Weak annihilation in the rare radiative $B \rightarrow \rho \gamma$ decay

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The amplitude of the $B \rightarrow \rho \gamma$ decay induced by the flavor-changing neutral current contains the penguin contribution and the weak annihilation contribution generated by the 4-quark operators in the effective Hamiltonian. The penguin contribution is known quite well. We analyze the weak annihilation which is suppressed by the heavy-quark mass compared to the penguin contribution. In the factorization approximation, the weak annihilation amplitude is represented in terms of the leptonic decay constants and the meson-photon matrix elements of the weak currents. The latter contain the $B \gamma$, $\rho \gamma$ transition form factors and contact terms determined by the equations of motion. We calculate the $B \gamma$ and $\rho \gamma$ form factors within the relativistic dispersion approach and obtain numerical estimates for the weak annihilation amplitude.

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I. INTRODUCTION

The investigation of rare semileptonic $B$ decays induced by the flavor-changing neutral current transitions $b \rightarrow s$ and $b \rightarrow d$ represents an important test of the standard model or its extensions. Rare decays are forbidden at the tree level in the standard model and occur through loop diagrams. Thus they provide the possibility to probe the structure of the electroweak sector at large mass scales from contributions of virtual particles in the loop diagrams. Interesting information about the structure of the theory is contained in the Wilson coefficients in the effective Hamiltonian which describes the $b \rightarrow s, d$ transition at low energies. These Wilson coefficients take different values in different theories with testable consequences in rare $B$ decays.

Among rare $B$ decays the radiative decays $b \rightarrow s \gamma$ and $b \rightarrow d \gamma$ have the largest probabilities. The $b \rightarrow s \gamma$ transitions are Cabibbo-Kobayashi-Maskawa (CKM) favored and have larger branching ratios than the $b \rightarrow d \gamma$ transitions. The $b \rightarrow s \gamma$ transition has been observed by CLEO in the exclusive channel $B \rightarrow K^* \gamma$ in 1993 and measured inclusively in 1995. The $B \rightarrow \rho \gamma$ decay will be extensively studied by BaBar and BELLE.

The main uncertainty in the theoretical analysis of $B$ decays is connected with long-distance QCD effects arising from the presence of hadrons in initial and final states. In inclusive decays these effects are under better control, however, from inclusive measurements it is more difficult to obtain precise results.

The decay amplitude contains two different contributions: one arising from the electromagnetic penguin operator and another from the 4-fermion operators in the effective Hamiltonian. One of the effects generated by the 4-fermion operators is weak annihilation (WA). Further details about short- and long-distance effects in the radiative decays can be found in recent publications [1–4] and references therein.

In the $B \rightarrow K^* \gamma$ decay the weak annihilation is negligible compared to the penguin effect: it is suppressed by two powers of the small parameter $\lambda \approx 0.2$ of the Cabibbo-Kobayashi-Maskawa matrix. In $B \rightarrow \rho \gamma$ both effects have the same order in $\lambda$ and must be taken into account.

The penguin contribution has been analyzed in several ways and is known quite well. On the other hand, the WA in $B \rightarrow \rho \gamma$ has been studied in less detail: the relevant form factors were analyzed within sum rules [5,6] and perturbative QCD [7]. However, some contributions to these form factors were neglected. These contributions may be relevant if precise measurements become available.

The aim of this paper is to analyze the weak annihilation for the $B \rightarrow \rho \gamma$ decay more closely.

In the factorization approximation, the weak-annihilation amplitude can be represented as the product of meson lepton decay constants and matrix elements of the weak current between meson and photon. The latter contain the meson-photon transition form factors and contact terms which are determined by the equations of motion. The photon can be emitted from the loop containing the $b$ quark which is described by $B \gamma$ transition form factors. It can also be emitted from the loop containing only light quarks described by the $\rho \gamma$ transition form factors (Fig. 1).

We calculate the $B \gamma$ form factors within the relativistic dispersion approach which expresses these form factors in terms of the $B$ meson wave function. We demonstrate that the form factors calculated by the dispersion approach behave in the limit $m_\rho \rightarrow \infty$ in agreement with perturbative QCD. The $B$ meson wave function was previously tested in the $B \rightarrow \ell^+ \ell^-$ decay and is known quite well, allowing us to provide reliable numerical estimates for the $B \gamma$ form factors.

FIG. 1. Diagrams describing the weak annihilation process for $B \rightarrow \rho \gamma$ in the factorization approximation: (a) The photon is emitted from the loop containing the $b$ quark, (b) the photon is emitted from the loop containing only light quarks.
The $\rho \gamma$ transition form factor is related to the divergence of the vector and axial-vector currents. In the case of the axial-vector current it is proportional to the light-quark masses if the classical equation of motion is applied. Because the quark momenta in the loop are high, these masses have to be identified with current quark masses. For this reason the corresponding $\rho\gamma$ form factor was neglected in previous analyses [1,5]. We find however that this argument is not correct: this form factor remains finite in the limit of vanishing light quark mass $m \to 0$ and behaves like $\sim M_f^2/M_b^2$ which means the violation of the classical equation of motion. We present the result for the $\rho\gamma$ transition form factor but leave a detailed discussion of the anomaly appearing for the matrix element $\langle \rho \gamma | \partial_\mu A_\nu(0) \rangle$ for a special publication.

