How do the polarisations look like?

Example: GW that propagates in z-direction

$\Rightarrow$ $p^\mu = (p, 0, 0, p)$ (can make different choices, this is a specific one)

we also choose $u^\mu = (1, 0, 0, 0)$

$$\Rightarrow \quad \Pi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_+ & e_x & 0 \\ 0 & e_x & -e_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

check this!

e_+ and e_x are the amplitudes of the two polarisations

↳ metric :

$$ds^2 = dx^\mu dx^\nu (\eta_{\mu\nu} + h_{\mu\nu})$$

$$= -dt^2 + dx^2\left[1 + e_+ \underbrace{\left(e^{ip(z-t)} + e^{-ip(z-t)}\right)}\right]$$

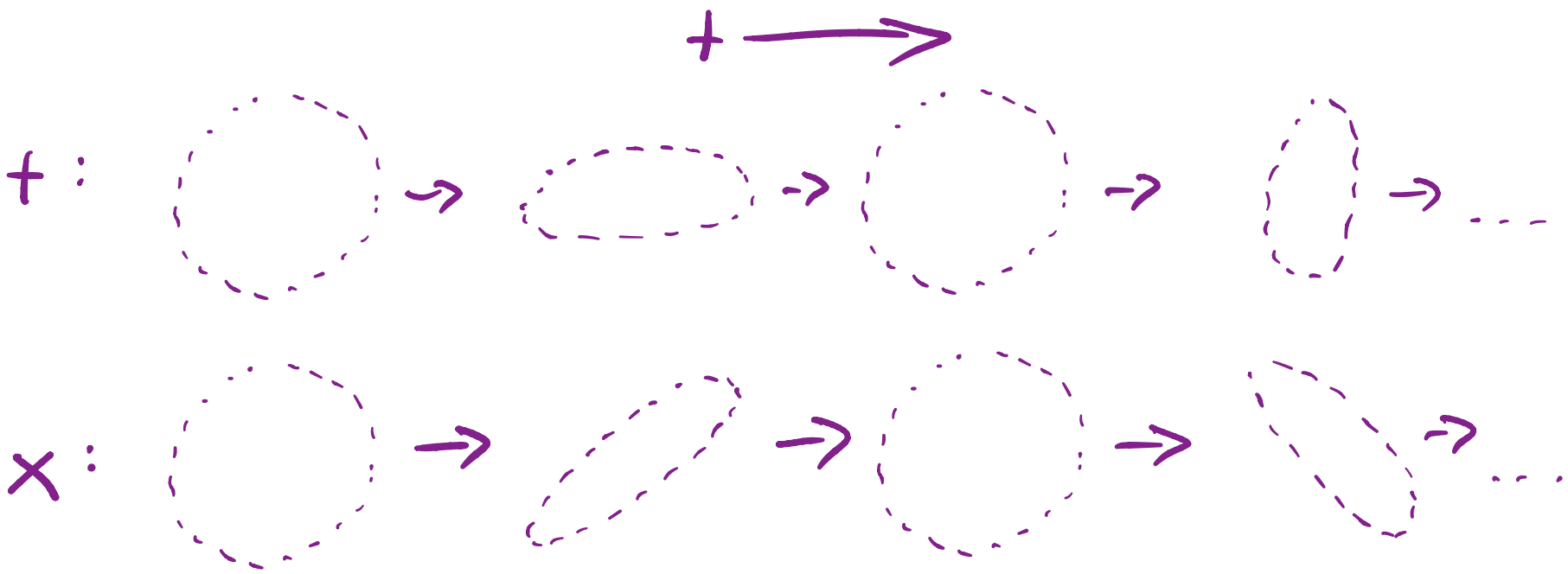
$$+ dy^2\left[1 - e_+ \underbrace{\left(e^{ip(z-t)} + e^{-ip(z-t)}\right)}\right]$$

$$+ 2dx\,dy\, e_x \underbrace{\left(e^{ip(z-t)} + e^{-ip(z-t)}\right)}$$

$$+ dz^2$$

$$2\cos[p(z-t)]$$

↳ two independent oscillation modes in x-y-plane



what are the dots?

