Quantum Gravity and the Renormalization Group

Assignment 9 – Dec 15

Exercise 18: Computing an RG flow

Motivation: Stuff is getting real now – we will derive the beta function for the potential that you saw in Exercise 17 to learn how the Wetterich equation works in practice. We will also introduce some general concepts that will come in handy when going back to gravity.

In this exercise, we will learn how to evaluate the right-hand side of the Wetterich equation:

$$k\partial_k\Gamma_k = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)^{-1} \cdot k\partial_k\mathfrak{R}_k\right]. \tag{18.I}$$

Here, we used the "supertrace" (STr) to indicate a generalisation of the loop momentum integral that we will explain soon. This will be important later when we look at gravity.
Our system will be the $O(N)$-symmetric vector model in three dimensions. Our ansatz for the effective average action Γ_k is the same as in Exercise 17:

$$\Gamma_k \simeq \int \mathrm{d}^3x \left[\frac{1}{2}(\partial_\mu\Phi_a)(\partial^\mu\Phi^a) + V_k(\rho)\right]. \tag{18.II}$$

Recall that $\rho = \Phi_a\Phi^a/2$. The goal is to plug this ansatz into (18.I) and find the expression for $k\partial_k V_k$.

a) Let us start with the easiest bit – the left-hand side. Evaluate the left-hand side of (18.I) for the ansatz (18.II).

Next, let us construct the right-hand side step by step. We will be very careful in the notation at every step – only once you have understood what is going on, you can become sloppy (like everybody working on it)!

b) The first ingredient that we need is the second variation of the effective average action. Compute

$$\frac{\delta^2\Gamma_k}{\delta\Phi^a(x)\,\delta\Phi^b(y)}. \tag{18.III}$$

For this, recall that by definition,

$$\frac{\delta\Phi^a(x)}{\delta\Phi^b(y)} = \delta_b{}^a\,\delta(x-y). \tag{18.IV}$$

Here, the delta function is three-dimensional.

Next, we have to choose some form of regularisation. Recall that in the path integral, we added a term to the action in the form of

$$\Delta S_k = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\tilde{\Phi}^a(-p)\,\mathfrak{R}_{k,ab}(p^2)\,\tilde{\Phi}^b(p). \tag{18.V}$$

This is in momentum space. Often, we actually work in position space, and then this reads

$$\Delta S_k = \frac{1}{2} \int \mathrm{d}^3x \, \Phi^a \, \mathfrak{R}_{k,ab}(-\partial^2) \, \Phi^b \, . \tag{18.VI}$$

Note how the regulator $\mathfrak{R}_k$ actually needs to carry ($O(N)$) indices, since the field also carries indices. In our case, we can choose the regulator diagonal in field space,

$$\mathfrak{R}_{k,ab}(-\partial^2) = \delta_{ab} \, \mathcal{R}_k(-\partial^2) \, . \tag{18.VII}$$

Now, $\mathcal{R}_k$ is a simple function (that happens to have an operator as its argument). Note the different type setting to make the difference (the literature often does not differentiate this).

c) Compute how the above regulator gets added to the two-point function $\Gamma_k^{(2)}$ by taking the second variation of ΔS_k in position space.

d) In the same language, what do we actually mean by the $k\partial_k\mathfrak{R}_k$ in (18.I)?

The easy part is over now – the next step is to invert the regularised two-point function. First, we discuss some conceptual aspects, then the actual computation.

e) The two-point function found in b) depends on two points x and y. You should have found however that it is local, that is, it can be written as

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)(x, y) = \mathcal{O}(x) \, \delta(x - y) \, . \tag{18.VIII}$$

Here, $\mathcal{O}(x)$ is some differential operator. The "local" refers to the delta function. Now, its inverse – the propagator $G_k(x, y)$ – is defined by

$$\int \mathrm{d}^3y \, G_k(x, y) \, \left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)(y, z) = \delta(x - z) \, . \tag{18.IX}$$

