Anderson Localization

Seminar Talk

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This is the summary of a seminar talk given in the Statistical Physics seminar by Prof. Mielke at Heidelberg University. This talk is meant to give a basic introduction to the localization phenomena which appear when studying disordered systems. We will talk about some notable results and will try to demonstrate in detail why localization appears in disordered systems in the one-dimensional case. In the second part we make some remarks on the scaling behaviour of conductivity and localization, and show the existence of a metal-insulator phase-transition in three dimensions.

Contents

1 Introduction

Calculations in solid state physics tend to heavily make use of the symmetry or translation invariance of systems to calculate various properties. The most prominent example is probably that for a periodic potential one finds that all solutions of the Schrödinger equation take the form of Bloch waves, i.e. they can be written as a periodic part times an exponential phase factor. These solutions are obviously also highly symmetric and extended over the whole solid. Anderson Localization is essentially what appears when one drops these symmetry and ordering requirements.

In this talk we will be focussing on the tight-binding model on a lattice in d-dimensions. We associate a state $|l\rangle$ with each lattice point, and allow for next-neighbour hopping with hopping strength t. The Hamiltonian then takes the form

$$H = \sum_l \varepsilon_l \, |l\rangle\langle l| + \sum_{\langle l,m\rangle} t_{lm} \, |l\rangle\langle m| \tag{1.1}$$

For now we will assume the lattice to be finite, with edge length L, but generally we are interested in the limit $L \to \infty$. For this introduction let us revisit the results one finds for a perfectly ordered system. We will set $\varepsilon_l = \varepsilon = const$ and $t_{lm} = t = const$. Since we are free to set the zero point of our energy scale, we will assume $\varepsilon = 0$. This makes the Hamiltonian very easy to solve. We will demonstrate this for d = 1, but the generalization to higher dimensions is straightforward.

With our simplifications the Hamiltonian now reads:

$$H = \sum_l t \, |l\rangle\langle l+1| + h.c. \tag{1.2}$$

we solve this by introducing plane-wave states:

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_l \exp(ikl) \, |l\rangle \tag{1.3}$$

$$|l\rangle = \frac{1}{\sqrt{L}} \sum_k \exp(-ikl) \, |k\rangle \tag{1.4}$$

We insert (1.4) into (1.2) and get

$$H = \frac{t}{N} \sum_l \sum_{k,k'} \exp(-ikl) \exp\big(ik'(l+1)\big) \, \big|k\big\rangle\big\langle k'\big| + h.c. \tag{1.5}$$

$$= \frac{t}{N} \sum_{k,k'} \exp\big(ik'\big) \sum_l \exp\big(-il(k-k')\big) \, \big|k\big\rangle\big\langle k'\big| + h.c. \tag{1.6}$$

Making use of the relation

$$\sum_l \exp\big(-il(k-k')\big) = N\delta_{kk'} \tag{1.7}$$

this simplifies to

$$H = t \sum_k \exp(ik) \, |k\rangle\langle k| + h.c. \tag{1.8}$$

$$= 2t \sum_k cos(k) \, |k\rangle\langle k| \tag{1.9}$$

Therefore our hamiltonian is diagonal in k-space with the eigenvalues $2t\cos(k)$. This implies that the total spectrum is the interval $[-2t, 2t]$. As mentioned previously the eigenstates are plane waves and therefore not localized in any way.

Anderson showed in his famous paper in 1958 [1] that these two properties of the solution (continuous spectrum and non-localized eigenfunctions) will cease to be the case for (sometimes even only slightly) disordered systems. He introduced disorder by adding a random contribution to the on-site energies of the hamiltonian, which he argued to be equivalent to adjusting the hopping elements.

Anderson's results have lead to intensive research into the subject of disordered systems in the last fifty years, bringing up a variety of interesting results ([2], [3]).
In this talk I want to give a brief overview and introduction to localization phenomena in disordered systems. I will start of by giving a brief (and highly selective) overview over some notable theorems which have been proven rigorously.