⇒ ring of particles, GW passes through and displaces them in the above pattern

back to quantisation

→ quantise plane waves by promoting coefficients

↳ creation/annihilation operators:

$$h_{\mu\nu}(x) = \sum_{\sigma=\pm} \int d^3\vec{p} \left[a_{\vec{p},\sigma}\, \pi_{\mu\nu}(\sigma)\, e^{ip\cdot x} + c.c. \right]$$

↳ $e_\pm = \frac{1}{\sqrt{2}}(e_+ \pm i e_\times)$ ← these are helicity eigenstates = eigenstates of rotation matrix around z-axis

→ $a_{\vec{p},\sigma}$ and $a^*_{\vec{p},\sigma}$ are quantum-mechanical operators with a standard set of commutation relations:

$$[a, a] = [a^\dagger, a^\dagger] = 0 \,, \quad [a_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}] = i\,\delta_{\sigma\sigma'}\,\delta^3(\vec{p}-\vec{p}')$$

→ standard procedure to build Fock space

(vacuum, single graviton states, etc.)

↳ are we done? is this QG yet? of course not!

⇝ interaction

this is the point where we will need some QFT for the first time

to make a QFT out of this, we need to compute loop corrections ⇝ need to know the Feynman rules

strategy: minimise pain and use de Donder gauge

⇝ expand Einstein-Hilbert Lagrangian to higher orders in h

$$\left(\mathcal{L} = \frac{\sqrt{-g}}{16\pi G_N} R\right)$$

⇝ propagator

3-graviton vertex

$$\hookrightarrow \quad \mathcal{L} \simeq -\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \frac{1}{4}(\partial_\mu h)^2 + V_3^{\mu\nu\sigma\kappa\lambda} h_{\mu\nu} h_{\sigma\sigma} h_{\kappa\lambda} + \dots$$

↑ up to rescaling, effectively $h_{\mu\nu} \to \sqrt{32\pi G_N}\, h_{\mu\nu}$

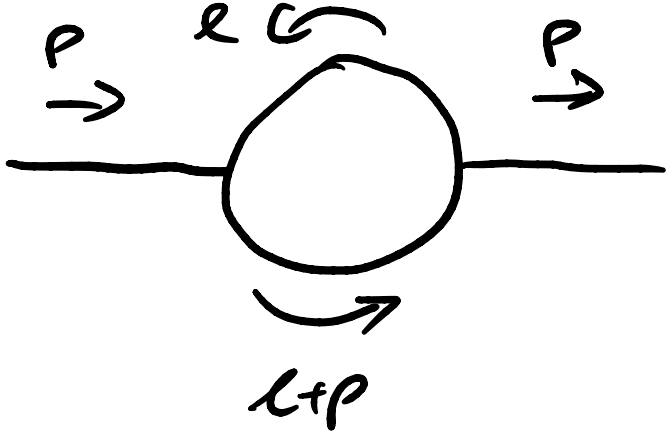
⇒ this is useful as it canonically normalises $h_{\mu\nu}$ and makes it that the limit $G_N \to 0$ gives the free theory

⇒ similar to gauge theory: can put coupling in front of F^2 term, or into the gauge field

side note: in a proper treatment, we have to use the Faddeev-Popov method to implement the gauge fixing; for now we do not need it as it doesn't change the power counting, but you will need it later

↳ Feynman rules for GR

We will discuss the loop correction to the propagator, i.e. the relevant diagram is



there is in principle also a tadpole diagram, but in dimensional regularisation it vanishes

↳ we need the graviton propagator and the 3-graviton vertex

in the following we will neglect factors of "i" as they do not change the scaling with momentum

Propagator: = inverse two-point function

$$[P^{-1}]^{\mu\nu\sigma\sigma}(p) \propto p^2 \Big[\frac{1}{2}\big(\eta^{\mu\sigma}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\sigma}\big) \textcircled{1} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\sigma} \Big]$$

(2)

in a more general gauge, there will also be terms of the

from $(p^\mu p^\nu \chi^{\rho\sigma} + \chi^{\mu\nu} p^\rho p^\sigma)$, ③

$(p^\mu \chi^{\nu\rho} p^\sigma + p^\nu \chi^{\mu\rho} p^\sigma + p^\mu \chi^{\nu\sigma} p^\rho + p^\nu \chi^{\mu\sigma} p^\rho)$, ④

and if higher curvature terms are there as well, $p^\mu p^\nu p^\rho p^\sigma$ ⑤

→ these ⑤ elements form a basis for two-point functions of a symmetric field

to compute the propagator, we have to find the inverse – but what is the identity?

