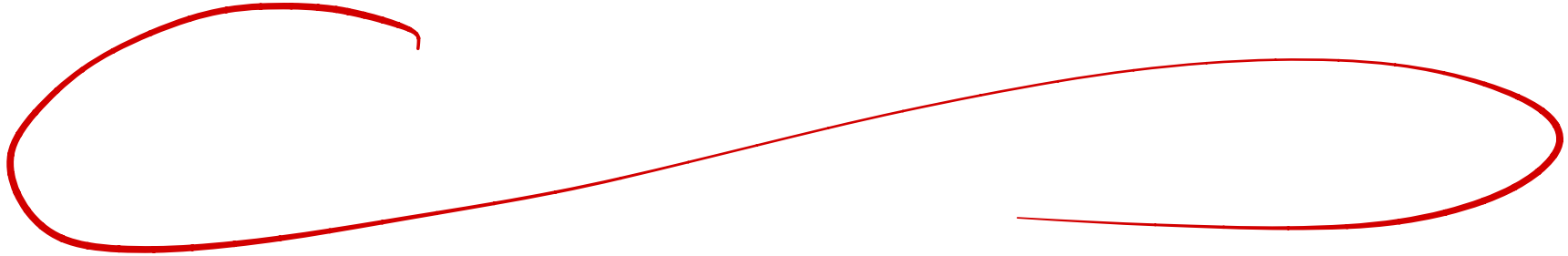
Quantum gravity and the renormalisation group



45 min + 15 min break + 45 min

lectures: Fr 11:15 – 13:00

Phil 19 seminar room (8080 EG 010)

tutorials: Mo 16:15 – 18:00

Phil 12 nHS (8022 EG 005)

central question: what is the quantum structure of spacetime?

how: Functional Renormalisation Group (FRG)

≙ "mathematical microscope for quantum field theories (QFTs)"

prerequisites:

- General Relativity (GR)
- QFT I (helpful: II)

you can try QFT I in parallel but it will be challenging

also helpful (but optional):

- QFT on curved spacetimes
- Standard Model / Particle Physics / ...
- Intro to FRG

content:

- perturbative quantisation of GR and breakdown of predictivity
- quadratic gravity and stability challenges
- asymptotic safety in QFTs
- FRG for gravity
- advanced topics (let's see how far we get)

useful material:

- lecture notes of last years ← significant overlap
- Lectures in Quantum Gravity [SciPost Phys. Lect. Notes 98, YT recordings @Quantumgravity.nordita]
  partial overlap, includes many more details + ST, quantum BHs, EFT, ...
- Asymptotically safe quantum gravity and its phenomenology – a review [Eichhorn 2606.21522]
  up-to-date review →

two great books with many details

- Introduction to Covariant Quantum Gravity and Asymptotic Safety [Percacci 2017]
- Quantum Gravity and the FRG: The Road towards Asymptotic safety [Reuter, Saueressig 2019]
- Handbook of Quantum Gravity [2024]

→ the QG bible, "handbook" = 4300 pages

<u>caution</u>: this is a course close to active research
→ there might be no answer to some Qs (yet), or they might be ambiguous!

tutorials: Mo 16:15-18:00 with Paul Spreyer (from Nov on)

→ where you really learn the material

→ weekly exercises + occasional presence exercises

no hand-in ☺

exercises, lecture notes + other info will be posted

on the course webpage

→ th.physik → people → professors ... lecturers → BK

exam: oral, 20 mins

- ask all your questions - there are no stupid questions!
- provide feedback if something about the lecture is not ideal for you to learn the material



knorr @ thphys.uni-heidelberg.de

sprenger @ — " —



Phil 12 326

Why quantum gravity?

→ GR describes classical gravity...

... but matter is quantum!

↳ Einstein's equations: $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$)

$G_{\mu\nu}$: functions on spacetime

$T_{\mu\nu}$: operators

note: we use natural units, i.e. $c = 1$, $\hbar = 1$ (mostly)

→ mathematical inconsistency

→ physical inconsistency: RHS exhibits

uncertainty (e.g. position vs. momentum), but LHS has no uncertainty

⇒ what is the gravitational field of a "fuzzy" source?

