Quantum Gravity and the Renormalization Group

Assignment 6 – Nov 24

Exercise 13: Avoiding Ostrogradski?

Motivation: In the lecture, we discussed the Ostrogradski problem of quadratic gravity. In this exercise, we will try to find ways to avoid the negative aspects without sacrificing the positive aspects of the higher-derivative terms.

The spin-two propagator in quadratic gravity has the structural form

$$\frac{4\lambda}{p^4 - \frac{\lambda M_{\text{Pl}}^2}{2} p^2} \,. \tag{13.I}$$

As we discussed in some detail, the fall-off with p^4 for large momenta (together with the fact that vertices also scale like the fourth power of momentum) make the theory renormalisable, and in particular asymptotically free. However, the partial fraction decomposition of the propagator,

$$\frac{8}{M_{\text{Pl}}^2} \left[\frac{1}{-p^2} - \frac{1}{-p^2 + \frac{\lambda}{2} M_{\text{Pl}}^2} \right] , \tag{13.II}$$

indicates that we have a massive ghost. Let us investigate different potential scenarios in how we could avoid it.

a) First, consider the propagator of a general quartic theory. Does it always propagate a ghost? Why?

b) **[hard question]** Suppose now that we start with a propagator that is the sum of two modes that both come with a positive prefactor, so that we do not propagate a ghost. Try to reconstruct the action that gives rise to such a propagator. Are there any problems with such a theory? Is this a local theory? What is its UV behaviour?

c) **[hard question]** Suppose that instead of quadratic gravity, we now add sixth-order derivative terms. What is propagated in such a theory? Can we avoid ghosts? If yes, what are the conditions for this? What are the renormalisation properties of sixth-derivative gravity?

Exercise 14: Extra modes in higher derivative gravity

Motivation: In this exercise, we will make the extra degrees of freedom that the curvature-squared terms introduce explicit. The general idea is the following: we first write down an action with an auxiliary (that is, non-dynamical) field, making sure that the equations of motion are equivalent to higher-derivative gravity. We then perform a shift of the metric to bring the action into a form of GR plus "extra stuff". For the R^2-term, we can do this exactly, whereas for the $R^{\mu\nu}R_{\mu\nu}$-term, we will restrict ourselves to the kinetic term of the new field.

Consider the following "weird" action:

$$S^{\text{weird}} = \int \mathrm{d}^4x\, \sqrt{-g}\, \left[\Phi\, R - \frac{1}{4\alpha}\, (\Phi - 1)^2\right] . \tag{14.I}$$

Here, α is a constant and Φ is a scalar field.

a) Derive the equations of motion, both for the auxiliary field Φ and for the metric. Plug the solution for the scalar field into the equation of motion for the metric. What is this equation of motion equivalent to? Also, plug the solution into the action. What do you get?

b) Perform a conformal transformation on S^{weird} (don't insert any solution to the equations of motion here!), this time with

$$g_{\mu\nu} \mapsto \Phi^{-1}\, \tilde{g}_{\mu\nu} . \tag{14.II}$$

How does the action look like?

c) Finally, reparameterise the scalar field by

$$\Phi = e^{c\,\tilde{\Phi}} , \tag{14.III}$$

where c is a suitable constant. Compute the action as a function of $\tilde{g}$ and $\tilde{\Phi}$ and choose the constant c cleverly (you will see what this means). What is this theory? What do we learn from this whole computation?

d) **[hard question]** Try to generalise this to $f(R)$ gravity. Does the same method go through? If so, how does the scalar potential look like?

Now shift your attention to this "super-weird" action:

$$S^{\text{super-weird}} = \int \mathrm{d}^4x\, \sqrt{-g}\, \left[R + \left(R^{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu}\right) f_{\mu\nu} - \frac{1}{4\beta}\, \left(f^{\mu\nu} f_{\mu\nu} - f^{\mu}{}_{\mu} f^{\nu}{}_{\nu}\right)\right] . \tag{14.IV}$$

Here, $f_{\mu\nu}$ is a symmetric rank-two tensor field, and β is a constant.

e) **[hard question]** Perform the same tasks as in a), but now for the action $S^{\text{super-weird}}$.

f) **[hard question]** Now shift the metric in $S^{\text{super-weird}}$ in the following way:

$$g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} + c\, f_{\mu\nu} . \tag{14.V}$$

Doing so, only keep terms up to second order in f, and fix c such that there is no kinetic mixing — that is, ensure that there is no term that is linear in both curvature and f. Once again, what is this theory, and what do we learn from this?