Quantum Gravity and the Renormalization Group

Assignment 7 – Dec 01

Exercise 15: Analysing beta functions

Motivation: The purpose of this exercise is to familiarise yourself with the study of beta functions: finding fixed points, computing critical exponents, and investigating the phase diagram. These are the bread-and-butter (or margarine/olive oil/whatever you fancy) skills when analysing renormalisation group flows.

In the following, we will consider two different sets of beta functions for two couplings G and Λ that depend on a reference scale k. They have mass dimensions -2 and 2, respectively, so that the dimensionless couplings are given by $g = Gk^2$ and $\lambda = \Lambda k^{-2}$.[a]

Consider first the following set of beta functions for the two couplings g, λ:

$$\begin{aligned} \beta_g &= k\partial_k g = 2g - \frac{135g^2}{72\pi - 5g}\,, \\ \beta_\lambda &= k\partial_k \lambda = -\left(2 + \frac{135g}{72\pi - 5g}\right)\lambda - g\left(\frac{43}{4\pi} - \frac{810}{72\pi - 5g}\right)\,. \end{aligned} \tag{15.I}$$

a) Compute all fixed points of this set of beta functions.

b) Compute the critical exponents at each of these fixed points. If there is an interacting fixed point, how many relevant parameters does it have? What does that mean for its predictivity?

c) We can actually solve the beta functions implicitly. Show in a first step that $g(k)$ implicitly defined via

$$G_N\, k^2 = \frac{g(k)}{\left(1 - \frac{145}{144\pi} g(k)\right)^{27/29}} \tag{15.II}$$

is a solution to the first beta function, where G_N is a constant. What is the meaning of G_N? *Hints:* Take a $k\partial_k$ derivative of this equation, solve for β_g, and check that you get back the original beta function. For the meaning of G_N, it might be helpful to expand to leading order in powers of k.

d) In a second step, show that $\lambda(k)$ defined by

$$\lambda(k) = \frac{162}{25} - \frac{43}{16\pi} g(k) + \ell\left(144\pi - 145g(k)\right)^{25/29} - \frac{144\pi}{3625g(k)}\left[87 + 25\ell\left(144\pi - 145g(k)\right)^{25/29}\right] \tag{15.III}$$

is a solution to the second beta function. Here, the constant ℓ can be written as

$$\ell = -\frac{29}{86400}\left(2^{-13}3^{-21}\pi^{-54}\right)^{1/29}\left(432\pi + 125G_N\,\Lambda_0\right)\,, \tag{15.IV}$$

with Λ_0 being yet another constant. What is the meaning of Λ_0? *Hint:* For the meaning of Λ_0, it might be helpful to expand to leading order in powers of k.

e) **[hard question]** Expand both solutions in powers of $1/k$ to leading non-trivial order. How does this relate to part b)? Could you have gotten this result more easily?

f) Instead of using λ, it can be useful to use the dimensionless combination $\tau = G\Lambda = g\lambda$. What is the interpretation of this coupling, and what are the potential IR ($k \to 0$) values that can be reached from the fixed point(s)?

g) Sketch the *phase diagram* in terms of g and τ (Mathematica's *StreamPlot* might come in handy, if you don't want to do it by hand). The phase diagram, or RG flow diagram, is a plot of the integral curves of the beta functions.

Your phase diagram should contain all fixed points and any potential other global features. If there are features other than fixed points, explain them.

Let us now briefly analyse a second set of beta functions:

$$\begin{aligned}\beta_g &= \left(2 - \frac{2}{3\pi}\frac{\lambda^2}{(1-2\lambda)^4}g\right)g\,,\\ \beta_\lambda &= -\left(2 + \frac{2}{3\pi}\frac{\lambda^2}{(1-2\lambda)^4}g\right)\lambda + \frac{g}{4\pi}\frac{1}{1-2\lambda}\left[1 + \frac{g}{3}\frac{2}{3\pi}\frac{\lambda^2}{(1-2\lambda)^4}\right]\,.\end{aligned} \tag{15.V}$$

h) Compute all real fixed points and their critical exponents. Is there anything unusual here? If so, try to explain what's going on.

i) **[hard question]** Sketch the phase diagram. What is the new feature in these beta functions? Which implications does this have for the admissible IR values of G and Λ? Can the relevant fixed point be used as a UV completion? Why/why not?