Semiclassical gravity

"let's close our eyes and pretend there is no issue here"

idea: maybe gravity couples to expectation value of stress-energy

$$T_{\mu\nu} \rightarrow \langle T_{\mu\nu} \rangle$$

which state?
finite? still tensor?

$$\hookrightarrow \quad G_{\mu\nu} \overset{?}{=} 8\pi G_N \langle T_{\mu\nu} \rangle$$

corresponds to: matter described by QFT on curved spacetime + gravitational field sourced by $\langle\,\rangle$ of energy-momentum

Q: is this enough to consistently describe quantum matter on classical spacetime?

short answer: probably not.

→ the Standard Model of particle physics has its own problems, e.g. triviality

Classical curvature singularities

- even GR signals its own breakdown, no QFT needed

  → curvature singularities
  → incomplete geodesics

  ] black holes, big bang

Example: Schwarzschild BH

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right)dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

is a vacuum solution of GR, with $R_{\mu\nu} = 0$

however: $R_{\mu\nu\sigma\rho} \neq 0$, and in particular

$$K = R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} = 48\frac{G_N^2 M^2}{r^6}$$

Kretschmann scalar

→ infinite curvature at $r=0$

⇒ infinite tidal forces

similarly, the proper time τ_0 of a massive particle to reach the centre when starting at $r=r_0$ is

$$\tau_0 = \pi \sqrt{\frac{r_0^3}{8M}} < \infty$$

→ geodesics terminate in finite time

→ more complete theory need not be quantum per se,

but it is a minimal assumption

everything else is quantum, why not gravity?

Perturbative QG

- try what worked for the Standard Model

  ↳ quantise free theory (gravitational waves)

    ↳ decompose into gravitons
    (similar to photons in QED)

then add interaction perturbatively

→ learn why we are still working on QG despite many decades of research

Recap of gravitational waves

recall Einstein's equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$

highly non-linear in $g_{\mu\nu}$

- GW = solutions to linearized equations

→ use background field method:

we split the metric $g_{\mu\nu}$ into an arbitrary background metric $\bar{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

variations exist, for us this is enough

→ you can define background curvatures, cov. derivatives etc.

→ indices are raised & lowered with $\bar{g}$

→ for GW, we consider $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ (flat background)

→ as long as $|h_{\mu\nu}| \ll 1$ (= grav. field created by perturbation is small, i.e. self-interactions can be neglected)

linearisation should work well

↳ linearise Christoffel symbol & curvature tensors: sheet 1

$\simeq$ means equality up to terms outside of our interest ← defined in context

$$\Gamma^{\mu}_{\alpha\beta} \simeq \frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta}\right)$$

$$R_{\mu\nu} \simeq \frac{1}{2}\left[\partial_\mu \partial^\alpha h_{\nu\alpha} + \partial_\nu \partial^\alpha h_{\mu\alpha} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu \underline{h}\right]$$

($\underline{h} = h^\alpha{}_\alpha$)

$$R \simeq \partial^\mu \partial^\nu h_{\mu\nu} - \partial^2 h$$

note: expanding to higher order gives interaction terms of $h_{\mu\nu}$

→ important in strong-field regime, e.g. when GWs are created via black hole mergers

→ in propagation, these interactions are negligible because the GW amplitudes are small

Plug into EE:

$$\partial^2 h_{\mu\nu} - (\partial_\mu \partial^\sigma h_{\nu\sigma} + \partial_\nu \partial^\sigma h_{\mu\sigma}) + \partial_\mu \partial_\nu h$$

$$+ \eta_{\mu\nu} \partial^\sigma \partial^\rho h_{\sigma\rho} - \eta_{\mu\nu} \partial^2 h = -16\pi G_N \tilde{T}_{\mu\nu}$$

technically here $g = \eta$

this can be rewritten by introducing $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$

$$\Rightarrow \partial^2 \bar{h}_{\mu\nu} - 2\left[\underbrace{\tfrac{1}{2}(\delta_\mu{}^\alpha \partial_\nu + \delta_\nu{}^\alpha \partial_\mu)}_{\text{Symmetrisation: } \delta_{(\mu}{}^\alpha \partial_{\nu)}} - \tfrac{1}{2}\eta_{\mu\nu}\partial^\alpha\right]\partial^\beta \bar{h}_{\alpha\beta} = -16\pi G \tilde{T}_{\mu\nu}$$