Finally, we provide numerical estimates of the weak-annihilation amplitude taking into account the $B\gamma, \rho\gamma$ transition form factors, and the contact term contributions.

In Sec. II the effective Hamiltonian for the $b \to d$ transition and the general structure of the amplitude are presented. In Sec. III we discuss the photon emission from the $B$ meson loop and obtain the $B\gamma$ transition form factors within the dispersion approach. Section IV contains results for the $\rho\gamma$ transition form factors. In Sec. V the numerical estimates are given. The concluding section summarizes our results.

II. THE EFFECTIVE HAMILTONIAN, THE AMPLITUDE AND THE DECAY RATE

The amplitude of the weak radiative $B \to \rho$ transition is given by the matrix element of the effective Hamiltonian for the $b \to d$ transition

$$A(B \to \rho \gamma) = \langle \gamma(q_1) \rho(q_2) | H_{\text{eff}}(b \to d) | B(0) \rangle,$$

where $p = B$ is the B momentum, $q_2$ is the photon momentum, $q_1 = q_2 + q_2' = 0$, and $q_2^2 = M_B^2 - p^2 = M_B^2$. The effective weak Hamiltonian has the structure [8]

$$H_{\text{eff}}(b \to d) = \frac{G_F}{\sqrt{2}} \xi_C \gamma_\mu \gamma_\nu O_{\gamma \nu} - \frac{G_F}{\sqrt{2}} \xi_C \gamma_\mu O_{\gamma \nu},$$

where only operators relevant for the penguin and weak annihilation effects are listed. $G_F$ is the Fermi constant, $\xi_C$ reads $\xi_C = V_{tb}^* V_{ts}$, $C_i$'s are the Wilson coefficients and $O_i$'s are the basis operators

$$O_{\gamma \nu} = \frac{e}{8 \pi^2} \bar{d}_b \sigma_{\mu \nu} m_b(1 + \gamma_5) b_a F_{\mu \nu},$$

$$O_1 = \bar{d}_b \gamma_\mu (1 - \gamma_5) u_a \bar{u}_\nu \gamma_\nu (1 - \gamma_5) b_\beta,$$

$$O_2 = \bar{d}_b \gamma_\mu (1 - \gamma_5) u_\beta \bar{u}_\nu \gamma_\nu (1 - \gamma_5) b_\alpha,$$

with the following notation: $e = \sqrt{\frac{4}{3} \pi a_{em}}$, $\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$, $\sigma_{\mu \nu} = i [\gamma_\mu, \gamma_\nu]/2$. $e^{0123} = 1$ and $\text{Sp}(\gamma^2 \gamma^2 \gamma^2 \gamma^2)$

$$= 4!e^{\alpha \beta \gamma \delta}, \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$
It is convenient to isolate the parity-conserving contribution which emerges from the product of the two equal-parity currents, and the parity-violating contribution which emerges from the product of the two opposite-parity currents.

1. The parity-violating amplitude

The parity-violating amplitude has the form

\[
A_{\text{PV}}(B^- \rightarrow \rho^- \gamma) = \frac{G_F}{\sqrt{2}} \xi \bar{x}_1 \langle (\rho^- \gamma | \bar{d} \gamma_{\gamma} u | 0) \rangle \langle 0 | \bar{u} \gamma_{\gamma} \gamma_{5} b | B^- \rangle + \langle \rho^- | \bar{d} \gamma_{\gamma} u | 0 \rangle \langle \gamma | \bar{u} \gamma_{\gamma} b | B^- \rangle. \tag{11}
\]

Here \( \alpha_1 \) is an effective Wilson coefficient, which we take as \( \alpha_1 = C_1 + C_2 / N_c \) at the scale \( \sim 5 \) GeV.

The first term is a contact term which can be calculated using the equations of motion for the quark fields [1]. Setting \( m_u = m_d \), we find

\[
\langle \rho^- \gamma | \bar{d} \gamma_{\gamma} u | 0 \rangle \langle 0 | \bar{u} \gamma_{\gamma} \gamma_{5} b | B^- \rangle = i p_{\rho} \bar{q} (\rho^- \gamma | \bar{d} \gamma_{\gamma} u | 0) = p_{\rho} \bar{q} (\rho^- \gamma | \bar{d} \gamma_{\gamma} u | 0) = \epsilon \epsilon^{\mu}_{\rho \lambda} \epsilon^{\nu}_{\mu \lambda} g_{\mu \nu} M_{\rho \lambda} \gamma_{\gamma} . \tag{12}
\]

The \( B^- \rightarrow \gamma \) amplitude in the second term of Eq. (11) can be parametrized as follows:

\[
\langle \gamma | \bar{u} \gamma_{\gamma} \gamma_{5} b | B^- (p) \rangle = \epsilon \epsilon^{\mu}_{\rho \lambda} \left( g_{\nu \mu} p_{\lambda} - p_{\mu} q_{\lambda} \right) \frac{2 F_A}{M_B} - \frac{2 F_B}{M_B} . \tag{13}
\]

It contains the form factor \( F_A(q_2^2 = M_B^2) \), and the contact term proportional to \( f_B \) (the derivation of this relation is given in the Appendix).