From this definition, argue that the propagator must also be local in the above sense, and in particular formally of the structure

$$G_k(x, y) = [\mathcal{O}(x)]^{-1} \, \delta(x - y) \, . \tag{18.X}$$

Our next task is thus to find $\mathcal{O}^{-1}$. For this, we need an *expansion scheme*, since we will not be able to do this without approximations. To decide on what we really need, consider again a). We really only need terms that contribute to the potential – terms with derivatives acting on fields contribute to other couplings. An expansion in derivatives like this is called . . . wait for it . . . *derivative expansion*. In going forward, we will thus assume that we can neglect $\partial_\mu \Phi^a \simeq 0$.

f) In the lowest order derivative expansion where we set $\partial_\mu \Phi^a \simeq 0$ (that is, Φ^a is approximately constant, $\Phi^a(x) \simeq \Phi^a$), compute the regularised propagator. *Hints:* Write the regularised two-point function in the form

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)_{ab}(x, y) = \left\{ A(-\partial^2, \Phi^2) \left(\delta_{ab} - \frac{\Phi_a \Phi_b}{\Phi^2}\right) + B(-\partial^2, \Phi^2) \frac{\Phi_a \Phi_b}{\Phi^2} \right\} \delta(x - y) \, , \tag{18.XI}$$

and determine the functions A, B. Do the two operators multiplying A and B look somewhat familiar? (Try to take their sum and products!) Conclude that you can also write the propagator as

$$G^{ab}(x,y) = \left\{ \left(\delta^{ab} - \frac{\Phi^a \Phi^b}{\Phi^2} \right) C(-\partial^2, \Phi^2) + \frac{\Phi^a \Phi^b}{\Phi^2} D(-\partial^2, \Phi^2) \right\} \delta(x-y) \,, \qquad \text{(18.XII)}$$

and compute C and D from A and B.

The last step is to make sense out of the STr. First of all, you might have noticed the $\cdot$ that I slipped into (18.I). Since both the propagator and the (k-derivative of the) regulator depend on two points, what we once again mean by this is

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k \right)^{-1} \cdot k\partial_k \mathfrak{R}_k \equiv \int \mathrm{d}^3 y \, \left(\Gamma_k^{(2)} + \mathfrak{R}_k \right)^{-1} (x,y) \; k\partial_k \mathfrak{R}_k(y,z) \,. \qquad \text{(18.XIII)}$$

g) Reinstate the $O(N)$ indices and compute (18.XIII) by using the results of the previous parts.

Last but not least, we define the supertrace via the integral over the coincidence limit, plus taking any "standard" index trace (tr):

$$\mathrm{STr}\, F(x,y) = \int \mathrm{d}^3 x \, \mathrm{tr} \lim_{y \to x} F(x,y) \,. \qquad \text{(18.XIV)}$$

This of course is more complicated in gravity, where all hell breaks loose, but this is a problem for future-you, not present-you.

h) Use the definition of the supertrace to finally evaluate the right-hand side of the flow equation! *Hints:* It could be helpful to represent the delta function in terms of its Fourier representation,

$$\delta(x-y) = \int \frac{\mathrm{d}^3 \ell}{(2\pi)^3} \, e^{-i\,\ell \cdot (x-y)} \,. \qquad \text{(18.XV)}$$

Use this to convert any $-\partial^2$ into momenta, and take the continuum limit. Finally, the tr is the trace (aka contraction) over the two remaining $O(N)$ indices. Recall for this that $\delta_a{}^a = N$.