[a] Guess what G and Λ should represent.

a) These beta functions are taken from arXiv:1507.08859. The equations are partially decoupled, which means that we can first look for zeros of β_g. There are two, located at

$$g_*^{\text{GFP}} = 0\,, \qquad g_*^{\text{NGFP}} = \frac{144\pi}{145}\,. \tag{15.1}$$

Since β_λ is linear in λ, we can simply solve it and find

$$\lambda_*^{\text{GFP}} = 0\,, \qquad \lambda_*^{\text{NGFP}} = \frac{48}{145}\,. \tag{15.2}$$

We thus find a Gaussian fixed point (GFP, aka the free theory), and an interacting, or non-Gaussian fixed point (NGFP).

b) To compute the critical exponents, we need to first compute the stability matrix. For this, we need

$$
\begin{aligned}
\frac{\partial \beta_g}{\partial g} &= 2 - \frac{270g}{72\pi - 5g} - \frac{675g^2}{(72\pi - 5g)^2}\,,\\
\frac{\partial \beta_g}{\partial \lambda} &= 0\,,\\
\frac{\partial \beta_\lambda}{\partial g} &= -\left(\frac{135}{72\pi - 5g} + \frac{675g}{(72\pi - 5g)^2}\right)\lambda - \left(\frac{43}{4\pi} - \frac{810}{72\pi - 5g}\right) + \frac{4050g}{(72\pi - 5g)^2}\,,\\
\frac{\partial \beta_\lambda}{\partial \lambda} &= -\left(2 + \frac{135g}{72\pi - 5g}\right)\,.
\end{aligned}
\tag{15.3}
$$

We can arrange this into the stability matrix

$$
M_{\text{stab}} = \begin{pmatrix} \frac{\partial \beta_g}{\partial g} & \frac{\partial \beta_g}{\partial \lambda} \\ \frac{\partial \beta_\lambda}{\partial g} & \frac{\partial \beta_\lambda}{\partial \lambda} \end{pmatrix}\,, \tag{15.4}
$$

and compute its eigenvalues at the fixed points. At the GFP, it reads

$$
M_{\text{stab}}^{\text{GFP}} = \begin{pmatrix} 2 & 0 \\ \frac{1}{2\pi} & -2 \end{pmatrix}\,, \tag{15.5}
$$

thus the critical exponents are

$$
\theta_1^{\text{GFP}} = 2\,, \qquad \theta_2^{\text{GFP}} = -2\,. \tag{15.6}
$$

In turn, at the NGFP, we have

$$
M_{\text{stab}}^{\text{NGFP}} = \begin{pmatrix} -\frac{58}{27} & 0 \\ \frac{245}{162\pi} & -4 \end{pmatrix}\,, \tag{15.7}
$$

so the critical exponents are

$$
\theta_1^{\text{NGFP}} = 4\,, \qquad \theta_2^{\text{NGFP}} = \frac{58}{27}\,. \tag{15.8}
$$

Both are positive, that is we have two relevant directions. This means that we cannot predict one coupling from the other, rather both correspond to free parameters.

c) To show that the implicit equation for $g(k)$ solves the beta function, we first divide by k^2, then take a logarithmic k-derivative, and multiply again by k^2. Since G_N is a constant by assumption, we find

$$
0 = \frac{1}{(1 - \frac{145}{144\pi}g(k))^{27/29}}\left[kg'(k) - 2g(k) + \frac{15}{16\pi}\frac{g(k)}{1 - \frac{145}{144\pi}g(k)}\,kg'(k)\right]\,. \tag{15.9}
$$

Solving this for $kg'(k)$, we find

$$
kg'(k) \equiv \beta_g = 2g(k) - \frac{135g(k)^2}{72\pi - 5g(k)}\,, \tag{15.10}
$$

which is the original beta function, so indeed the implicit equation solves it.