Summing the contributions of the photon emission from the \( B \)-meson loop and the \( \rho \)-meson loop gives the amplitude \( A_{\text{PV}} \) which can be represented in the form \( A_{\text{PV}} = \epsilon^{\mu}_{\rho \lambda} T_{\mu} \) with \( q_{1 \rho}^{\mu} T_{\mu} = 0 \) as required by gauge invariance. Thus, the weak-annihilation contribution to the form factor \( F_{\text{PV}} \) for the \( B^- \rightarrow \rho^- \gamma \) decay is\(^1\)

\[
F_{\text{PV}}^{\text{WA}} = \xi a_1 f_{\rho} M_{\rho} \frac{2 F_A}{M_B} + \frac{2 F_B}{M_B - M_{\rho}^2} . \tag{14}
\]

The two contact terms which are present in the amplitudes of the photon emission from the \( B \) meson loop and from the \( \rho \)-meson loop do not cancel each other (we disagree here with the claim of Ref. [6]) but lead to a nonvanishing contribution proportional to \( f_B \).

C. The parity-conserving amplitude

This amplitude reads

\[
A_{\text{PC}}(B^- \rightarrow \rho^- \gamma) = - \frac{G_F}{\sqrt{2}} \xi a_1 \langle \rho^- | \bar{d} \gamma_{\gamma} u | 0 \rangle \langle \gamma | \bar{u} \gamma_{\gamma} b | B^- \rangle + \langle \gamma | \bar{u} \gamma_{\gamma} \gamma_{5} b | B^- \rangle . \tag{15}
\]

The \( B^- \rightarrow \gamma \) amplitude from the first term in the brackets contains the form factor \( F_V(q_2^2 = M_B^2) \):

\[
\langle \gamma | \bar{u} \gamma_{\gamma} b | B^- (p) \rangle = \epsilon \epsilon^{\mu}_{\rho \lambda} \epsilon^{\nu}_{\mu \lambda} g_{\mu \nu} M_{\rho \lambda} \gamma_{\gamma} . \tag{16}
\]

The second term in Eq. (15) can be reduced to the divergence of the axial-vector current and contains another form factor, \( G_V \): namely,

\[
\langle 0 | \bar{u} \gamma_{\gamma} \gamma_{5} b | B^- \rangle \langle \gamma | \bar{d} \gamma_{\gamma} \gamma_{5} u | 0 \rangle = \epsilon \epsilon^{\mu}_{\rho \lambda} \epsilon^{\nu}_{\mu \lambda} \epsilon^{\mu}_{\rho \lambda} \epsilon^{\nu}_{\mu \lambda} g_{\mu \nu} M_{\rho \lambda} \gamma_{\gamma} . \tag{17}
\]

Thus, the weak annihilation contribution to \( F_{\text{PC}} \) reads

\[
F_{\text{PC}}^{\text{WA}} = \xi a_1 M_{\rho} \frac{2 F_V}{M_B} f_B G_V M_{\rho} f_{\rho} . \tag{18}
\]

Summing up, within the factorization approximation the weak annihilation amplitude can be expressed in terms of the three form factors \( F_A, F_V, \) and \( G_V \).

III. THE FORM FACTORS \( F_A \) AND \( F_V \)

In this section we derive the formulas for the form factors \( F_{A,V} \) within the dispersion approach to the transition form factors. This approach has been formulated in detail in [10] and applied to the weak decays of heavy mesons in [11]. Recall that the form factors \( F_{A,V} \) describe the transition of the \( B \)-meson to the photon with the momentum \( q_1 \), \( q_2^2 = 0 \), induced by the axial-vector (vector) current with the momentum transfer \( q_2 \), \( q_2^2 = M_{\rho}^2 \). For technical reasons, it is convenient to treat the form factor \( F_{A(V)}^{\rho} \) as describing the amplitude of the photon-induced transition of the \( B \)-meson into a \( b \bar{u} \) axial-vector (vector) virtual particle with the corresponding factor \( 1/(s - q_2^2) \) in the dispersion integral. Then we can directly apply the equations obtained in [10] for the meson-meson transition form factors.

A. The form factor \( F_A \)

The form factor \( F_A \) is given by the diagrams of Fig. 2. Figure 2(a) shows \( F_A^{\rho} \), the contribution to the form factor of the process when the \( b \) quark interacts with the photon; Fig. 094006-3
Here \( \lambda(a,b,c) = (a-b-c)^2 - 4bc \) is the triangle function.

It is convenient to change the direction of the quark line in the loop diagram of Fig. 2(b). This is done by performing the charge conjugation of the matrix element and leads to a sign change for the \( \gamma_5 \gamma_5 \) vertex.