Don't perform the momentum integrals yet (you can't anyway without specifying the shape of the regulator). Rather, your result should look like

$$k\partial_k V_k(\rho) = (N-1) I_1(\rho) + I_2(\rho) \,, \qquad \text{(18.XVI)}$$

where $I_{1,2}$ are two different integrals.

i) To evaluate the integrals, use the Litim regulator,

$$\mathcal{R}_k(z) = (k^2 - z)\, \theta \left(1 - \frac{z}{k^2} \right) \,, \qquad \text{(18.XVII)}$$

where θ is the Heaviside function.

j) As the very last step, let us reproduce (17.IV). For this, we rescale the field and the potential via

$$\Phi^a \mapsto c\sqrt{N}\,\hat{\Phi}^a\,, \qquad V(\rho) \mapsto c^2\,N\,\hat{V}(\hat{\rho})\,, \tag{18.XVIII}$$

for a suitable constant c. Then take a ρ-derivative of the flow equation (since (17.IV) is the flow for the derivative of the potential) and take the limit $N \to \infty$. What is the right value for c to achieve a match?

k) **[hard question]** Think about systematic extensions of our approximation. You can take the following questions as an orientation:

– Where does the statement come from that the Wetterich equation generates all terms compatible with symmetries?

– Related to this, how can we systematically extend our approximation? For example, how would we take into account terms that include factors of $\partial_\mu \Phi^a$, and which beta functions would we be able to compute by keeping these terms?

– Do you think everything is going through in the same way with gravity? If not, what could go wrong?

a) Since the field itself is independent of k, we directly get

$$k\partial_k \Gamma_k \simeq \int \mathrm{d}^3x\, k\partial_k V_k(\rho)\,. \tag{18.1}$$

b) Let us do this in two steps. Taking the first derivative (and being mindful about indices and arguments), we have

$$\begin{aligned}
\frac{\delta \Gamma_k}{\delta \Phi^a(x)} &= \int \mathrm{d}^3z\, \frac{\delta}{\delta \Phi^a(x)} \left[\frac{1}{2}(\partial_\mu \Phi_c(z))(\partial^\mu \Phi^c(z)) + V_k(\rho(z))\right] \\
&= \int \mathrm{d}^3z \left[\frac{1}{2}\left(\partial_\mu \frac{\delta \Phi_c(z)}{\delta \Phi^a(x)}\right)(\partial^\mu \Phi^c(z)) + \frac{1}{2}(\partial_\mu \Phi_c(z))\left(\partial^\mu \frac{\delta \Phi^c(z)}{\delta \Phi^a(x)}\right) + \frac{\delta V_k(\rho(z))}{\delta \Phi^a(x)}\right] \\
&= \int \mathrm{d}^3z \left[\frac{1}{2}\left(\partial_\mu \delta_{ca}\delta(z-x)\right)(\partial^\mu \Phi^c(z)) + \frac{1}{2}(\partial_\mu \Phi_c(z))\left(\partial^\mu \delta_a{}^c \delta(z-x)\right) + V_k'(\rho(z))\frac{\delta \rho(z)}{\delta \Phi^a(x)}\right] \\
&= \int \mathrm{d}^3z \left[\frac{1}{2}\left(\partial_\mu \delta_{ca}\delta(z-x)\right)(\partial^\mu \Phi^c(z)) + \frac{1}{2}(\partial_\mu \Phi_c(z))\left(\partial^\mu \delta_a{}^c \delta(z-x)\right) + V_k'(\rho(z))\frac{\delta \frac{\Phi^c(z)\Phi_c(z)}{2}}{\delta \Phi^a(x)}\right] \\
&= \int \mathrm{d}^3z \left[\frac{1}{2}\left(\partial_\mu \delta_{ca}\delta(z-x)\right)(\partial^\mu \Phi^c(z)) + \frac{1}{2}(\partial_\mu \Phi_c(z))\left(\partial^\mu \delta_a{}^c \delta(z-x)\right) + V_k'(\rho(z))\,\Phi^c(z)\delta_{ac}\delta(z-x)\right] \\
&= \int \mathrm{d}^3z \left[-(\partial^2 \Phi_a(z)) + V_k'(\rho(z))\,\Phi_a(z)\right]\delta(z-x) \\
&= -(\partial^2 \Phi_a(x)) + V_k'(\rho(x))\,\Phi_a(x)\,.
\end{aligned} \tag{18.2}$$

Here, we used the definition of the functional derivative, and used partial integration to remove derivatives acting on delta functions. All derivatives **before integration** act on z, whereas **after integration** they act on x.