Expanding (15.II) in powers of k, we notice that the left-hand side starts at k^2. To be consistent, the right-hand side has to start at k^2 as well. This directly entails that

$$
g(k) \sim G_N\,k^2 + \mathcal{O}(k^4)\,, \qquad k \to 0\,. \tag{15.11}
$$

Recalling the definition of the dimensionful coupling G, we find

$$
G_N = \lim_{k\to 0}\frac{g(k)}{k^2} = \lim_{k\to 0} G(k)\,, \tag{15.12}
$$

so that G_N is the IR value of $G(k)$.

d) This is a bit lengthy, but the strategy is the same as for c): take the logarithmic k-derivative of the solution and compare it with the beta function. After finitely many steps, you will find that this indeed solves the beta function (I trust your calculus skills).

Expanding the solution in powers of k (and using (15.II)), to leading order we find

$$\lambda(k) \sim -\frac{144\pi}{3625 G_N\, k^2}\left(87 + 600 \times 2^{13/29}\, 3^{21/29}\, \pi^{25/29}\, \ell\right) + \mathcal{O}(k^0)\,, \qquad k \to 0\,. \tag{15.13}$$

If we now plug in the expression for ℓ, we find

$$\lambda(k) \sim \frac{\Lambda_0}{k^2} + \mathcal{O}(k^0)\,, \qquad k \to 0\,. \tag{15.14}$$

We can once again compare with the dimensionful coupling, and find in complete analogy to c) that

$$\Lambda_0 = \lim_{k\to 0} k^2\, \lambda(k) = \lim_{k\to 0} \Lambda(k)\,. \tag{15.15}$$

So the meaning of Λ_0 is that it is the IR value of $\Lambda(k)$.

e) Let us first consider the equation for $g(k)$. When $k \to \infty$, the left-hand side diverges. How can we achieve the same on the right-hand side? One way would be to have $g(k) \to \infty$ since the right-hand side scales as $g(k)^{2/29}$ in this limit, however that is not too interesting. The other possibility is to have a zero of the denominator. Let us thus make the ansatz

$$g(k) \sim \frac{144\pi}{145}\left(1 - c_1\, k^{-c_2}\right)\,, \qquad k \to \infty\,, \tag{15.16}$$

where $c_{1,2}$ are constants and $c_2 > 0$. We notice that this is actually an expansion about the fixed point! If we insert this into the solution, we get

$$G_N\, k^2 \simeq \frac{\frac{144\pi}{145}\left(1 - c_1\, k^{-c_2}\right)}{\left(c_1 k^{-c_2}\right)^{27/29}} \simeq \frac{144\pi}{145}\frac{1}{c_1^{27/29}} k^{27 c_2/29}\,, \qquad k \to \infty\,. \tag{15.17}$$

Here we only kept the leading-order term. Matching both sides, we find

$$c_2 = \frac{58}{27}\,, \qquad c_1 = \left(\frac{144\pi}{145 G_N}\right)^{29/27}\,. \tag{15.18}$$

The value for c_2 is simply the critical exponent! This should not surprise you — after all, we expanded the solution about the interacting fixed point. This is just an explicit confirmation that we indeed get the right asymptotic behaviour.

Throwing this into the solution for $\lambda(k)$ and expanding, we find

$$\lambda(k) \sim \frac{48}{145} - \frac{588}{725} c_1 k^{-58/27}\,, \qquad k \to \infty\,. \tag{15.19}$$

Did you expect a k^{-4} now? Ha, gotcha! Let us explain what is going on here, by going back to the linear analysis around a fixed point.