Now both diagrams in Figs. 2(a) and 2(b) are reduced to the diagram of Fig. 3 which defines the form factor \( F_A^{(1)}(m_1,m_2) \): Setting \( m_1 = m_b, m_2 = m_u \) gives \( F_A^{(b)} = F_A^{(b)} = Q_B F_A^{(1)}(m_b, m_u) \). Similarly, setting \( m_1 = m_u, m_2 = m_b \) gives \( F_A^{(a)} = -Q_B F_A^{(1)}(m_u, m_b) \). For the diagram of Fig. 3 (quark 1 emits the photon, quark 2 is the spectator) the trace reads

\[
-\text{Sp} \gamma_5 (m_2 - k_2) \gamma_\mu \gamma_5 (m_1 + k'_1) \gamma_\mu (m_1 + k_1) = 4i (k_1 + k'_1) \gamma_\mu (m_1 + k_2 - m_2 + k'_2) + 4i (g_{\mu\nu} q_a - g_{\mu a} q_v)
\]

\[
\times (m_1 k_2 + m_2 k_1)_a.
\]

The spectral density of the form factor \( F_A^{(1)}(m_1,m_2) \) in the variable \( p^2 = k_1 + k_2 \), is the coefficient of the structure \( g_{\mu\nu} \)

\[
\times (m_1 k_2 + m_2 k_1)_a.
\]

The form factor \( F_A^{(1)}(m_1,m_2) \) is

\[
\frac{2}{M_B} F_A^{(1)}(m_1,m_2) = \sqrt{\frac{N_c}{4\pi^2}} \int_{(m_b + m_u)^2}^{\infty} ds \phi_B(s) \left\{ m_1 \log \frac{s + m_1^2 - m_2^2 + \lambda^{1/2}(s,m_1^2,m_2^2)}{s + m_1^2 - m_2^2 - \lambda^{1/2}(s,m_1^2,m_2^2)}
\right. \\
+ \frac{1}{m_2 - m_1} \frac{\lambda^{1/2}(s,m_2^2,m_u^2)}{2} \\
\left. + \frac{1}{p q_1} \frac{\lambda^{1/2}(s,m_u^2,m_2^2)}{2} \\
- \frac{m_2^2}{m_2^2 - \lambda^{1/2}(s,m_1^2,m_2^2)} \log \frac{s + m_1^2 - m_2^2 + \lambda^{1/2}(s,m_1^2,m_2^2)}{s + m_1^2 - m_2^2 - \lambda^{1/2}(s,m_1^2,m_2^2)} \right\}.
\]

For \( q_1^2 = 0 \) one has \( p q_1 = (M_B^2 - M_\rho^2)/2 \).

Now, let us analyze the behavior of the form factor in the limit \( m_b \to \infty \). To this end it is convenient to rewrite the spectral representation (21) in terms of the light-cone variables as follows (see [12] for details):

\[
\frac{2}{M_B} F_A^{(1)}(m_1,m_2) = \frac{\sqrt{N_c}}{4\pi^2} \int dx_1 dx_2 dk_2^2 \delta(1-x_1-x_2)
\]

\[
\times \phi_B(s) \frac{m_1 x_2 + m_2 x_1 - (m_1 - m_2)}{s - M_\rho^2} \\
\times k_2^2 (p q_1).
\]

Here \( x_i \) is the fraction of the B-meson light-cone momentum carried by the quark \( i \), and \( s = m_1^2 x_1 + m_2^2 x_2 + k_2^2 / x_1 x_2 \). For the form factors \( F_A^{(a)} \) and \( F_A^{(b)} \) one obtains

\[
\frac{2}{M_B} F_A^{(a)} = -Q_B \frac{\sqrt{N_c}}{4\pi^2} \int dx_2 dk_2^2 \phi_B(s) \frac{m_1 x_2 + m_2 x_1 + 2(m_1 - m_2) k_2^2}{s - M_\rho^2} \\
\times \left\{ m_b x_b + m_u x_u + \frac{2(m_b - m_u) k_2^2}{M_B^2 - M_\rho^2} \right\},
\]

\[
\frac{2}{M_B} F_A^{(b)} = Q_B \frac{\sqrt{N_c}}{4\pi^2} \int dx_2 dk_2^2 \phi_B(s) \frac{m_1 x_2 + m_u x_u + 2(m_u - m_b) k_2^2}{s - M_\rho^2} \\
\times \left\{ m_b x_b + m_u x_u + \frac{2(m_u - m_b) k_2^2}{M_B^2 - M_\rho^2} \right\},
\]

with

\[
\frac{m_1^2}{x_1} + \frac{m_2^2}{x_2} + \frac{k_2^2}{x_1 x_2}.
\]
Let us recall that the $B$-meson decay constant has the following representation in terms of the wave function [10]:

$$f_B = \frac{\sqrt{N_c}}{4\pi^2} \int \frac{dx dk^2}{x_u x_b} \phi_B(s) \{m_u x_b + m_b x_u\}.$$  

(24)

