Some hints on solving differential equations

Dmitri Melikhov

I give hints on solving differential equations in some special cases.

I. FIRST-ORDER DIFFERENTIAL EQUATIONS

The general form of first-order equation is

$$\frac{dN}{dt} = F(N,t), \tag{1}$$

where F is an arbitrary function of two variables $N(t)$ and t. To make solution of this equation unique we must add the 'initial' condition

$$N(t=0) = N_0. \tag{2}$$

In some cases (i.e. for some special functions F) it is possible to formulate an algorithm for solving Eq. (1).

A. A separable function $F(N,t) = f_1(N)f_2(t)$

$$\frac{dN}{dt} = f_1(N)f_2(t). \tag{3}$$

This equation may be solved by separating variables. Namely, we write

$$\frac{dN}{f_1(N)} = f_2(t)\cdot dt. \tag{4}$$

Integrating this equation we find

$$\int \frac{dN}{f_1(N)} = \int f_2(t)\cdot dt. \tag{5}$$

Don't forget to add a constant C when calculating the integral.

B. Linear first-order equation with coefficient functions depending on t

$$\frac{dN}{dt} + b(t)N = f(t). \tag{6}$$

First step:
Find solution of the homogeneous equation (i.e. with zero right-hand side)

$$\frac{dn}{dt} + b(t)n = 0. \tag{7}$$

This equation may be solved by separating variables. The result reads

$$\frac{dn}{n} = -b(t)dt. \tag{8}$$

Integration of this equation gives

$$\log(|n|) = -\int b(t)dt + \tilde{c}. \tag{9}$$

Exponentiating this equation, we get

$$|n(t)| = exp(\tilde{c})exp\{-\int b(t)dt\}. \tag{10}$$

Equaivalently, we can write this equation in the form

$$n(t) = n_0\ \exp\left(-\int_0^t b(\tau)d\tau\right), \tag{11}$$

where now

$$n_0 = n(t=0). \tag{12}$$

Second step:
Seek solution to Eq. (7) in the form

$$N(t) = C(t)n(t), \tag{13}$$

where $n(t)$ is given by (11) and $C(t)$ is to be further determined from the differential equation. To obtain this differential equation for $C(t)$ let us insert our Ansatz (13) into (7). Take into account that

$$\frac{dN}{dt} = \frac{dC}{dt}n + C\frac{dn}{dt}, \tag{14}$$

and rewrite Eq (6) as

$$\frac{dC}{dt}n + C\left\{\frac{dn}{dt} + b(t)n\right\} = f(t). \tag{15}$$

Using (7), the expression in curly brackets vanish and we find

$$\frac{dC(t)}{dt}n(t) = f(t). \tag{16}$$

It is now clear what was the reason to seek solution in the form $N(t) = C(t)n(t)$ where $n(t)$ is the solution of the homogeneous equation: terms proportional to C cancel each other. The final equation contains only dC/dt and the known functions $f(t)$ and $n(t)$. So

$$C(t) = \int_0^t d\tau\ \frac{f(\tau)}{n(\tau)} + \bar{C}_0. \tag{17}$$

Gathering all things together we find

$$N(t) = C(t)n(t) = \left[\bar{C} + \int_0^t d\tau\frac{f(\tau)}{n(\tau)}\right]n(t) = \bar{C}\ n(t) + \int_0^t d\tau f(\tau)\frac{n(t)}{n(\tau)}. \tag{18}$$

Now take into account that (see Eq. (11)

$$n(t)/n(\tau) = \exp\left(-\int_\tau^t b(\tau')d\tau'\right), \tag{19}$$

(the constant c drops out from this relation). Using (19), we find for $N(t)$

$$N(t) = C_0 \cdot \exp\left(-\int_0^t b(\tau)d\tau\right) + \int_0^t d\tau f(\tau)\exp\left(-\int_\tau^t b(\tau')d\tau'\right). \tag{20}$$

The constant $C_0 = c\bar{C}$ can be determined by setting $t = 0$ in both sides, which leads to the relation

$$C_0 = N(t=0) = N_0.$$

Notice the following property of the solution (20): the first term is the (general) solution of the homogeneous equation [which does not know anything about the function $f(t)$], and the second term is a special solution of the inhomogeneous equation which depends on $f(t)$.

II. SECOND-ORDER DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Let us consider the following inhomogeneuos equation

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \beta x(t) = f(t) \tag{21}$$

There are also two initial conditions

$$x(t=0) = x_0 \tag{22}$$

$$\frac{dx}{dt}(t=0) = v_0. \tag{23}$$

If t is time, and $x(t)$ is coordinate, then $v(t) = dx/dt$ is velocity, and $a(t) = d^2x/dt^2$ is acceleration. The initial conditions are coordinate and velocity at $t = 0$.

The solution of Eq. (21) is the sum of the general solution of the homogeneous equation and special solution of the inhomogeneous equation.

Our question now is: how to find this special solution?

There is a general algorithm for this based on the **Fourier transform**.