2 Notable Rigorous results

The Anderson Model

Before we actually come to the theorems, we have to make precise what system we are looking at. All subsequent results apply to the d-dimensional tight binding Hamiltonian with random on-site energies. Precisely:
We are looking at a d-dimensional lattice $\mathbb{Z}^d$ with lattice positions $l \in \mathbb{Z}^d$ and corresponding orthonormal states $|l\rangle$. Our Hamiltonian shall contain on-site energies which are statistically independent between sites. For this talk we want to assume, that the energies are uniformly distributed in an interval $[-\lambda, \lambda]$. Formally:

$$P(\{\varepsilon_l\}) = \prod_l P(\varepsilon_l) \tag{2.1}$$

with

$$P(\varepsilon_l) = \frac{1}{2\lambda} \mathbf{1}_{[-\lambda,\lambda]} \tag{2.2}$$

Then the Hamiltonian acts as:

$$H = \sum_l \varepsilon_l \, |l\rangle\langle l| + \sum_{\langle l,m \rangle} |l\rangle\langle m| \tag{2.3}$$

Compared to 1.1 we have set the hopping strength t to 1, which we can do by just rescaling all of our energies.

The following is a (of course highly selective) list of notable result which have been proven rigorously. Proofs of these theorems can be found in [4] and [5].

Theorem 1. *(Localization for high disorder)*
$\exists \lambda_0(d): \quad \forall \lambda > \lambda_0:$
H has discrete spectrum and all eigenfunctions are exponentially localized in the sense that:
For eigenfunctions $\varphi : \mathbb{Z}^d \to \mathbb{C}$ with:

$$H\varphi = E\varphi \tag{2.4}$$

it holds

$$\big|\varphi(x)\big| \leq A \exp\big(-k|x - x_0|\big) \tag{2.5}$$

for some $A, k > 0$, $x_0 \in \mathbb{Z}^d$ with k independent of the realization of $\{\varepsilon_l\}$.

Theorem 2. *(Localization for high energies)*
For arbitrary λ, $\exists U(d, \lambda)$ so that in the region $|E| > U$ the spectrum of H is discrete and the eigenfunctions are localized as in theorem 1.

It should be noted that the theorem does not imply that there exist any eigenfunctions in the given energy region.

Theorem 3. *For $d = 1$ and $d = 2$ theorem 1 holds with $\lambda_0 = 0$.*

I.e. for $d \leq 2$ localization appears even for arbitrarily small disorder. In this talk I want to give some motivation for some of these effects, which may seem quite surprising after all. Therefore in the following section I want to give a somewhat heuristic proof of exponential localization for arbitrary disorder in one dimension.

3 The one-dimensional case

The following is largely based on the respective sections dealing with the one-dimensional case in [6] and [7].
We rewrite our Hamiltonian for the Anderson model (2.3) in one dimension as follows:

$$H = \sum_l \varepsilon_l \, |l\rangle\langle l| + |l\rangle\langle l+1| + |l\rangle\langle l-1| \tag{3.1}$$

The basic procedure will be to get conditions on the possible eigenstates of this Hamiltonian and then argue that all physically sensible eigenstates are localized. For a general state

$$|\varphi\rangle = \sum_l a_l \, |l\rangle \tag{3.2}$$

we require

$$H \, |\varphi\rangle = E \, |\varphi\rangle \tag{3.3}$$

and thereby get:

$$\sum_l (\varepsilon_l a_l + a_{l+1} + a_{l-1}) |l\rangle = E \sum_l a_l |l\rangle \tag{3.4}$$

This gives us a way to iteratively compute a_{l+1} given a_l and a_{l-1}:

$$a_{l+1} = (E - \varepsilon_l)a_l - a_{l-1} \tag{3.5}$$

We can rewrite this as a Matrix equation:

$$\begin{pmatrix} a_{l+1} \\ a_l \end{pmatrix} = \begin{pmatrix} E - \varepsilon_l & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_l \\ a_{l-1} \end{pmatrix} \tag{3.6}$$