$h_{\mu\nu}$ is a <u>symmetric</u> tensor

maps symmetric tensors onto themselves

$$\Rightarrow \quad \mathbb{1}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = h_{\mu\nu}$$

$$\Rightarrow \quad \mathbb{1}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\sigma \delta_\nu{}^\rho\right)$$

$$\rightarrow \quad \mathcal{P}_{\alpha\beta\mu\nu}\left[\mathcal{P}^{-1}\right]^{\mu\nu\rho\sigma} \overset{!}{=} \mathbb{1}_{\alpha\beta}{}^{\rho\sigma}$$

in general, you would now proceed by spanning $\mathcal{P}$ in the above basis ① - ⑤

in our gauge, we only need ① and ② (why?)

$$\hookrightarrow \text{ansatz}: \quad \mathcal{P}_{\alpha\beta\mu\nu} = c_1 \cdot \overbrace{\frac{1}{2}\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu}\right)}^{\mathbb{1}_{\alpha\beta\mu\nu}} + c_2\, \eta_{\alpha\beta}\eta_{\mu\nu}$$

compute the product:

$$\left(P \cdot P^{-1}\right)_{\alpha\beta}{}^{\sigma\delta} = c_1 \left[p^2 \mathbb{1}_{\alpha\beta}{}^{\sigma\delta} - \frac{p^2}{2} \eta_{\alpha\beta} \zeta^{\sigma\delta} \right]$$

$$+ c_2 \left[p^2 \eta_{\alpha\beta} \zeta^{\sigma\delta} - \frac{p^2}{2} \eta_{\alpha\beta} \underbrace{\eta_{\mu\nu} \zeta^{\mu\nu}}_{=4} \zeta^{\sigma\delta} \right]$$

$$= c_1 p^2 \mathbb{1}_{\alpha\beta}{}^{\sigma\delta}$$

$$+ p^2 \underbrace{\left[-\frac{c_1}{2} + c_2 - 2c_2 \right]}_{-\frac{c_1}{2} - c_2 \overset{!}{=} 0} \eta_{\alpha\beta} \zeta^{\sigma\delta}$$

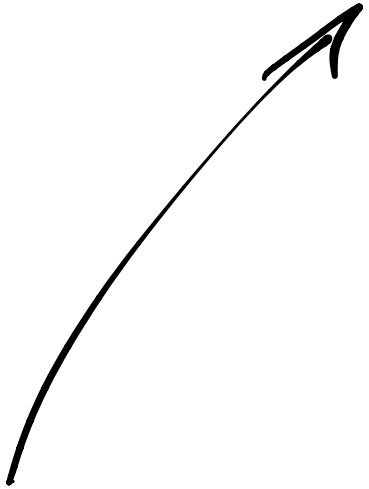
$\overset{!}{=} 1$

$\Rightarrow c_1 = 1/p^2$

$\Rightarrow \quad c_2 = -\frac{c_1}{2} = -\frac{1}{2p^2}$

$$\Rightarrow \quad P_{\alpha\beta\mu\nu} = \frac{1}{p^2}\left[\mathbb{1}_{\alpha\beta\mu\nu} - \frac{1}{2}\mathbb{P}_{\alpha\beta}\mathbb{P}_{\mu\nu}\right]$$

note: this is up to the proportionality factor that we dropped for the two-point fct. (essentially ν!)



propagator in de Donder gauge

3-graviton vertex: this is where things start to be painful

↗ tutorial 3

→ general structure of V_3: combinations of two momenta (out of the three) and metrics, each with 6 indices ↑ why two?

