allowed spins under $SO(3)$ are

$$j=|1-1|,|1-1|+1,\ldots,1+1 \quad\Rightarrow\quad j=0,1,2\,. \tag{12.38}$$

Therefore, $(1,1)$ contains three $SO(3)$ irreps:

$$(1,1)\;\longrightarrow\;2\oplus1\oplus0\,. \tag{12.39}$$

The scalar $(0,0)$ remains a singlet:

$$(0,0)\;\longrightarrow\;0\,. \tag{12.40}$$

Putting both together,

$$(1,1)\oplus(0,0)\;\longrightarrow\;2\oplus1\oplus0\oplus0\,. \tag{12.41}$$

The dimension of a spin-j representation of $SO(3)$ is $2j+1$. Thus:

$$\dim(2)=5\,,\qquad \dim(1)=3\,,\qquad \dim(0)=1\,. \tag{12.42}$$

Since there are two distinct spin zero pieces (from $(1,1)$ and $(0,0)$), we have two scalars. Hence, the entries of the symmetric Lorentz two-tensor decomposes into representations of $SO(3)$ as

$$10=5+3+1+1\,, \tag{12.43}$$

corresponding to the dimensions of the four subspaces onto which the projectors $\mathcal{P}^{(2)}$, $\mathcal{P}^{(1)}$, $\mathcal{P}^{(0,s)}$, and $\mathcal{P}^{(0,w)}$ project.

g) The set of projectors does not form a basis. The missing terms are

$$\mathcal{Q}^{(0,sw)}{}_{\mu\nu}{}^{\rho\sigma}=\frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega^{\rho\sigma}\,, \tag{12.44}$$

$$\mathcal{Q}^{(0,ws)}{}_{\mu\nu}{}^{\rho\sigma}=\frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta^{\rho\sigma}\,. \tag{12.45}$$

If you look sharply, you can see that these are a mixture of the two scalar projectors – and this is indeed what these basis elements do, they mix the spin zero sector. The $\mathcal{Q}$ are not projectors though – their trace is zero, and they do not appear in the decomposition of unity. This is because they are off-diagonal terms in the decomposition. Following the same strategy as above, we find

$$\mathcal{P}^{(2)} \cdot \mathcal{Q}^{(0,sw)} = \mathcal{Q}^{(0,sw)} \cdot \mathcal{P}^{(2)} = 0\,, \tag{12.46}$$

$$\mathcal{P}^{(2)} \cdot \mathcal{Q}^{(0,ws)} = \mathcal{Q}^{(0,ws)} \cdot \mathcal{P}^{(2)} = 0\,, \tag{12.47}$$

$$\mathcal{P}^{(1)} \cdot \mathcal{Q}^{(0,sw)} = \mathcal{Q}^{(0,sw)} \cdot \mathcal{P}^{(1)} = 0\,, \tag{12.48}$$

$$\mathcal{P}^{(1)} \cdot \mathcal{Q}^{(0,ws)} = \mathcal{Q}^{(0,ws)} \cdot \mathcal{P}^{(1)} = 0\,, \tag{12.49}$$

$$\mathcal{P}^{(0,s)} \cdot \mathcal{Q}^{(0,sw)} = \mathcal{Q}^{(0,sw)} \cdot \mathcal{P}^{(0,w)} = \mathcal{Q}^{(0,sw)}\,, \tag{12.50}$$

$$\mathcal{P}^{(0,s)} \cdot \mathcal{Q}^{(0,ws)} = \mathcal{Q}^{(0,sw)} \cdot \mathcal{P}^{(0,s)} = 0\,, \tag{12.51}$$

$$\mathcal{P}^{(0,w)} \cdot \mathcal{Q}^{(0,ws)} = \mathcal{Q}^{(0,ws)} \cdot \mathcal{P}^{(0,s)} = \mathcal{Q}^{(0,ws)}\,, \tag{12.52}$$

$$\mathcal{P}^{(0,w)} \cdot \mathcal{Q}^{(0,sw)} = \mathcal{Q}^{(0,ws)} \cdot \mathcal{P}^{(0,w)} = 0\,, \tag{12.53}$$

$$\mathcal{Q}^{(0,sw)} \cdot \mathcal{Q}^{(0,ws)} = \mathcal{P}^{(0,s)}\,, \tag{12.54}$$

$$\mathcal{Q}^{(0,ws)} \cdot \mathcal{Q}^{(0,sw)} = \mathcal{P}^{(0,w)}\,, \tag{12.55}$$

$$\mathcal{Q}^{(0,ws)} \cdot \mathcal{Q}^{(0,ws)} = \mathcal{Q}^{(0,sw)} \cdot \mathcal{Q}^{(0,sw)} = 0\,. \tag{12.56}$$

All products can be written in terms of the introduced projectors and $\mathcal{Q}$s, so this forms a basis. We could summarise this as "projectors do projector things" and "people give names to objects that sometimes make sense".

h) For the first term we can use the decomposition of unity. Thus we only have to decompose $\eta_{\mu\nu}\eta_{\rho\sigma}$. Evidently, this comes from the spin zero sector. We can write

$$\mathcal{P}^{(0,s)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{3}\left(\eta_{\mu\nu} - \omega_{\mu\nu}\right)\left(\eta^{\rho\sigma} - \omega^{\rho\sigma}\right) = \frac{1}{3}\eta_{\mu\nu}\eta^{\rho\sigma} - \frac{1}{3}\left(\eta_{\mu\nu}\omega^{\rho\sigma} + \omega_{\mu\nu}\eta^{\rho\sigma}\right) + \frac{1}{3}\omega_{\mu\nu}\omega^{\rho\sigma}\,. \tag{12.57}$$

We thus can write

$$\eta_{\mu\nu}\eta_{\rho\sigma} = 3\mathcal{P}^{(0,s)}{}_{\mu\nu\rho\sigma} + \sqrt{3}\left(\mathcal{Q}^{(0,ws)}{}_{\mu\nu\rho\sigma} + \mathcal{Q}^{(0,sw)}{}_{\mu\nu\rho\sigma}\right) - \mathcal{P}^{(0,w)}{}_{\mu\nu\rho\sigma}\,. \tag{12.58}$$

Inserting this into the propagator, we finally find

$$\mathcal{P} \simeq \frac{1}{p^2}\left[\mathcal{P}^{(2)} + \mathcal{P}^{(1)} + \frac{3}{2}\mathcal{P}^{(0,w)} - \frac{1}{2}\mathcal{P}^{(0,s)} - \frac{\sqrt{3}}{2}\left(\mathcal{Q}^{(0,ws)} + \mathcal{Q}^{(0,sw)}\right)\right]. \tag{12.59}$$

We can see that all spin components appear in the propagator in this gauge. This is because the propagator is an off-shell object, and thus includes also the gauge degrees of freedom that eventually are cancelled by the Faddeev-Popov ghosts. To see that these extra modes are unphysical, one can use a non-covariant gauge. If you want to learn more, have a look at the Lectures in Quantum Gravity, section 2.2.