The Fourier transform

Namely, for any function $x(t)$ we define the function $\hat{x}(\omega)$ as follows

$$\hat{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} x(t) dt. \tag{24}$$

Then $x(t)$ is related to its 'image' $\hat{x}(\omega)$ as follows

$$x(t) = \frac{1}{\sqrt{2\pi}} \int e^{-i\omega t} \hat{x}(\omega) d\omega. \tag{25}$$

The Fourier transform for derivatives of the function $x(t)$ can be easily found

$$x(t) \to \hat{x}(\omega); \tag{26}$$

$$\frac{dx(t)}{dt} \to -i\omega\hat{x}(\omega); \tag{27}$$

$$\frac{d^2x(t)}{dt^2} \to (-i\omega)^2\hat{x}(\omega) = -\omega^2\hat{x}(\omega). \tag{28}$$

Using the Fourier transform the differential equation for $x(t)$ is reduced to the algebraic equation for $\hat{x}(\omega)$:

$$-\omega^2 \cdot \hat{x}(\omega) - i\alpha\omega \cdot \hat{x}(\omega) + \beta \cdot \hat{x}(\omega) = \hat{f}(\omega), \tag{29}$$

where $\hat{f}(\omega)$ is the tranform of the function $f(t)$. The solution of tbis equation is simple

$$\hat{x}(\omega) = \frac{\hat{f}(\omega)}{-\omega^2 - \alpha i\omega + \beta}. \tag{30}$$

We can now reconstruct $x(t)$ using (25). The final result reads

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{-\omega^2 - i\alpha\omega + \beta} \int_{-\infty}^{\infty} d\tau \cdot f(\tau)\, e^{i\omega\tau}. \tag{31}$$

III. DIFFERENTIAL EQUATIONS AND CONSERVATION LAWS

Let us consider the following special case: namely, let our system be described by the differential equation

$$m\frac{d^2x}{dt^2} = f(x), \tag{32}$$

where $x(t)$ is coordinate, d^2x/dt^2 is acceleration, m is mass and $f(x)$ is force.

Notice the two essential features of this equation:

1. There is no term proportional to velocity $v = dx/dt$ in the l.h.s. (left-hand side)
2. The 'force' on the r.h.s. (right-hand side) *explicitly* depends only on coordinate, and does not depend *explicitly* on time. (Of course, force depends on time *implicitly*, because x depends on time. But it is importnat for us that there is no explicite dependence).

It is quite a general dynamical equation for a system without dissipation (which corresponds to linear term in v). In other words, there is no *kinetic* friction and no viscosity in our system.

It is convenient to introduce a potential by the following relation

$$f(x) = -\frac{dU(x)}{dx}. \tag{33}$$

Then our initial equation takes the form

$$m\frac{d^2x}{dt^2} + \frac{dU(x)}{dx} = 0. \tag{34}$$

Let us now multiply both sides of this equation by the factor $v(t) = dx/dt$, and take into account that $a(t) = \frac{d^2x}{dt^2} = \frac{dv}{dt}$. Then we find

$$m \cdot v\frac{dv}{dt} + \frac{dU(x)}{dx}\frac{dx}{dt} = 0. \tag{35}$$

Notice that

$$m \cdot v\frac{dv}{dt} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) \tag{36}$$

$$\frac{dU(x)}{dx}\frac{dx}{dt} = \frac{d}{dt}U(x(t)). \tag{37}$$

Therefore the initial equation (32) takes the form

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) + \frac{d}{dt}U(x(t)) = 0, \tag{38}$$

that means[1]

$$\frac{mv^2}{2} + U(x) = \text{const} = E. \tag{39}$$

The last equation is just the *energy conservation*: in a system without dissipation and without external time-dependent forces the sum of the kinemtic and potential energy is conserved.

Energy is the integral of our equation of motion. From the initial condition we find

$$E = \frac{mv_0^2}{2} + U(x_0). \tag{40}$$

[1] In mathematics energy is called in this case the 'integral of the equation of motion', and v is called the 'integrating multiplier'.

We can proceed further and find $x(t)$ from the equation

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U(x))}. \tag{41}$$

The latter can be solved by separating variables

$$\frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = dt. \tag{42}$$

Integrating this equation we find

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}\{E - U(x)\}}} = t. \tag{43}$$

Here we have taken into account that $x = x_0$ for $t = 0$.

Attention: One should look carefully at points where the direction of motion might be reversed!

The candidate points are those for which $v = 0$ or equivalently

$$E - U(x) = 0.$$

This *might be* a signal that the direction of motion is reversed. But not necessarily!

The final conclusion depends on details of the potential.

Now, what happens if there is a linear term in the differential equation?

Let us consider equation of the form

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \frac{dU}{dx} = 0, \tag{44}$$

where $r > 0$.

By the same trick as before we find

$$\frac{d}{dt}\left(\frac{mv^2}{2} + U(x)\right) = -r \cdot v^2 < 0. \tag{45}$$

Therefore the energy in this case is **not** conserved and **decreases** with time.

One can find out that energy is also **not** conserved if one applies *external* force which depends *explcitly* on time.[2]

[2]We speak in these cases about the energy of an object which moves in a dissipative medium (say in a viscous liquid) or under an external time-dependent force. If we include into consideration the source of this external force, the medium in which the particle moves, etc, then the full energy of this broader system is conserved.