→ with momentum conservation ($p_1+p_2+p_3=0$), there are 161 terms (not the most compact way of writing this)

relevant info: $V_3 \sim$ "p^2" ↖ meaning all possible combinations

↳ can evaluate  now

learn in the tutorial how to not waste hours on this

↳ doing so and performing the loop integral, we encounter divergences of the form

$$[\text{div. prefactor}] \times \underbrace{[\partial^\mu \partial^\nu h_{\mu\nu} - \partial^2 h]^2}$$

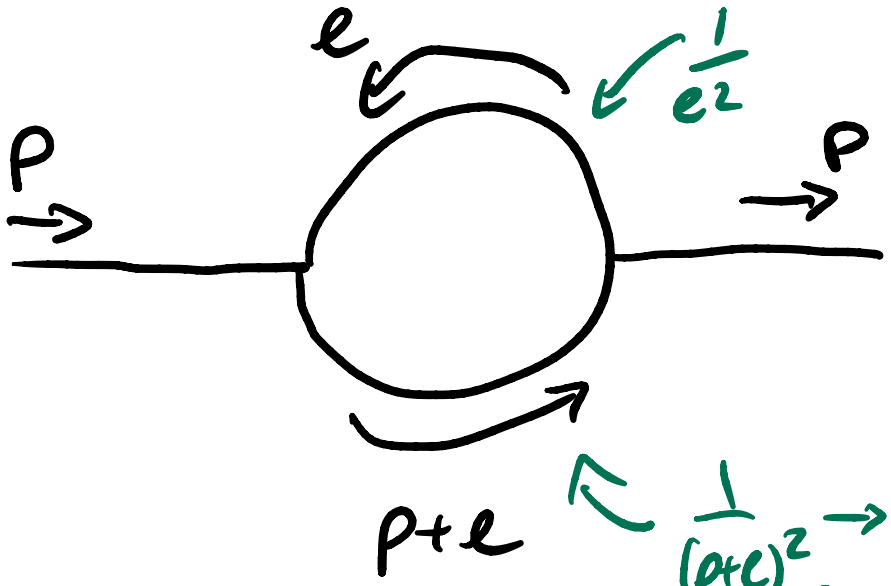
↑ this is the expansion of R^2 to second order in h

and other terms

⇝ how do these terms arise and what do they mean?

① the divergence

for this, we count powers of the loop momentum ℓ



vertices could either be p^2, $p\cdot\ell$ or ℓ^2

⇒ in the least divergent case, $\sim \int d^4\ell \left(\frac{1}{\ell^2}\right)^2 p^4$ at large ℓ

→ logarithmic divergence × p^4

(there are others as well)

② the tensor structure

the resulting terms must be proportional to expansions of curvature invariants — otherwise, diffeomorphism invariance would be broken

at fourth orders in derivatives — p^4 above —

candidates are ↗ sheet 2

$$R^2,\ R_{\mu\nu}R^{\mu\nu},\ R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta},\ D^2R$$

↗

total derivative

⇝ can be neglected

in $d=4$, the combination

$$\epsilon = \sqrt{g}\left(R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

is a total derivative – it is called the

Gauss-Bonnet invariant

$\rightsquigarrow$ its integral $\int d^4x\,\epsilon$ is a topological invariant and measures the number of "handles" of spacetime

$\hookrightarrow$ only 2 terms are linearly independent and can contribute to the action (in the absence of boundary terms): R^2 and $R_{\mu\nu}R^{\mu\nu}$ this is a choice

Consequence: to absorb these divergences, $\sim \int d^4x \mathcal{L}$ counterterms have to be added to the action

$$\rightsquigarrow \int d^4x \sqrt{g} \left[a R^2 + b R_{\mu\nu} R^{\mu\nu} \right]$$

these are **not** of the form of the original Einstein-Hilbert action

$\Rightarrow$ two new couplings appear in the action

Fun fact: in the absence of matter, one can perform a field redefinition

$g_{\mu\nu} \to g_{\mu\nu} + c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R$ so that for suitable c_1, c_2, the counterterms vanish

to see this, we plug in this shift into the action and expand to linear order in c_1, c_2 [this is enough because we look at the one-loop term]

$$\rightsquigarrow \quad S[g_{\mu\nu} + c_1 R_{\mu\nu} + c_2 R g_{\mu\nu}]$$

$$\simeq S[g_{\mu\nu}] + \int d^4x \underbrace{\frac{\delta S[g]}{\delta g_{\mu\nu}}}_{-\sqrt{-g}\, G_{\mu\nu}} (c_1 R_{\mu\nu} + c_2 R g_{\mu\nu})$$

$$\simeq S[g] + \int d^4x \sqrt{-g} \left[-c_1 R_{\mu\nu} R^{\mu\nu} + \left(\tfrac{c_1}{2} + c_2\right) R^2 \right]$$