We can now take a second derivative and get

$$\begin{aligned}
\frac{\delta^2\Gamma_k}{\delta\Phi^a(x)\delta\Phi^b(y)} &= \frac{\delta}{\delta\Phi^b(y)}\left[-(\partial^2\Phi_a(x)) + V_k'(\rho(x))\,\Phi_a(x)\right] \\
&= -\partial^2\frac{\delta\Phi_a(x)}{\delta\Phi^b(y)} + \frac{\delta\Phi_a(x)}{\delta\Phi^b(y)}V_k'(\rho(x)) + \Phi_a(x)\frac{\delta V_k'(\rho(x))}{\delta\Phi^b(y)} \\
&= -\partial^2\delta_{ab}\delta(x-y) + V_k'(\rho(x))\,\delta_{ab}\delta(x-y) + V_k''(\rho(x))\Phi_a(x)\Phi_b(x)\delta(x-y) \\
&= \left[\left(-\partial^2 + V_k'(\rho(x))\right)\delta_{ab} + V_k''(\rho(x))\Phi_a(x)\Phi_b(x)\right]\,\delta(x-y)\,.
\end{aligned} \tag{18.3}$$

Once again, all derivatives act on x.

c) Repeating the same steps as above, we have the addition

$$\mathcal{R}_k(-\partial^2)\delta_{ab}\delta(x-y)\,. \tag{18.4}$$

d) What is meant there really is

$$k\partial_k\mathfrak{R}_{k,ab}\,. \tag{18.5}$$

e) Using the definition of $\mathcal{O}$ and of the propagator, we find

$$\int \mathrm{d}^3y\, G_k(x,y)\,\mathcal{O}(y)\delta(y-z) = \int \mathrm{d}^3y\,\left[\mathcal{O}^\dagger(y)G_k(x,y)\right]\,\delta(y-z) = \mathcal{O}^\dagger(z)G_k(x,z)\,. \tag{18.6}$$

To complete the proof, we note that $\mathcal{O}^\dagger$ is simply $\mathcal{O}$ (this is because $\mathcal{O}$ comes out of a second variation, which makes the operator hermitian), and that $\mathcal{O}$ has only even powers of derivatives so that acting on z is the same as acting on x.

f) We have found that

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)_{ab}(x,y) = \left[\left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho(x))\right)\delta_{ab} + V_k''(\rho(x))\Phi_a(x)\Phi_b(x)\right]\delta(x-y)\,. \tag{18.7}$$

We will now assume that $\Phi^a(x) \simeq \Phi^a$ is constant. Using the hint, we can split the regularised two-point function into

$$\begin{aligned}
\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)_{ab}(x,y) &= \Bigg\{\left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho)\right)\left(\delta_{ab} - \frac{\Phi_a\Phi_b}{\Phi^2}\right) \\
&\qquad + \left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho) + \Phi^2V_k''(\rho)\right)\frac{\Phi_a\Phi_b}{\Phi^2}\Bigg\}\delta(x-y) \\
&= \Bigg\{\left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho)\right)\left(\delta_{ab} - \frac{\Phi_a\Phi_b}{\Phi^2}\right) \\
&\qquad + \left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho) + 2\rho V_k''(\rho)\right)\frac{\Phi_a\Phi_b}{\Phi^2}\Bigg\}\delta(x-y)\,.
\end{aligned} \tag{18.8}$$

We can thus read off

$$A(-\partial^2,\rho) = -\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho)\,, \tag{18.9}$$

$$B(-\partial^2,\rho) = -\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho) + 2\rho V_k''(\rho)\,. \tag{18.10}$$