The key point here is that this allows for a convenient way to write the iterative application of the relation (3.5). If we define

$$T_l \coloneqq \begin{pmatrix} E - \varepsilon_l & -1 \\ 1 & 0 \end{pmatrix} \tag{3.7}$$

then we have:

$$\begin{pmatrix} a_{l+1} \\ a_l \end{pmatrix} = T_l \cdot T_{l-1} \cdot \ldots \cdot T_1 \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \tag{3.8}$$

We want to show exponential localization of all possible eigenstates. Therefore we would like to see a behaviour similar to:

$$\begin{pmatrix} a_{l+1} \\ a_l \end{pmatrix} \propto \exp(-\gamma l) \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \tag{3.9}$$

at least in the limit $l \to \infty$. Therefore let us look at the following quantity:

$$\lim_{l \to \pm\infty} \frac{1}{|l|} \log\big(\|T_l \cdot \ldots \cdot T_1\|\big) \tag{3.10}$$

Where we used the operator norm:

$$\|A\| = \sup_{|x| \neq 0} \frac{|Ax|}{|x|} \tag{3.11}$$

for some given norm $|\cdot|$ on $\mathbb{R}^2$. For a precise treatment one has to deal with the following two questions: does this series converge, and is the limit independent of the specific realization of $\{\epsilon_l\}$? Before referencing the precise statement, we want to give heuristic motivation why the answer to both of these questions is yes. Considering convergence we will prove at least the boundedness of the given series. We have:

$$\log\big(\|T_2 \cdot T_1\|\big) \leq \log\big(\|T_2\|\|T_1\|\big) = \log\big(\|T_2\|\big) + \log\big(\|T_2\|\big) \tag{3.12}$$

Since the ε_l are bounded, the same holds for $\|T_l\|$. The constant entries in (3.7) also set a minimal bound on $\|T_l\|$ and therefore also $\log\|T_l\|$ is bounded. Since we divide by $\frac{1}{|l|}$ in (3.10) therefore also the given series (3.10) is bounded.

Considering the second question on whether the given limit is independent of the specific choice of $\{\varepsilon_l\}$, this can be shown by making use of an ergodic theorem. Without going into detail, the given limit corresponds to some sort of "time-average" which can be replaced by an ensemble average using the ergodic theorem.

All this is made precise in the following Theorem of Fürstenberg.

Theorem 4. *For the given probability distribution of $\{\varepsilon_l\}$ the limit*

$$\gamma(E) \coloneqq \lim_{l\to\pm\infty} \frac{1}{|l|} \log\big(\|T_l \cdot \ldots \cdot T_1\|\big) > 0 \tag{3.13}$$

exists with probability 1 and is independent of the realization of $\{\varepsilon_l\}$.

Since $\gamma > 0$ in theorem 4, this does not prove localization, and might seem of little use as of now. However, we did not yet take into account the dependency on the vector $\binom{a_1}{a_0}$. One expects the upper bound given by the operator norm in (3.13) to be strict for some initial vector $u \in \mathbb{R}^2$ (u represents what we called $\binom{a_1}{a_0}$ before). The question is what happens if we set u to the other "direction" in $\mathbb{R}^2$. One finds the following result:

Theorem 5.

$$\exists u_\pm(E, \{\varepsilon_l\}) \in \mathbb{R}^2 : \tag{3.14}$$

$$\lim_{j\to\infty} \frac{1}{|l|} \log\big(|T_l \cdot \ldots \cdot T_1 u_+|\big) = -\gamma(E) \tag{3.15}$$

$$\lim_{j\to\infty} \frac{1}{|l|} \log\big(|T_l \cdot \ldots \cdot T_1 u|\big) = +\gamma(E) \quad \textit{for } u \notin L(u_+) \tag{3.16}$$

$$\lim_{j\to-\infty} \frac{1}{|l|} \log\big(|T_l \cdot \ldots \cdot T_1 u_-|\big) = -\gamma(E) \tag{3.17}$$

$$\lim_{j\to-\infty} \frac{1}{|l|} \log\big(|T_l \cdot \ldots \cdot T_1 u|\big) = +\gamma(E) \quad \textit{for } u \notin L(u_-) \tag{3.18}$$

where $L(u)$ denotes the linear subspace spanned by the vector u.