Recall that to compute the stability matrix, we expanded the vector of beta functions around a fixed point:

$$\vec{\beta}(\vec{g}) \simeq \underbrace{\vec{\beta}(\vec{g}_*)}_{=0} + \left.\frac{\partial \vec{\beta}(\vec{g})}{\partial \vec{g}}\right|_{\vec{g}=\vec{g}_*} (\vec{g} - \vec{g}_*) + \dots\,. \tag{15.20}$$

The stability matrix is the gradient of the vector of beta functions with respect to all couplings evaluated at the fixed point,

$$M_{\text{stab}} = \left. \frac{\partial \vec{\beta}(\vec{g})}{\partial \vec{g}} \right|_{\vec{g}=\vec{g}_*} . \tag{15.21}$$

We can now solve the linearised beta functions, which amounts to a solution close to the fixed point. It is a set of linear differential equations, solved by

$$\vec{g} \simeq \vec{g}_* + \sum_i c_i \, \vec{e}_i \, k^{-2\theta_i} . \tag{15.22}$$

In this, the c_i are the integration constants, and $\vec{e}_i$ is the i-th eigenvector of M_{stab} with eigenvalue θ_i. Convince yourself that this is indeed the solution to the linearised equation.

Now let us come back to our system. From b), we know that the critical exponents are $\theta_1 = 4$ and $\theta_2 = 58/27$. Let us also compute the eigenvectors. For θ_1, the eigenvector is $(0,1)$, that is, it points purely into the λ-direction. By contrast, an eigenvector for θ_2 is $(60\pi, 49)$ (for simplicity, we don't normalise it – this is not necessary since we anyhow have the free parameters c_i). This means that, close to the NGFP, we can write the general solution as

$$\begin{aligned} g(k) &\simeq \frac{144\pi}{145} + c_1 \times 0 \, k^{-4} + c_2 \times 60\pi \, k^{-58/27} , \\ \lambda(k) &\simeq \frac{48}{145} + c_1 \times 1 \, k^{-4} + c_2 \times 49 \, k^{-58/27} . \end{aligned} \tag{15.23}$$

In this, we coloured the eigenvectors red. For large enough k, the k^{-4}-term is sub-leading compared to the $k^{-58/27}$-term – this explains the above asymptotic expansion, we simply didn't go far enough in the expansion. We can furthermore check that the ratio of the sub-leading terms in the two couplings came out right:

$$\frac{g(k) - g_*}{\lambda(k) - \lambda_*} \simeq \frac{-\frac{144\pi}{145} c_1}{-\frac{588}{725} c_1} = \frac{60\pi}{49} . \tag{15.24}$$

f) This coupling, in the context of gravity, would correspond to the value of the cosmological constant in Planck units. The potential IR values lie on the whole real line, since there are no restrictions on G_N and Λ_0 in the above solutions.

g) For this, let us first write down the beta function for τ. By definition,

$$\beta_\tau = k\partial_k (g\lambda) = -\frac{43}{4\pi} g^2 + \frac{810 g^2}{72\pi - 5g} - \frac{270 g \, \tau}{72\pi - 5g} . \tag{15.25}$$

In terms of τ and g, the phase diagram has the following features:

- There is a fixed *line* at $g = 0$, i.e., every point on the τ-axis is a fixed point. How does this fit to the idea that we only found two fixed points for g and λ? To understand this, look at the expansion of λ for small k: in almost all cases, it actually diverges like k^{-2}! For τ however, the flow reaches a finite value as $k \to 0$. This is what typically happens with positive mass dimension terms (think: mass terms) – if you want to have a finite physical mass when $k \to 0$, you need the dimensionless mass to diverge.
- Then there is the NGFP that we have found above. Actually, since our equations partially decouple, the line $g = g_*^{\text{NGFP}}$ splits the phase diagram into a weak-gravity regime $g < g_*$, where trajectories start at the NGFP and flow down towards a finite value of τ as $k \to 0$. On the line, only τ is flowing, not g. Above the line, we go away from a sensible IR limit.