The two operators are actually a complete set of projectors in the space of constant Φ^a:

$$\begin{aligned} \left(\delta_a^{\ b} - \frac{\Phi_a \Phi^b}{\Phi^2}\right)\left(\delta_b^{\ c} - \frac{\Phi_b \Phi^c}{\Phi^2}\right) &= \left(\delta_a^{\ c} - \frac{\Phi_a \Phi^c}{\Phi^2}\right), \\ \left(\delta_a^{\ b} - \frac{\Phi_a \Phi^b}{\Phi^2}\right)\left(\frac{\Phi_b \Phi^c}{\Phi^2}\right) &= 0 \\ \left(\frac{\Phi_a \Phi^b}{\Phi^2}\right)\left(\frac{\Phi_b \Phi^c}{\Phi^2}\right) &= \frac{\Phi_a \Phi^c}{\Phi^2}, \\ \left(\delta_a^{\ b} - \frac{\Phi_a \Phi^b}{\Phi^2}\right) + \left(\frac{\Phi_a \Phi^b}{\Phi^2}\right) &= \delta_a^{\ b}. \end{aligned} \tag{18.11}$$

They are a little bit like transverse and longitudinal projectors, but in field space instead of in momentum space. Since they form a complete basis, we can indeed span the propagator with them, and the functions C and D are simply the inverses of the functions A and B, respectively,

$$C(-\partial^2, \rho) = \left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho)\right)^{-1}, \tag{18.12}$$

$$D(-\partial^2, \rho) = \left(-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho) + 2\rho V_k''(\rho)\right)^{-1}. \tag{18.13}$$

Don't worry too much about the definition of the inverse, we will make this more concrete in a moment. Note however already how the choice of a constant field helps here, since $[\partial_\mu, \Phi^a] \simeq 0$.

g) Reinstating the indices, we actually have

$$\left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)^{-1} \cdot k\partial_k \mathfrak{R}_k = \int \mathrm{d}^3 y \left(\Gamma_k^{(2)} + \mathfrak{R}_k\right)^{-1\,ac}(x, y)\; k\partial_k \mathfrak{R}_{k,cb}(y, z). \tag{18.14}$$

Since our regulator is proportional to the identity in field space, and multiplies a delta function, we find

$$\left\{ \left(\delta_b^{\ a} - \frac{\Phi^a \Phi_b}{\Phi^2}\right) C(-\partial^2, \rho)\, k\partial_k \mathcal{R}_k(-\partial^2) + \frac{\Phi^a \Phi_b}{\Phi^2} D(-\partial^2, \rho)\, k\partial_k \mathcal{R}_k(-\partial^2) \right\} \delta(x - y). \tag{18.15}$$

h) Let us first insert the Fourier representation for the delta function:

$$\begin{aligned} &\left\{ \left(\delta_b^{\ a} - \frac{\Phi^a \Phi_b}{\Phi^2}\right) C(-\partial^2, \rho)\, k\partial_k \mathcal{R}_k(-\partial^2) + \frac{\Phi^a \Phi_b}{\Phi^2} D(-\partial^2, \rho)\, k\partial_k \mathcal{R}_k(-\partial^2) \right\} \delta(x - y) \\ &= \int \frac{\mathrm{d}^3 \ell}{(2\pi)^3} \left\{ \left(\delta_b^{\ a} - \frac{\Phi^a \Phi_b}{\Phi^2}\right) C(-\partial^2, \rho)\, k\partial_k \mathcal{R}_k(-\partial^2) + \frac{\Phi^a \Phi_b}{\Phi^2} D(-\partial^2, \rho)\, k\partial_k \mathcal{R}_k(-\partial^2) \right\} e^{-i\ell\cdot(x-y)} \\ &= \int \frac{\mathrm{d}^3 \ell}{(2\pi)^3} \left\{ \left(\delta_b^{\ a} - \frac{\Phi^a \Phi_b}{\Phi^2}\right) C(\ell^2, \rho)\, k\partial_k \mathcal{R}_k(\ell^2) + \frac{\Phi^a \Phi_b}{\Phi^2} D(\ell^2, \rho)\, k\partial_k \mathcal{R}_k(\ell^2) \right\} e^{-i\ell\cdot(x-y)}. \end{aligned} \tag{18.16}$$