$\leftarrow$ includes $\frac{1}{2}$ (= normalisation to unit strength)

compare to QED:

$$\partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu = J_\mu$$

wave equation, but has gauge redundancy (= non-trivial kernel)

$$A_\mu \to \tilde{A}_\mu + \partial_\mu \lambda$$

$\Rightarrow$ have to gauge-fix, e.g. with Lorenz gauge $\partial^\mu A_\mu = 0$

the same is true in gravity:

if $\bar{h}'_{\mu\nu}$ is a solution, so is

$$\bar{h}^2_{\mu\nu} = \bar{h}'_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \eta_{\mu\nu}\, \partial^\lambda \varepsilon_\lambda$$

for any ε

$\Rightarrow$ not all modes of $\bar{h}_{\mu\nu}$ are <u>physical</u> !

↳ the "gauge redundancy" in gravity is

diffeomorphism invariance:

→ physics does not depend on your choice of coordinates

Conclusion:

$$x^\alpha \to x'^\alpha = x^\alpha + \varepsilon^\alpha$$

$$\Rightarrow \quad g_{\mu\nu} \to \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta}$$

$$\hookrightarrow \quad \eta_{\mu\nu} + h_{\mu\nu} \to (\delta^\alpha_\mu + \partial_\mu \varepsilon^\alpha)(\delta^\beta_\nu + \partial_\nu \varepsilon^\beta)(\eta_{\alpha\beta} + h'_{\alpha\beta})$$

$$\hookrightarrow \quad h_{\mu\nu} \to h'_{\mu\nu} + \underline{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu} + \mathcal{O}(\varepsilon^2)$$

linearised diffeo. trafo.

- as in "normal" gauge theories, this is **not** a physical symmetry, but rather a **redundancy** in our description: $h_{\mu\nu}$ and $h'_{\mu\nu}$ are **physically equivalent** *

- to get rid off this, we have to

  gauge-fix

  = impose condition on $\bar{h}_{\mu\nu}$

can be used to simplify your life

→ remember Lorentz gauge in QED

* conditions apply, you can have an entire course about this

e.g. [Stromberger 1703.05448]

"large" diffeos are "fun"

popular choice: de Donder gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

Sanity check: Is this a valid condition, i.e. can we always find an ξ_μ such that $\partial^\mu \bar{h}_{\mu\nu} = 0$ without restricting physical configurations?

→ given some $h_{\mu\nu}$ which does not satisfy $\partial^\mu \bar{h}_{\mu\nu} = 0$,

search for $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \xi_\mu$ so that

$$\partial^\mu \bar{h}'_{\mu\nu} = 0$$

recall $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

$$= \partial^\mu \left[h'_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h' \right]$$

$$= \partial^\mu \Big[h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \tfrac{1}{2}\eta_{\mu\nu}(h + 2\partial^\alpha \varepsilon_\alpha) \Big]$$

(the $\partial_\mu \varepsilon_\nu$ and $\partial_\nu \varepsilon_\mu$ terms: cancel!)

$$0 = \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \varepsilon_\nu$$

→ this is a wave equation for ε_ν and "always" admits a solution

⟶ de Donder gauge is an <u>incomplete</u> gauge fixing: we can still shift ε_ν by solutions to the homogeneous wave eq. (as in QED!)

this happens pretty universally in gauge theories and will eventually come to bite you

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- back to linearised EE: in deDonder gauge,

$$\partial^2 \bar{h}_{\mu\nu} = -16\pi G_N \tilde{T}_{\mu\nu}$$

→ wave equation yay!