– Furthermore, there is a divergent line at $g = 72\pi/5$. This is a pole in the beta functions, and marks where our approximation that was used to derive the beta functions breaks down. The trajectories that emanate upward from the NGFP end in this divergent line at a finite scale.

g) These beta functions are taken from arXiv:1205.4218. They have multiple complex fixed points, but once again only two real ones. As always, there is the GFP, with the same critical exponents as before. The NGFP is now at

$$g_* = \frac{24\pi}{49}\left(6 + \left(\frac{2}{5}\right)^{1/3} - 8\left(\frac{2}{5}\right)^{2/3}\right) \approx 3.683\,, \qquad \lambda_* = \frac{1}{7}\left(\frac{5}{2} + \left(\frac{5}{2}\right)^{1/3} - \left(\frac{5}{2}\right)^{2/3}\right) \approx 0.288\,. \tag{15.26}$$

The critical exponents are

$$\theta_{1,2} = \frac{17}{5} - 3\left(\frac{2}{5}\right)^{2/3} \pm \frac{\mathbf{i}}{5}\sqrt{311 - 90\left(\frac{2}{5}\right)^{1/3} + 2010\left(\frac{2}{5}\right)^{2/3}} \approx 1.77 \pm 7.31\,\mathbf{i}\,. \tag{15.27}$$

Huh, we get a complex-conjugate pair of critical exponents! What does that mean? For this, let us recall that the critical exponents determine the power law running of couplings close to the fixed point. An imaginary part in the exponent then means an oscillation – and since the real part of the critical exponents is positive, it is a damped oscillation. The two couplings thus approach the fixed point in a spiral. To show that, we have to realise that the couplings have to stay real, and thus the two integration constants must be complex-conjugate as well (since also the eigenvectors are complex-conjugates). In particular, for our system this means around the NGFP

$$\begin{aligned} g(k) &\simeq g_* + c\,z\,k^{-\theta_1} + \bar{c}\,\bar{z}\,k^{-\theta_2}\,, \\ \lambda(k) &\simeq \lambda_* + c\,k^{-\theta_1} + \bar{c}\,k^{-\theta_2}\,, \end{aligned} \tag{15.28}$$

where we normalised the eigenvectors to have unit strength in λ-direction, c is the *complex* integration constant, and z is the g-component of the eigenvector. Since we have a complex integration constant, this is equivalent to two real integration constants, as expected from the real parts of the critical exponents.

h) The new feature in this phase diagram is that there is a *limit cycle* around the NGFP which is IR-attractive. The limit cycle is a closed curve in the phase diagram. This means that trajectories emanating from the NGFP will approach this limit cycle, so that the dimensionless couplings will be (approximately) cyclic; in particular, they stay finite for all k. This immediately means that the dimensionful Newton's coupling G diverges for $k \to 0$, so we do *not* get a sensible IR limit. This means that such a fixed point is unsuitable to serve as a UV completion.

Exercise 16: *[Presence]* Analysing beta functions, part 2

Motivation: Let us study yet another set of beta functions.

Consider two coupling G, G_{C^3} with dimensionless couplings $g = Gk^2$ and $g_{C^3} = G_{C^3}k^4$. For these, look at the following set of beta functions:

$$\begin{aligned} \beta_g &= 2g - \frac{8}{3}g^2\,, \\ \beta_{g_{C^3}} &= 4g_{C^3} + \frac{g^2}{1890}\,. \end{aligned} \tag{16.I}$$

Find fixed points and critical exponents. Solve the differential equations to get $g(k)$ and $g_{C^3}(k)$ – you should get two free parameters for the general solution. Fix one of them by requiring $g(k) \sim G_N k^2$ as $k \to 0$. Can you fix the other constant by requiring that you hit the fixed point as $k \to \infty$?
Investigate the IR limit $(k \to 0)$ of $g_{C^3}(k)$ – is there anything unusual here? If so, explain!

The fixed points can simply be read off. We will skip the GFP and go right to the NGFP:

$$g_* = \frac{3}{4}, \qquad g_{C^3 *} = -\frac{1}{13440}. \tag{16.1}$$

The critical exponents are

$$\theta_1 = 2, \qquad \theta_2 = -4. \tag{16.2}$$

So we finally have an irrelevant direction, and we can fix a coupling!