Due to the wave function $\phi_B(s)$, the integral in the heavy quark limit is dominated by the region $x_u = \bar{A}/m_b$, $x_b = 1 - \bar{A}/m_b$, where $\bar{A}$ is a constant of order $M_B - m_b$. This leads to the following expansion of the form factors in the $1/m_b$ series:

$$\frac{2}{M_B} F_A^{(a)} = -Q_u \frac{f_b}{\bar{A} m_b} + \ldots$$

(25)

$$\frac{2}{M_B} F_A^{(b)} = Q_b \frac{f_b}{m_b^2} + \ldots.$$

Clearly, the dominant contribution in the heavy quark limit comes from the process when the light quark emits the photon, whereas the emission of the photon from the heavy quark gives only a $1/m_b$ correction. The expressions (25) for the form factor $F_A^{(a)}$ agree with the result of Ref. [7], while we have found a different sign for $F_A^{(b)}$.

**B. The form factor $F_V$**

The consideration of the form factor $F_V$ is very similar to the form factor $F_A$. $F_V$ is determined by the two diagrams shown in Fig. 4: Fig. 4(a) gives $F_V^{(b)}$, the contribution of the process when the $b$ quark interacts with the photon; Fig. 4(b) describes the contribution of the process when the quark $u$ interacts. One has

$$F_V = F_V^{(b)} + F_V^{(u)}.$$  

(26)

It is again convenient to change the direction of the quark line in the loop diagram of Fig. 4(b) describing $F_V^{(u)}$ by performing the charge conjugation of the matrix element. For the vector current $\gamma_\mu$ in the vertex the sign does not change (in contrast to the $\gamma_\mu\gamma_5$ case considered above).

Then both diagrams in Figs. 4(a) and 4(b) are reduced to the diagram of Fig. 5 which gives the form factor $F_V^{(1)}(m_1, m_2)$: Setting $m_1 = m_b$, $m_2 = m_u$ gives $F_V^{(b)}$: $F_V^{(b)} = Q_b F_V^{(1)}(m_b, m_u)$; setting $m_1 = m_u$, $m_2 = m_b$ gives $F_V^{(u)}$: $F_V^{(u)} = Q_u F_V^{(1)}(m_u, m_b)$. The trace corresponding to the diagram of Fig. 4 (1, active quark; 2, spectator) reads

$$\frac{2}{M_B} F_V^{(1)}(m_1, m_2) = -Q_u \frac{f_b}{\bar{A} m_b} + \ldots$$

(28)
The dominant contribution in the heavy quark limit again comes from the process when the light quark emits the photon. Now both form factors $F_{V}^{(u)}$ and $F_{V}^{(d)}$ in Eqs. (28) perfectly agree with the expansions obtained in [7].

As seen from Eqs. (25) and (28), one finds $F_{A} = F_{V}$ in the heavy quark limit, in agreement with the large-energy effective theory [13].

**IV. THE FORM FACTOR $G_{V}$**

The form factor $G_{V}$ is determined by the divergence of the matrix element of the charged current between the vacuum and the $\rho^{-}\gamma$ state,

\[ \langle \gamma(q_{1})\rho^{-}(q_{2})|\bar{d}\gamma_{\nu}\gamma_{5}u|0\rangle = eG_{V}\varepsilon_{q_{1}\nu}\varepsilon_{q_{2}}. \]  

The corresponding diagrams are shown in Fig. 6. If the classical equations of motion are applied, the form factor $G_{V}$ is proportional to the light-quark masses. This is the reason why this form factor was neglected in previous analyses [1,5]. However, a proper calculation shows that this argument is not correct: in fact the classical equations of motions do not hold and the divergence contains the anomaly.

The anomalous behavior of the divergence of the axial-vector current in the chiral limit is a well-known phenomenon discovered in the two-photon amplitude $\langle \gamma\gamma|\bar{d}\gamma_{\nu}\gamma_{5}u|0\rangle$ [14]. A very clear way to demonstrate the anomaly is to start with the matrix element of the axial-vector current and to calculate the spectral representations for the relevant form factors. The anomaly is then obtained by performing the divergence at the final stage of the calculation [15]. A similar treatment applied to the matrix element $\langle \gamma\rho^{-}|\bar{d}\gamma_{\nu}\gamma_{5}u|0\rangle$ leads to the form factor $G_{V}$ which does not vanish in the limit $m \rightarrow 0$. We will present a detailed discussion of the spectral representation in the variable $q_{2}^{2}$ and introducing the appropriate $\rho$-meson radial wave function $\psi_{\rho}(s)$, one finds the following expression for the form factor $G_{V}$:

\[ G_{V} = \sqrt{N_{c}}(Q_{u} + Q_{d}) \left[ -\frac{M_{B}^{2}}{4\pi^{2}} \int_{0}^{s} ds \frac{\psi_{\rho}(s)(s - M_{B}^{2})}{(s - M_{B}^{2} - i0)^{2}} \right]. \]  

Equation (30) describes the anomaly which takes place for the $\gamma\rho$ final state as well as for the $\gamma\gamma$ one. There is however an important difference between the two cases: For the $\gamma\gamma$ final state the divergence remains finite in the limit $m \rightarrow 0$ and $p^{2} = M_{B}^{2} \rightarrow \infty$. For the $\rho\gamma$ final state the divergence is finite for $m \rightarrow 0$ but decreases as $1/p^{2}$ for $p^{2} \rightarrow \infty$. It is convenient to introduce the parameter $\kappa$ such that

\[ G_{V} = (Q_{u} + Q_{d})\kappa \frac{M_{B}^{2}}{M_{B}^{2} - M_{B}^{2} - i0^{2}}, \]  

with $\kappa$ staying finite for $m = 0$ and $M_{B} \rightarrow \infty$.