Here we used that the operator $-\partial^2$ acting on the exponential gives the squared momentum ℓ^2. The next step is to take the coincidence limit, which simply removes the exponential. The tr is the contraction of the two $O(N)$ indices, so that we get

$$\int \frac{\mathrm{d}^3 \ell}{(2\pi)^3} \left\{ (N - 1)\, C(\ell^2, \rho)\, k\partial_k \mathcal{R}_k(\ell^2) + D(\ell^2, \rho)\, k\partial_k \mathcal{R}_k(\ell^2) \right\}. \tag{18.17}$$

The final step is to integrate over x. This gives the volume of the spacetime, and the same factor appears on the left-hand side.[2] We can thus finally read off the beta function for the potential and insert our expressions for C and D:

$$k\partial_k V_k(\rho) = \frac{1}{2}\left[(N-1)\int \frac{\mathrm{d}^3\ell}{(2\pi)^3}\frac{k\partial_k\mathcal{R}_k(\ell^2)}{\ell^2+\mathcal{R}_k(\ell^2)+V_k'(\rho)} + \int \frac{\mathrm{d}^3\ell}{(2\pi)^3}\frac{k\partial_k\mathcal{R}_k(\ell^2)}{\ell^2+\mathcal{R}_k(\ell^2)+V_k'(\rho)+2\rho V_k''(\rho)}\right]. \tag{18.18}$$

The factor of 1/2 is from the Wetterich equation itself.

i) Let us evaluate the threshold integrals. For this, we need to compute

$$k\partial_k\mathcal{R}_k(\ell^2) = 2k^2\theta\left(1-\frac{\ell^2}{k^2}\right) + 2\frac{\ell^2}{k^2}(k^2-\ell^2)\delta\left(1-\frac{\ell^2}{k^2}\right). \tag{18.19}$$

Note that for our case, the second term will integrate to zero. This is because the delta function will always multiply its argument in the integrals, so that the contribution vanishes. Also note that the Heaviside function will have no non-trivial effect in the denominator: since it also appears in the numerator, the integration range is restricted to $\ell^2 \leq k^2$. In that range, the Heaviside theta is always one. This entails that the denominators are simply made ℓ-independent!

To evaluate the integrals, we also use spherical coordinates. With this, we have

$$\begin{aligned}&\int \frac{\mathrm{d}^3\ell}{(2\pi)^3}\frac{k\partial_k\mathcal{R}_k(\ell^2)}{\ell^2+\mathcal{R}_k(\ell^2)+V_k'(\rho)}\\ =&\frac{1}{(2\pi)^3}\int_{-k}^{k}\mathrm{d}\ell\,\ell^2\int_0^\pi \mathrm{d}\theta\,\sin\theta\int_0^{2\pi}\mathrm{d}\varphi\,\frac{2k^2}{k^2+V_k'(\rho)}\\ =&\frac{1}{3\pi^2}\frac{k^5}{k^2+V_k'(\rho)},\end{aligned} \tag{18.20}$$

and

$$\begin{aligned}&\int \frac{\mathrm{d}^3\ell}{(2\pi)^3}\frac{k\partial_k\mathcal{R}_k(\ell^2)}{\ell^2+\mathcal{R}_k(\ell^2)+V_k'(\rho)+2\rho V_k''(\rho)}\\ =&\frac{1}{(2\pi)^3}\int_{-k}^{k}\mathrm{d}\ell\,\ell^2\int_0^\pi \mathrm{d}\theta\,\sin\theta\int_0^{2\pi}\mathrm{d}\varphi\,\frac{2k^2}{k^2+V_k'(\rho)+2\rho V_k''(\rho)}\\ =&\frac{1}{3\pi^2}\frac{k^5}{k^2+V_k'(\rho)+2\rho V_k''(\rho)}.\end{aligned} \tag{18.21}$$