We will look at this analytically.[1] The solution for g reads

$$g(k) = \frac{k^2}{c_1 + \frac{4}{3}k^2}. \tag{16.3}$$

With this, we can solve the second equation, and get

$$g_{C^3}(k) = c_2\, k^4 + \frac{k^4}{3780 c_1^2}\left[\frac{c_1}{c_1 + \frac{4}{3}k^2} + \ln\frac{k^2}{3c_1 + 4k^2}\right]. \tag{16.4}$$

Let us fix the constants. For small k, we want

$$g(k) = G_N\, k^2 = \frac{k^2}{c_1}, \qquad \Rightarrow \qquad c_1 = 1/G_N. \tag{16.5}$$

With this, we can write

$$g(k) = \frac{G_N k^2}{1 + \frac{4}{3}G_N k^2}. \tag{16.6}$$

This has the nice interpretation that we measure k^2 in units of G_N (i.e., Planck units if G_N is Newton's constant). Inserting this into the solution for g_{C^3}, we have

$$g_{C^3}(k) = c_2 k^4 + \frac{(G_N k^2)^2}{3780}\left[\frac{1}{1 + \frac{4}{3}G_N k^2} + \ln\left(\frac{1}{3}\frac{G_N k^2}{1 + \frac{4}{3}G_N k^2}\right)\right]. \tag{16.7}$$

For large k, g_{C^3} behaves as

$$g_{C^3}(k) \sim \left(c_2 - \frac{\ln 2}{1890}G_N^2\right) k^4 - \frac{1}{13440} + \mathcal{O}(k^{-2}), \qquad k \to \infty. \tag{16.8}$$

Aha! Here we see that if we want to hit the fixed point when $k \to \infty$, we *have* to choose

$$c_2 = \frac{\ln 4}{3780} G_N^2. \tag{16.9}$$

[1] This is not the standard way to do this – we can do it here because the system is so simple that it admits analytic solutions. Usually, you have to 1) find the fixed point, 2) compute the critical exponents and eigenvectors, 3) set to zero the integration constants attached to irrelevant directions, 4) pick values for the remaining free parameters to fix initial conditions to solve the RG flow numerically.

This is precisely how a coupling is predicted from an irrelevant direction: for any other choice of c_2, the trajectory will diverge away from the fixed point as $k \to \infty$. Only for the above choice, we are on the safe trajectory. This trajectory is given by

$$g_{C^3}(k) = \frac{(G_N k^2)^2}{3780} \left[\frac{1}{1 + \frac{4}{3} G_N k^2} + \ln \left(\frac{4}{3} \frac{G_N k^2}{1 + \frac{4}{3} G_N k^2} \right) \right] . \tag{16.10}$$

We can even express this as

$$g_{C^3}(g) = \frac{g^2}{3780} \frac{1}{1 - \frac{4}{3} g} \left[1 + \frac{1}{1 - \frac{4}{3} g} \ln \frac{4g}{3} \right] . \tag{16.11}$$

It doesn't get more explicit than this, that we can express one coupling in terms of the other. In this case, it works *at all* k, which is unusual.

Now, we can just take this and read off the IR value ($k \to 0$) of G_{C^3}, right? Right??? Let's expand:

$$g_{C^3}(k) \sim \frac{(G_N k^2)^2}{3780} \left[1 + \ln \left(\frac{4}{3} G_N k^2 \right) \right] , \qquad k \to \infty , \tag{16.12}$$

sooooo

$$G_{C^3} = \lim_{k \to 0} \frac{g_{C^3}(k)}{k^4} = \lim_{k \to 0} \frac{(G_N)^2}{3780} \left[1 + \ln \left(\frac{4}{3} G_N k^2 \right) \right] ? \tag{16.13}$$

Welcome to the world of IR divergences originating from ultrasoft graviton radiation. So what in the everpresent Lambda is going on? The short story is that massless particles SU... are more difficult to handle than massive particles. The problem is that in any scattering process, one can radiate a large number of infinitely soft massless particles "for free", that is, without impacting momentum conservation. This leads to IR divergences that pop up everywhere, and it is a complete mess. However, as soon as you compute an *actual observable*, that is a gauge-invariant expression, this stuff should drop out, leaving behind a finite expression. At least that's the theory. If you want to read a bit more about this, then the Weinberg soft graviton theorem is a good starting point.