Loosely speaking there exist two "directions" of initial vectors which will lead to exponentially decaying eigenfunctions in the two limits $j \to \pm\infty$ respectively. These directions will depend on the energy E for which we are looking for an eigenfunction. If the two vectors $u_\pm$ happen to coincide (or be linearly dependent) this will define a starting vector $\binom{a_1}{a_0}$ which through application of (3.8) defines an eigenfunction which is indeed exponentially localized. In the other case for which u_+ and u_- are not scalar multiples of each other, theorem 5 implies that for all vectors u the solution will be exponentially increasing for either $l \to \pm\infty$. Physically it is quite obvious that this will not lead to normalizable and physically sensible solutions. Even though normalizability will not be the necessary criterion in the end (otherwise plane waves would not be a

physical solution as well), a proper treatment using spectral theory on infinitely dimensional vector spaces shows that in the given case the unbounded solutions are indeed not eigenfunctions in the correct sense.

The discreteness of the spectrum follows directly from the above considerations. It is only for specific (discrete) values of E that $u_\pm$ will coincide and indeed yield valid solutions.

4 Conductivity and Scaling

In the second part of this talk I want to talk about the so-called "scaling argument" ([6], [2]) which demonstrates the existence of a metal-insulator phase transition in three dimensions. For that, we will have to take a look at the conductivity of our lattice. For the following section let us review the definitions of conductivity and how it relates to some other quantities.

Central in most practical cases is the resistance R of a solid. The conductance G is defined as its inverse:

$$G = \frac{1}{R} \tag{4.1}$$

Both these quantities depend on the material as well as the shape of the object. For a cuboid with surface A and length l (in the direction of flowing electricity) the following relation holds:

$$R = \rho\frac{l}{A} \Rightarrow G = \sigma\frac{A}{l} \tag{4.2}$$

with $\sigma = \frac{1}{\rho}$, the conductivity, a material constant.

For the following let us assume that our system is a perfect cube with side length L. (4.2) can then be rewritten as:

$$G = \sigma L^{d-2} \tag{4.3}$$

The conductivity σ can actually be explicitly calculated if one knows all the eigenstates $|\alpha\rangle$ of the system. For an electric field with frequency ω applied in the z-direction, and a system with temperature T and size L^d one finds (the so-called "Kubo-Formula"):

$$\sigma(T,\omega,L) = \frac{\pi e^2}{m^2\omega L^d}\sum_\alpha\sum_{\beta\neq\alpha}\left|\langle\alpha|p_z|\beta\rangle\right|^2[f(E_\alpha,T)-f(E_\beta,T)]\delta(E_\alpha-E_\beta-\hbar\omega) \tag{4.4}$$

Here $f(E,T)$ denotes the Fermi-function and p_z is the momentum-operator in z-direction. Explicit computation here yields that for localized eigenstates

$$\sigma(T=0,\omega=0,L\to\infty) = 0 \tag{4.5}$$

i.e. in the limit of an infinitely large system the dc-conductivity vanishes at zero temperature.

To stay in line with the literature we will, in the following, look at the dimensionless conductance $g = \frac{\hbar}{e^2}G$, however this does not heave any meaning for what we do here.

We now want to investigate the scaling behaviour of the conductance with respect to the system size L. In the end we are always interested in the limit $L \to \infty$. Just for convenience we add a logarithm to both g and L and thereby arrive at the quantity:

$$\frac{\mathrm{d}\ln(g)}{\mathrm{d}\ln(L)} \tag{4.6}$$

This can in general be a very complicated function which also depends on various other system parameters. However, we are going to make the assumption that the whole scaling behaviour can be described by only one scaling-parameter which we can choose to be the conductance itself. This is an assumption which will turn out not to be true in quite some cases, however it allows for an easy (and sometimes also quantitatively correct) description of actually visible phenomena. With this (4.6) becomes:

$$\frac{\mathrm{d}\ln(g)}{\mathrm{d}\ln(L)} =: \beta(g) \tag{4.7}$$

We are now going to calculate $\beta(g)$ in the limit of large and small g.
In the limit of large g we expect localization to be absent, and therefore (4.3) to be valid with a σ (nearly) constant in L (this can be also seen as an experimental result). This gives:

$$\text{for } g \text{ large}: \quad \beta(g) = d - 2 \tag{4.8}$$

For small g we expect localization to be present. Explicit calculation (using (4.4)) yields here that:

$$g \propto \exp\left(-\frac{L}{\xi}\right) \tag{4.9}$$

for some specific localization length ξ (which is related to γ from the previous section). The intuitive explanation here is that increasing the system size will generally move localized states further apart from each other, which leads to exponentially decreasing overlap and thereby also conductance. Taking the derivative with respect to L gives:

$$\text{for } g \text{ small}: \quad \beta(g) = -\frac{L}{\xi} = \ln(g) \tag{4.10}$$

We can now draw these two limiting cases into a diagram and somehow interpolate between them (see figure 1).

Assume that we have a system with some size $L = l$ and an associated conductance g_0. Now we are interested in the limit $L \to \infty$. Assuming the interpolation in the diagram is correct, we see that for $d \leq 2$, $\beta(g)$ is always negative, i.e. the conductance always decreases with increasing L and will eventually go to zero. This corresponds to our previous observation that for $d \leq 2$ all eigenstates are localized. For $d = 3$ however there exists a critical point where β goes through zero. This point is unstable (for $L \to \infty$), meaning that if g_0 happens to lie above it (e.g. $\beta(g_0) > 0$) for increasing L, g will also increase. For very large L we are then looking at a conducting metal. If however,

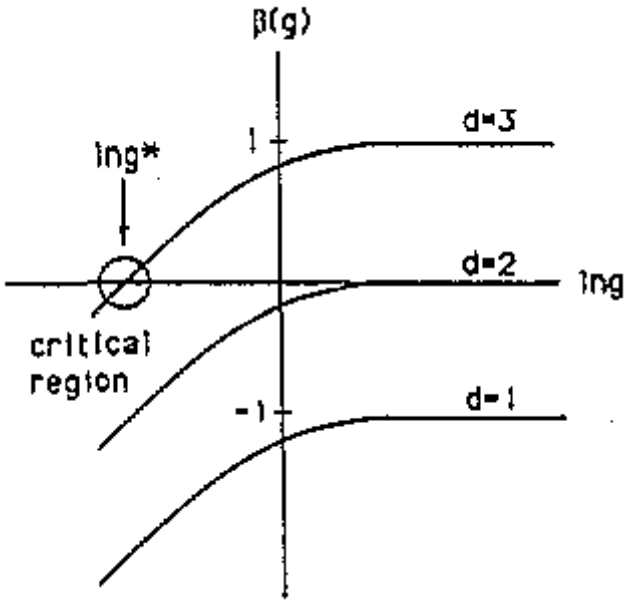

Figure 1: The β function for the zero-temperature conductance of a disordered system for dimensionality d = 1,...,3. g(L) increases with increasing L if $\beta > 0$. but decreases for $\beta < 0$. $\beta = 0$ defines the critical point corresponding to the Anderson transition. It is only achievable in d = 3. *Graphic taken from [6], original image caption*

$\beta(g_0) < 0$ the conductance will always decrease and again reach zero for $L \to \infty$, so our system is now an insulator. This precisely describes a metal-insulator phase-transition.

One may ask, what is the underlying physics behind this phase transition, e.g. what determines "on which side of the critical point" the system ends up? We have seen in section 2 that in three dimensions not all eigenstates are necessarily localized. However, if the band is wide enough localized states will in general exist for the highest and lowest energies. At $T = 0$ the system's behaviour will depend on the position of the Fermi energy (see figure 2). If the Fermi energy lies still in the localized (low-energy) region, then all populated states will be localized and there is no dc-conductance at zero temperature. If the Fermi energy is high enough to allow for non-localized states, then the dc-conductivity will not vanish. This is the physical intuition between the metal-isolator phase transition. The critical energy which marks the border between localized and extended states is called "mobility edge".