This culminates in

$$k\partial_k V_k(\rho) = \frac{N-1}{6\pi^2}\frac{k^5}{k^2+V_k'(\rho)} + \frac{1}{6\pi^2}\frac{k^5}{k^2+V_k'(\rho)+2\rho V_k''(\rho)}. \tag{18.22}$$

j) Note that under the rescaling, the denominators stay invariant:

$$V_k'(\rho)\mapsto \hat{V}_k'(\hat{\rho}), \qquad \rho V_k''(\rho)\mapsto \hat{\rho}\hat{V}_k''(\hat{\rho}). \tag{18.23}$$

Thus only the left-hand side changes under the rescaling, and we get

$$c^2\,N\,k\partial_k\hat{V}_k(\hat{\rho}) = \frac{N-1}{6\pi^2}\frac{k^5}{k^2+\hat{V}_k'(\hat{\rho})} + \frac{1}{6\pi^2}\frac{k^5}{k^2+\hat{V}_k'(\hat{\rho})+2\hat{\rho}\hat{V}_k''(\hat{\rho})}. \tag{18.24}$$

[2] Formally, this is infinite for constant fields and on standard flat space. However, since we can easily identify the source of this infinity, there is no problem here.

Taking the leading term as $N \to \infty$, we find

$$k\partial_k \hat{V}_k(\hat{\rho}) \simeq \frac{1}{6\pi^2 c^2} \frac{k^5}{k^2 + \hat{V}_k'(\hat{\rho})} \,. \tag{18.25}$$

To get to the equation that we discussed in the previous exercise, we have to choose

$$c^2 = \frac{1}{3\pi^2} \,. \tag{18.26}$$

To complete the computation, we have to take a $\hat{\rho}$-derivative (since we discussed the flow equation of the derivative of the potential),

$$k\partial_k \hat{V}_k'(\hat{\rho}) \simeq -\frac{1}{2} \frac{k^5 \, \hat{V}_k''(\hat{\rho})}{\left(k^2 + \hat{V}_k'(\hat{\rho})\right)^2} \,, \tag{18.27}$$

and change to dimensionless quantities. Note also how all factors of k come in such a way that all of them drop out when going to dimensionless quantities – as it must be.

k) To answer the questions:

- We can see from the flow equation of the potential that even if we would start with, e.g., just a ϕ^4-term, all other powers of the field are generated. The same is true if we go beyond the approximation of constant fields.
- A systematic extension is possible e.g. in the form of the derivative expansion. At the next order, the effective average action needs to include extra terms, namely

$$\Gamma_k \simeq \int \mathrm{d}^3 x \left[V_k(\rho) + \frac{1}{2} Z_k(\rho) \, (\partial_\mu \Phi_a)(\partial^\mu \Phi^a) + Y_k(\rho)(\partial_\mu \rho)(\partial^\mu \rho) + \dots \right] . \tag{18.28}$$

  The extra terms clearly add new contributions to $\Gamma_k^{(2)}$. Also in the inversion to compute the propagator, we cannot anymore assume a constant field. We can rather expand in derivatives of the field. Nevertheless, a few pitfalls appear. The first thing is that the field projection operators now do not commute anymore with derivatives, so the ordering of terms matters. Second, in the inversion, we have to be careful, since once again derivatives and fields do not commute. One way to perform an inversion of an expression with both derivatives and the field would be to add and subtract a constant term. Concretely, let us assume that we want to compute $C(-\partial^2, \rho)$ as previously, but without assuming $\partial_\mu \Phi^a \simeq 0$. We can do this by writing

$$-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho) = \left[-\partial^2 + \mathcal{R}_k(-\partial^2) + V_k'(\rho_0)\right] + \left[V_k'(\rho) - V_k'(\rho_0)\right] , \tag{18.29}$$

  where ρ_0 is an *arbitrary* constant value. For the inversion of the above, we can then use the geometric series to the desired order, taking the second bracket as the term expanded in.
- Of course, a lot of things are more complicated with gravity. Some include: we have to define the supertrace (next sheets), the diffeomorphism symmetry is broken by the regulator and gauge-fixing, we have to treat three different fields (graviton fluctuations, ghost and anti-ghost), and many more.