**V. NUMERICAL ESTIMATES**

Let us write once more the penguin and the weak annihilation contributions to the $B^{-} \rightarrow \rho^{-}\gamma$ amplitude:

\[ F_{P}^{\text{penn}} = F_{P}^{\text{PC}} = -\xi_{i}C_{\gamma} \frac{m_{B}}{2\pi^{2}}T_{1}(0), \]

\[ F_{P}^{\text{NA}} = \xi_{i}a_{1}\frac{M_{B}}{M_{B}^{2} - M_{B}^{2}} \left[ \frac{2F_{A}}{M_{B}^{2}} + \frac{2f_{B}}{M_{B}^{2} - M_{B}^{2}} \right], \]

\[ F_{P}^{\text{NA}} = \xi_{i}a_{1}\frac{M_{B}}{M_{B}^{2} - M_{B}^{2}} \left[ \frac{2F_{V}}{M_{B}^{2}} - \frac{f_{B}G_{V}}{M_{B}^{2} - M_{B}^{2}} \right]. \]

The scaling behavior of the form factors in the limit $M_{B} \rightarrow \infty$ reads

\[ T_{1}(0) \sim M_{B}^{-3/2} [17], \quad F_{A,B} \sim M_{B}^{-1/2}, \quad G_{V} \sim M_{B}^{-2} \]  

such that

\[ F_{P}^{\text{penn}} \sim M_{B}^{-1/2}, \quad F_{P}^{\text{PC}} \sim M_{B}^{-3/2}. \]

The terms proportional to $f_{B}$ ($f_{B} \sim 1/\sqrt{M_{B}}$) are $1/M_{B}$-suppressed compared with the terms containing $F_{A,V}$. As we see below numerically this leads to a suppression by a factor of $4 \sim 5$.

We now proceed to numerical estimates for the $B$-meson decay. The scale-dependent Wilson coefficients $C_{i}(\mu)$ and $a_{1}(\mu)$ take the following values at the renormalization scale $\mu \approx 5$ GeV [8]:

\[ \frac{1}{8\pi^{2}} \int_{0}^{\infty} ds |\psi_{\rho}(s)|^{2} = 1. \]
TABLE I. Results for the weak-annihilation form factors $F_A$, $F_V$ and $G_V$ for the $B^-\rightarrow\rho^-\gamma$ decay. The accuracy of our estimates is about 10%. The sum rule results are recalculated from [6] according to the relation $F_{A,V} = -f_{A,V}/f_\rho^-$. The results from [7] are recalculated according to $F_{A,V} = -\frac{1}{2}f_{A,V}$ [7] for $\Lambda=0.5$ GeV.

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<td>$-F_A$</td>
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<tr>
<td>$-G_V$</td>
<td>&lt;0.004</td>
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</table>

$C_1 = 1.1, \quad C_2 = -0.241, \quad C_7 = -0.312,$

$\quad a_1 = C_1 + C_2/N_c \approx 1.02.$  \hspace{1cm} (36)

The penguin form factor was previously calculated within the dispersion approach with the result $T_1^{B^-\rightarrow\rho^-}(0) = 0.27 \pm 0.3$ [11]. Using the same parameters and the $B$ meson wave function as in [11] we obtain the form factors $F_{A,B}$ shown in Table I. Our result for the form factor $F_V$ is in good agreement with the estimates from other approaches. The form factor $F_A$ agrees well with the constraints from perturbative QCD and turns out to be considerably larger than the corresponding sum rule estimate.

The value of the $G_V$ is sensitive to the details of the $\rho$ meson wave function $\psi_\rho$. The reason for that is the presence of the term $(s-M_\rho^2)$ in the integrand in Eq. (30) which changes sign in the integration region. Assuming $\psi_\rho(s) = \beta^2/(\beta^2 + s)^2$ and setting $\beta=0.8$ GeV, which gives a good description of the $\rho$ meson radius, leads to $\kappa = -1.8$. Conservatively, we take $-\kappa < 2.0$ and use this value for further estimates. The form factor $G_V$ does not contribute more than a few % to the full amplitude, but can sizably correct the weak-annihilation part.