↳ we wanted to quantise the free field in vacuum

→ solve $\partial^2 h_{\mu\nu} = 0$

check why

$\partial^2 \bar{h}_{\mu\nu} = 0 \iff \partial^2 h_{\mu\nu} = 0$

this has a general solution in terms of a superposition of plane waves

$$h_{\mu\nu}(x) = \int d^4q \; \tilde{h}_{\mu\nu}(q) \; e^{i q_\alpha x^\alpha} \qquad \text{with } q^2 = 0$$

we will consider a single Fourier mode with momentum p_α,

$$h_{\mu\nu}(x) = \Pi_{\mu\nu} \, e^{i p_\alpha x^\alpha} + \Pi^*_{\mu\nu} \, e^{-i p_\alpha x^\alpha}$$

↑ polarisation tensor

similar to polarisation vector of the photon

note: $p^2 = 0$ to solve the wave equation

↪ gravitons are **massless** in GR (and in Minkowski)

How to fix $\pi_{\mu\nu}$?

1) gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$

in Fourier space, this entails

$p^\mu \bar{\pi}_{\mu\nu} = 0$ ← same "bar" as for $\bar{h}_{\mu\nu}$

these are <u>4</u> conditions

2) residual gauge dependence $\partial^2 \varepsilon_\nu = 0$

⇝ superposition of plane waves

focus on mode with some momentum p,

$$\varepsilon_\mu(x) = \epsilon_\mu e^{ip_\alpha x^\alpha} + \epsilon_\mu^* e^{-ip_\alpha x^\alpha}$$

under this gauge transformation,

$$\bar{\Pi}'_{\mu\nu} = \bar{\Pi}_{\mu\nu} + i\left(p_\mu \epsilon_\nu + p_\nu \epsilon_\mu - \eta_{\mu\nu}\, p\cdot\epsilon\right)$$

(check this)

we now can impose

$$u^\mu \bar{\Pi}'_{\mu\nu} = 0 \qquad \text{for a constant vector } u^\mu$$

$$= u^\mu \bar{\Pi}_{\mu\nu} + i\left(p\cdot u\, \epsilon_\nu + p_\nu\, u\cdot\epsilon - u_\nu\, p\cdot\epsilon\right)$$

3 eqs. for 4 components of ϵ

→ always has (complex) sol.

→ this fixes $\overline{3}$ more components of $\bar{\Pi}_{\mu\nu}$

not 4 because $p^\mu \bar{\Pi}_{\mu\nu} u^\nu = 0$

linear combination of $\bar{\Pi}_{\mu\nu} u^\nu$

3) residual gauge part II

we can take the trace of the transformed $\bar{\Pi}_{\mu\nu}$:

$$\bar{\Pi}^\mu{}_\mu{}' = \bar{\Pi}^\mu{}_\mu + i(2p\cdot\epsilon - 4p\cdot\epsilon)$$
$$= \bar{\Pi}^\mu{}_\mu - 2i\, p\cdot\epsilon$$

↳ we can always make $\bar{\pi}_{\mu\nu}$ traceless

remember: we had one free component of ϵ left above !

→ fixes 1 component of $\bar{\pi}_{\mu\nu}$

In summary, the conditions on $\pi_{\mu\nu}$ are:

$$\pi^{\mu}{}_{\mu} = 0 \quad , \quad p^{\mu}\pi_{\mu\nu} = 0 \quad , \quad u^{\mu}\pi_{\mu\nu} = 0$$

traceless

transverse (to direction of propagation, like for photons)

check that 1), 2), 3) carry over from $\bar{\pi}$ to π !

$\Rightarrow$ in $d=4$, the graviton has <u>two</u> physically distinct polarisations

<u>Excursion</u>: general dimension d ($d-1$ spatial + 1 time)

- $\Pi_{\mu\nu}$ is symmetric $\rightarrow$ $\frac{d(d+1)}{2}$ components
- $p^\mu \Pi_{\mu\nu} = 0$ $\rightarrow$ d conditions
- $u^\mu \Pi_{\mu\nu} = 0$ $\rightarrow$ $d-1$ conditions
- $\Pi^\mu{}_\mu = 0$ $\rightarrow$ 1 condition

$\Rightarrow$ $\frac{d(d-3)}{2}$ independent polarisations

General wisdom: "gauge symmetry always hits twice"

above: d conditions from gauge fixing

but eom has non-trivial kernel

↳ common feature in gauge theories, related to secondary constraints ↗sheet 3

↳ in $d \leq 3$: no propagating gravitational waves

→ no two orthogonal directions in which to oscillate

⇒ $d=4$ is the minimal spacetime dimensionality for GWs to propagate ↗(why) is our universe 4d?