Strongly Correlated Bosons

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Definitions

1. Bosonic Hubbard model

$$H = \sum_{x,y} t_{x,y} b_x^\dagger b_y + \sum_x U_x b_x^\dagger b_x^\dagger b_x b_x$$
$$[b_x^\dagger, b_y] = \delta_{x,y}$$

2. Graph Theory

- Graph G: Set of Vertices $V(G)$ and edges $E(G)$. $e \in E(G), e = \{x, y\}, x, y \in V(G)$
- Walk $w = (x_1, e_1, \cdots, x_{n-1}, e_{n-1}, x_n), e_i = \{x_i, x_{i+1}\}$
- Cycle: Closed, self avoiding walk i.e. $x_1 = x_n$ and $x_i \neq x_j$ for $i \neq j$
- G connected: $\forall x, y \in V(G)$ exists a self avoiding walk from x to y
- G 2-connected: There is no edge, so that G decays into two unconnected subgraphs, if that edge is removed.
- G bibartite: $V(G) = V_1(G) \dot{\bigcup} V_2(G)$ so that each edge joins a vertex of V_1 to a vertex of V_2
- f: Bounded faces of G
- G planar: Can be drawn in the plane so that no edges intersect
- Linegraph $L(G)$ of G: Graph whose vertex set is the edge set $E(G)$ of G. Two vertices $e, e' \in E(G)$ of $L(G)$ are joined by an edge, if $|e \cap e'|$.
- Incidence matrix: $B = (b_{xe})_{x \in V(G), e \in E(G)} = \begin{cases} 1: & x \in e \\ 0: & \text{else} \end{cases}$
- From now on let G be a bibartite, planar and 2-connected graph.
- Consider the Hubbard model on $L(G)$

3. Hubbard Model on $L(G)$

$$H = \sum_{e,e'} t_{e,e'} b_e^\dagger b_{e'} + \sum_e U_e b_e^\dagger b_e^\dagger b_e b_e$$
$$t_{e,e'} = t \sum_{x \in V(G)} b_{xe} b_{xe'}$$
$$e, e' \in E(G)$$
$$U_e,\ t > 0$$

Construction of single particle states

- $B^T B$ is pos. semidefinite → non-negative eigenvalues.
- Bipartite, 2-connected graph → $B^T B$ has 0 as eigenvalue with multiplicity $|E(G)| - |V(G)| + 1$
- Orientation of faces: Boundary of face orientated clockwise
- Orientation of edges: From V_1 to V_2 (bipartite graph)
- Let f be a bounded face and $C(f)$ be the boundary cycle
- Define

$$S = (s)_{f \in F(G), e \in E(G)} = \begin{cases} 1 : & e \in C(f), e, f \text{ same orientation,} \\ -1 : & e \in C(f), e, f \text{ different orientation,} \\ 0 : & \text{else.} \end{cases}$$

$$BS^T = 0$$

$$dim(ker(B)) = |F(G)|$$

$$\Rightarrow \text{Columns of } S^T \text{ form basis of } ker(B)$$

$$b_f^\dagger := \sum_{e \in E(G)} s_{fe} b_e^\dagger$$

$$\Rightarrow b_f^\dagger \left|0\right\rangle \text{ is single particle ground state with energy 0}$$

- $b_f^\dagger \left|0\right\rangle$ localized only on the enclosing cycle of f

Construction of multi particle states

- Cycle set C: $\{c_i, i = 1, \ldots, N\}$, c_i: edge disjoint cycles
- Contraction C' of C: $F(c_i') \subset F(c_i) \forall i$ and $\cup_i F(c_i') \subset \cup_i F(c_i)$
- Define multi particle states via cycle sets:

$$\left|\Phi(C)\right\rangle = O^\dagger(C) \left|0\right\rangle$$

$$O^\dagger(C) := \prod_i \sum_{f \in F(c_i)} b_f^\dagger$$

- Basis of ground states for small enough densities:

  Theorem 1. *The states $\left|\Phi(C^{(u)})\right\rangle$ belonging to uncontractible cycle sets form a basis of the Fock space F_0 of the kernel of H.*

  Proof. See Theorem of Mortruk and Mielke [2] □

- Critical density reached, when the graph is close packed with uncontractible cycles.
- For higher densities, the construction above is no further applicable.

Exceeding the critical density

- Hard to get exact results. $\rightarrow$ variational methods and numeric calculation.
- Simplifications
  - Weak interaction: Construction of Wannier Basis, Mean-field treatment
  - Strong interaction: Hardcore limit $U \rightarrow \infty$
- Analytic result for 1-D systems: Pair formation. See Mielke [5]
- Variational results: Pair formation of localized pairs in some broad class of graphs.

References

[1] Drescher, M., Mielke, A.: Hard-core bosons in flat band systems above the critical density. Eur. Phy B90, 217 (2017)

[2] Motruk, J., Mielke, A.: Bose-Hubbard model on two-dimensional line graphs. J. Phys. A 45(22), 225,206 (2012)

[3] Pudleiner, P., Mielke, A.: Interacting bosons in two-dimensional flat band systems. Eur. Phys. J. B 88, 207 (2015)

[4] S.D. Huber, E. Altman, Phys. Rev. B 82, 184502 (2010)

[5] Mielke, A.: Pair Formation of Hard Core Bosons in Flat Band Systems. J. Stat. Phys. 171, 679-695 (2018)