Using the obtained form factors and the decay constants $f_B=0.18$ GeV, $f_\rho=0.21$ GeV we arrive at the following estimates [we show contributions of different terms in Eq. (33) separately]:

$F_{\text{penguin}}^{\text{penguin}} = 20\xi_t, \quad \text{MeV},$

$F_{\text{WA}}^{\text{WA}} = (-7.8 \text{ (cont. of } F_A) + 2.2 \text{ (cont. of } f_B)) \xi_u, \quad \text{MeV}$

$= -5.6\xi_u, \quad \text{MeV}$  \hspace{1cm} (37)

$F_{\text{PC}}^{\text{WA}} = (-6.0 \text{ (cont. of } F_V) + 0.7 \text{ (cont. of } G_V)) \xi_u, \quad \text{MeV}$

$= -5.3\xi_u, \quad \text{MeV}.$

Taking into account errors in the form factors, for the ratios of the weak-annihilation to the penguin amplitudes we find

parity-violating: $F_{\text{WA}}^{\text{WA}}/F_{\text{penguin}}^{\text{penguin}} = -(0.28 \pm 0.025)\xi_u/\xi_t,$  \hspace{1cm} (38)

parity-conserving: $F_{\text{PC}}^{\text{WA}}/F_{\text{penguin}}^{\text{penguin}} = -(0.25 \pm 0.025)\xi_u/\xi_t.$

In the standard model $|\xi_u/\xi_t| \approx 0.4$.

For comparison, sum rules reported the ratio $F_{\text{WA}}^{\text{WA}}/F_{\text{penguin}}^{\text{penguin}} = -0.3\xi_u/\xi_t$ [6]. Our results for both $F_{\text{WA}}$ and $F_{\text{WA}}^{\text{penguin}}$ agree well with the sum rule estimates [6]. We would like to notice, however, that the anatomy of $F_{\text{WA}}$ in our analysis is different. For instance, we have found $F_A$ to be considerably larger than the sum rule result. But after including the contact term $-f_B$, we have come to $F_{\text{penguin}}^{\text{penguin}}$ close to the sum rule estimate.

The calculated form factors and the ratios of the weak-annihilation to the penguin amplitudes is one of the necessary ingredients for the calculation of the branching ratios of the $B^-\rightarrow\rho^-\gamma$ decays, isospin and $CP$ asymmetries. In addition to the above ratios, these quantities contain the phase induced by the strong interactions and the $CP$-violating phase of the CKM matrix (see [3,4] for details). The corresponding analysis was done recently in [3] using the value $F_{\text{WA}}^{\text{WA}}/F_{\text{penguin}}^{\text{penguin}} = -0.3\xi_u/\xi_t.$

VI. CONCLUSION

We have analyzed the weak annihilation for the radiative decay $B^-\rightarrow\rho^-\gamma$ in the factorization approximation.

(i) We have calculated the form factors $F_A$ and $F_V$ describing the photon emission from the $B$ meson loop within the relativistic dispersion approach. We have performed the $1/m_B$ expansion of the form factors and demonstrated that the form factors of the dispersion approach exhibit a behavior in agreement with the large-energy limit of QCD.

(ii) We have analyzed the contribution to the weak annihilation amplitude from the diagram when the photon is emitted from the loop containing only light quarks. For the parity-conserving process this quantity is related to the divergence of the axial-vector current

$$\langle \gamma(q_1)\rho^-\gamma(q_2)|\bar{\psi}_1\gamma_\mu\gamma_5\psi_2|0\rangle = e\epsilon_{q_1}\epsilon_{q_2}\epsilon_{\gamma_\mu}(Q_{u}+Q_{\bar{d}})\kappa \gamma_\mu M_{\rho}/M_B^2$$  \hspace{1cm} (39)

with $\kappa$ staying finite in the chiral limit $m\rightarrow0$. This result means the violation of the classical equations of motion and represents an anomaly which has the same origin as the anomaly of the matrix element $\langle \gamma(q_1)\gamma(q_2)|\bar{\psi}_1A_\mu|0\rangle$. The value of $\kappa$ is found to be sensitive to subtle details of the $\rho$-meson wave function. Conservatively, we estimate $|\kappa| \lesssim 1$.

(iii) We have also included contact terms which were missed in some of the previous analyses. Numerical estimates for the weak annihilation contribution to the $B^-\rightarrow\rho^-\gamma$ amplitude are given in Eq. (38).

It was noticed in [5] that the weak annihilation mechanism is crucial for the $D\rightarrow\rho\gamma$ decays in which it dominates over the penguin mechanism. It is therefore very important to take into account all the contributions to the weak annihilation amplitude listed above for a proper description of the rare radiative $D$ decays.
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APPENDIX: TRANSVERSITY OF THE PARITY-VIOLATING AMPLITUDE

The parity-violating amplitude (11) is given by the sum of the two terms

\[ A_{1}^{PV}(B^{-}\rightarrow \rho^{-})=A_{1}^{PV}+A_{2}^{PV} \]  

where \( A_{1}^{PV}= \frac{\rho^{-}(q_{2})}{\bar{d}}\gamma_{\mu}u|0\rangle\langle \gamma(q_{1})|u\gamma_{\nu}\gamma_{5}b|B^{-}(p)\rangle \) and \( A_{2}^{PV}= \frac{\rho^{-}(q_{2})}{\bar{d}}\gamma_{\gamma}q_{1}|0\rangle\langle u\gamma_{\nu}\gamma_{5}b|B^{-}(p)\rangle \).