(up to boundary terms)

this pattern of new divergences persists at every loop order: at n loops, we need terms with $2n+2$ derivatives

note: at $n=2$, this includes both
cubic curvature terms as well as
quadratic curvature terms with two derivatives
→ the latter are not all total derivatives
e.g. $R D^2 R$

Sad fact: starting at $n=2$, you <u>cannot</u> absorb
the divergences with a (local) field redefinition

for $n=2$ the offending term in $d=4$ is

$$R_{\mu\nu}{}^{\sigma\tau} R_{\sigma\tau}{}^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu\nu}$$

this two-loop counterterm was first computed
by Goroff & Sagnotti in '85/'86

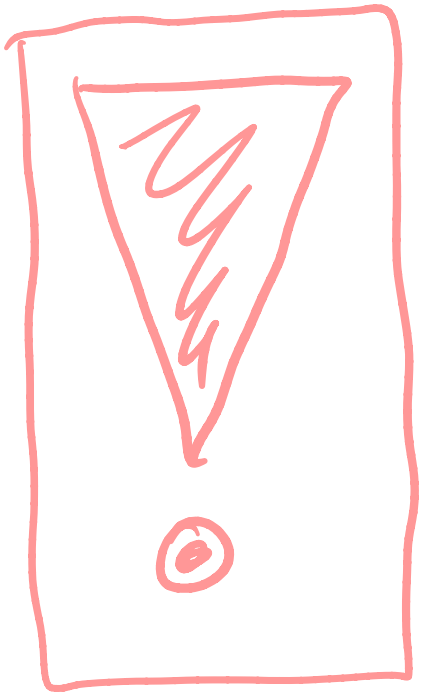
Phys. Lett. B 160 (1985) 81-86
Nucl. Phys. B 266 (1986) 709-736

⇝ think about this, given the argumentation we used at one-loop order – is it clear that the above terms cannot be absorbed?

⇝ the problem (likely) repeats at higher loop orders – why? ↑

likely, because the prefactor of a divergence could be zero; the point of the Goroff-Sagnotti computation was to show it is not

Conclusion: the perturbative quantisation of the Einstein-Hilbert action leads to a non-predictive theory which has infinitely many free couplings



↖ the finite prefactors of the new terms, once divergences have been absorbed

⇒ perturbative non-renormalisability

= breakdown of predictivity

let us argue for the above claim with $2n+2$ derivatives at n loops

$\rightsquigarrow$ compute superficial degree of divergence D in $\underline{d}$ dimensions

= expected (worst) divergence, barring any cancellations (e.g. through contractions of indices)

to count the divergence, we introduce a cutoff in the integrals over loop momenta, Λ_{UV} : $\int d^d\ell \to \int^{\Lambda_{UV}} d^d\ell$

recall: propagators scale like $1/p^2$
vertices scale like p^2
} in GR

G in a diagram with P propagators, L loops, and V vertices, we will have

$$\left[\frac{1}{\ell^2}\right]^P \left[\int d^d\ell\right]^L \left[\ell^2\right]^V \sim [\ell]^{\overbrace{-2P+dL+2V}^{=D}}$$

$$\Rightarrow \quad D = -2P + dL + 2V$$

we can simplify this by using

$$L = P - V + 1$$

which is a topological relation that holds for all loop diagrams

proof: consider arbitrary diagram with P propagators, L loops and V vertices

→ successively cut propagators without cutting diagrams apart

↳ every cut removes 1 propagator and 1 loop

↳ repeat this until no loop is left

⇒ $P-L$ propagators are left

→ we still have V vertices that are connected by the $P-L$ propagators; there is at most a single connection between any two vertices (otherwise there is a loop); vertices cannot be connected directly and via another vertex (otherwise there is a loop) → vertices form a connected line

⇒ we need $V-1$ lines $= P-L$ propagators

plugging in this relation, we get

$$D = (d-2)L + 2$$

→ in $d=2$, only quadratic divergences = can be absorbed!

gravity is actually <u>topological</u> in $d=2$

in $d=2$, the Gauss-Bonnet term is exactly the Einstein term

→ in $d>2$, the degree of divergence increases with loop order, as claimed

Q: is perturbative QG still useful? <u>YES</u>

↝ before we discuss possible solutions, we try to extract some physical insights from pert. QG