Quantum Gravity and the Renormalization Group

Assignment 7 – Dec 01

Exercise 15: Analysing beta functions

Motivation: The purpose of this exercise is to familiarise yourself with the study of beta functions: finding fixed points, computing critical exponents, and investigating the phase diagram. These are the bread-and-butter (or margarine/olive oil/whatever you fancy) skills when analysing renormalisation group flows.

In the following, we will consider two different sets of beta functions for two couplings G and Λ that depend on a reference scale k. They have mass dimensions -2 and 2, respectively, so that the dimensionless couplings are given by $g = Gk^2$ and $\lambda = \Lambda k^{-2}$.[a]

Consider first the following set of beta functions for the two couplings g, λ:

$$\begin{aligned} \beta_g &= k\partial_k g = 2g - \frac{135g^2}{72\pi - 5g}\,, \\ \beta_\lambda &= k\partial_k \lambda = -\left(2 + \frac{135g}{72\pi - 5g}\right)\lambda - g\left(\frac{43}{4\pi} - \frac{810}{72\pi - 5g}\right)\,. \end{aligned} \tag{15.I}$$

a) Compute all fixed points of this set of beta functions.

b) Compute the critical exponents at each of these fixed points. If there is an interacting fixed point, how many relevant parameters does it have? What does that mean for its predictivity?

c) We can actually solve the beta functions implicitly. Show in a first step that $g(k)$ implicitly defined via

$$G_N\, k^2 = \frac{g(k)}{\left(1 - \frac{145}{144\pi} g(k)\right)^{27/29}} \tag{15.II}$$

is a solution to the first beta function, where G_N is a constant. What is the meaning of G_N? *Hints:* Take a $k\partial_k$ derivative of this equation, solve for β_g, and check that you get back the original beta function. For the meaning of G_N, it might be helpful to expand to leading order in powers of k.

d) In a second step, show that $\lambda(k)$ defined by

$$\lambda(k) = \frac{162}{25} - \frac{43}{16\pi} g(k) + \ell\,(144\pi - 145g(k))^{25/29} - \frac{144\pi}{3625 g(k)}\left[87 + 25\ell\,(144\pi - 145g(k))^{25/29}\right] \tag{15.III}$$

is a solution to the second beta function. Here, the constant ℓ can be written as

$$\ell = -\frac{29}{86400}\left(2^{-13}3^{-21}\pi^{-54}\right)^{1/29}(432\pi + 125 G_N\,\Lambda_0)\,, \tag{15.IV}$$

with Λ_0 being yet another constant. What is the meaning of Λ_0? *Hint:* For the meaning of Λ_0, it might be helpful to expand to leading order in powers of k.

e) **[hard question]** Expand both solutions in powers of $1/k$ to leading non-trivial order. How does this relate to part b)? Could you have gotten this result more easily?

f) Instead of using λ, it can be useful to use the dimensionless combination $\tau = G\Lambda = g\lambda$. What is the interpretation of this coupling, and what are the potential IR ($k \to 0$) values that can be reached from the fixed point(s)?

g) Sketch the *phase diagram* in terms of g and τ (Mathematica's *StreamPlot* might come in handy, if you don't want to do it by hand). The phase diagram, or RG flow diagram, is a plot of the integral curves of the beta functions.

Your phase diagram should contain all fixed points and any potential other global features. If there are features other than fixed points, explain them.

Let us now briefly analyse a second set of beta functions:

$$\begin{aligned}
\beta_g &= \left(2 - \frac{2}{3\pi}\frac{\lambda^2}{(1-2\lambda)^4} g\right) g\,, \\
\beta_\lambda &= -\left(2 + \frac{2}{3\pi}\frac{\lambda^2}{(1-2\lambda)^4} g\right)\lambda + \frac{g}{4\pi}\frac{1}{1-2\lambda}\left[1 + \frac{g}{3}\frac{2}{3\pi}\frac{\lambda^2}{(1-2\lambda)^4}\right].
\end{aligned} \tag{15.V}$$

h) Compute all real fixed points and their critical exponents. Is there anything unusual here? If so, try to explain what's going on.

i) **[hard question]** Sketch the phase diagram. What is the new feature in these beta functions? Which implications does this have for the admissible IR values of G and Λ? Can the relevant fixed point be used as a UV completion? Why/why not?

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[a] Guess what G and Λ should represent.