The above-mentioned scaling argument allows for the calculation of various properties (e.g. critical exponents, etc.) of this phase-transition, given some careful improvement of the procedure we called "interpolation" by means of perturbation theory. This is not something directly suitable for this talk, but in the last decades there has been a lot of effort made to describe and understand these types of transitions ([2], [3]).

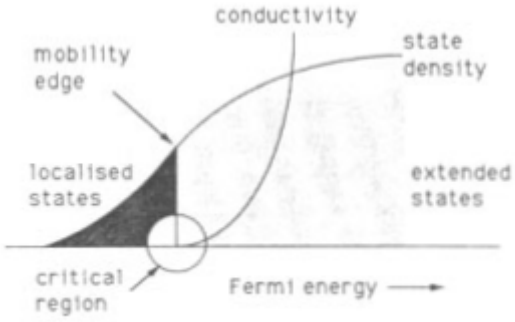

Figure 2: The concept of the mobility edge. Electronic states below and above the mobility edge are localized and extended, respectively. If the Fermi energy lies in the region of the localized states, the system is insulating at $T = 0$. In the extended-states region it is metallic. *Graphic taken from [6], original image caption*

5 Summary

Adding a small amount of disorder to a system might seem to show itself only in some slight perturbations to previously known solutions. However, we have shown that disorder can completely change the systems properties and generally lead to some very interesting effects such as localization and vanishing conductance. Localization appears to be a relatively general consequence of disordered systems, however rigorous proofs are difficult and tend to be limited to somewhat specific systems (as in our case, only on-site energies changed with an uniform probability distribution). Still in the last fife decades exact results together with a vast number of numerical simulations have lead to more than a handful of nobel prizes in this area (especially with the discovery and explanation of the quantum hall effect).

6 References

[1] P. W. Anderson. "Absence of Diffusion in Certain Random Lattices". In: *Phys. Rev.* 109 (5 Mar. 1958), pp. 1492–1505. DOI: 10.1103/PhysRev.109.1492. URL: https://link.aps.org/doi/10.1103/PhysRev.109.1492.

[2] Patrick A. Lee and T. V. Ramakrishnan. "Disordered electronic systems". In: *Rev. Mod. Phys.* 57 (2 Apr. 1985), pp. 287–337. DOI: 10.1103/RevModPhys.57.287. URL: https://link.aps.org/doi/10.1103/RevModPhys.57.287.

[3] Ferdinand Evers and Alexander D. Mirlin. "Anderson transitions". In: *Rev. Mod. Phys.* 80 (4 Oct. 2008), pp. 1355–1417. DOI: 10.1103/RevModPhys.80.1355. URL: https://link.aps.org/doi/10.1103/RevModPhys.80.1355.

[4] Michael Aizenman and Stanislav Molchanov. "Localization at large disorder and at extreme energies: An elementary derivations". In: *Communications in Mathematical Physics* 157.2 (Oct. 1993), pp. 245–278. ISSN: 1432-0916. DOI: 10.1007/BF02099760. URL: https://doi.org/10.1007/BF02099760.

[5] Jürg Fröhlich and Thomas Spencer. "A rigorous approach to Anderson localization". In: *Physics Reports* 103.1 (1984), pp. 9–25. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(84)90061-9. URL: http://www.sciencedirect.com/science/article/pii/0370157384900619.

[6] B Kramer and A MacKinnon. "Localization: theory and experiment". In: *Reports on Progress in Physics* 56.12 (1993), p. 1469. URL: http://stacks.iop.org/0034-4885/56/i=12/a=001.

[7] Reinhard Lang. *Spectral Theory of Random Schrödinger Operators*. Springer, 1991. ISBN: 978-3-540-54975-8. DOI: https://doi.org/10.1007/BFb0093929.