(i) We start with \( A_{1}^{PV} \). Let us write

\[ \langle \gamma(q_{1})|u\gamma_{\nu}\gamma_{5}b|B^{-}(p)\rangle = e_{1}^{\mu}p_{\rho}^{\nu}T_{\mu,\nu}^{B}(p,q) \]

where \( J_{\mu}(\gamma) = \frac{1}{2}u\gamma_{\mu}u - \frac{1}{2}d\gamma_{\mu}d - \frac{1}{2}b\gamma_{\mu}b \) is the electromagnetic quark current. The amplitude has the following Lorentz structure:

\[ T_{\mu,\nu}^{B}(p,q) = i \int dxe^{\mu\nu}(0|T(\gamma_{\mu}(x),\bar{u}\gamma_{\nu}\gamma_{5}b)|B^{-}(p),) \]

where \( C \) is the contact term. The contact term can be determined using the conservation of the electromagnetic current \( \partial_{\mu}J_{\mu} = 0 \), which leads to the relation

\[ q_{\mu}T_{\mu,\nu}^{B}(p,q) = -\langle 0||\bar{Q}_{B}(\bar{d}\gamma_{\nu}\gamma_{5}u)|B^{-}(p)\rangle \]

\[ = -if_{B}p_{\nu} \]  

for the \( B^{-} \) meson. This gives \( C = -f_{B} \). For the radiative decay \( q_{1}^{\mu} = 0 \), we find

\[ A_{1}^{PV} = i\epsilon_{\mu}^{\rho}p_{\mu}^{\nu}\epsilon_{\mu}^{\sigma}p_{\sigma}^{\pi}(g_{\rho\sigma}p_{\pi} - p_{\rho}q_{1\pi})F_{1A}(0) \]

\[ -p_{\rho}^{\nu}\left( \frac{2p_{B}}{M_{B}^{2} - M_{\rho}^{2}} \right) \]

\[ = i\epsilon_{\mu}^{\rho}p_{\mu}^{\nu}\epsilon_{\mu}^{\sigma}p_{\sigma}^{\pi}(g_{\rho\sigma}p_{\pi} - p_{\rho}q_{1\pi})F_{1A}(0) \]

\[ -p_{\rho}^{\nu}\left( \frac{2f_{B}}{M_{B}^{2} - M_{\rho}^{2}} \right) \]  

(A5)

(ii) Using the equation of motion for the quark fields \( Q_{q} = 2/3e, Q_{d} = -1/3e \)

\[ i\gamma_{\nu}\partial_{\nu}q(x) = m_{q}q(x) - Q_{q}A \gamma_{\nu}q(x), \]

\[ i\partial_{\nu}\bar{q}(x)\gamma_{\nu} = -m_{\bar{q}}(x) + Q_{\bar{q}}A \bar{q}(x)\gamma_{\nu}. \]  

(A6)

one obtains for \( A_{2}^{PV} \)

\[ A_{2}^{PV} = i\epsilon_{\mu}^{\rho}p_{\mu}^{\nu}\epsilon_{\mu}^{\sigma}p_{\sigma}^{\pi}(g_{\rho\sigma}p_{\pi} - p_{\rho}q_{1\pi}) \]

\[ = -\langle 0||Q_{B}(\bar{d}\gamma_{\nu}\gamma_{5}u)|0\rangle \]

\[ = e_{1}^{\mu}p_{\mu}^{\nu}f_{B}p_{\mu}M_{B}^{2} + O(m_{u}, m_{d}). \]  

(A7)

(iii) For the sum we find

\[ A_{1}^{PV} + A_{2}^{PV} = i\epsilon_{\mu}^{\rho}p_{\mu}^{\nu}\epsilon_{\mu}^{\sigma}p_{\sigma}^{\pi}(g_{\rho\sigma}p_{\pi} - p_{\rho}q_{1\pi}) \]

\[ \times \left[ F_{1A}(0) + \frac{2f_{B}}{M_{B}^{2} - M_{\rho}^{2}} \right] \]  

(A8)

and obtain the relation (14) after setting \( F_{1A} = 2F_{A}/M_{B} \).

(iv) In the general case, one obtains

\[ A^{PV}(B^{-}\rightarrow \rho^{-}\gamma) = i\epsilon_{\mu}^{\rho}p_{\mu}^{\nu}\epsilon_{\mu}^{\sigma}p_{\sigma}^{\pi}(g_{\rho\sigma}p_{\pi} - p_{\rho}q_{1\pi}) \]

\[ \times \left[ \frac{2F_{A}^{2}}{M_{B}^{2} - M_{\rho}^{2}} \right] \]  

(A9)

where \( Q_{B} \) is the charge of the \( B \) meson. Hence, for the \( B^{0} \) \( \rightarrow \rho^{0}\gamma \) decay the contact term is absent. Notice also that \( F_{A}^{\rho} = -F_{A}^{B} \) as can be obtained by the charge conjugation of the parity-violating amplitude \( A^{PV} \).
