Exercises for

Advanced Quantum Theory

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Exercise 1

A classically integrable system like the simple pendulum or the Kepler problem often becomes chaotic when an additional periodic driving force is added. The periodically driven pendulum is probably the most famous example. This is the reason why periodically driven quantum systems are often taken as a prototype to study 'quantum chaos'. We will discuss time dependent and esp. periodically driven quantum systems later in the course. Interesting is also the case, where the periodic driving is itself a quantum process. The most simple model of that kind is given by the Hamiltonian

$$H = -\frac{\Delta}{2}\sigma_x + \omega b^\dagger b + \frac{\lambda}{2}\sigma_z(b + b^\dagger)$$

which is similar to the spin-boson model discussed in the lecture but with only one boson. It couples a two-level system to a quantum harmonic oscillator. Despite its simplicity, this model has been discussed in the literature over decades.

1. Choose the first two parts of H as the diagonal part H_d of H and the third as H_r. Show that with the choice $\eta = [H_d, H]$ we get a generator of the continuous unitary transformation of the form
$$\eta = \frac{1}{2}\eta_z\sigma_z(b - b^\dagger) + \frac{i}{2}\eta_y\sigma_y(b + b^\dagger)$$
where η_y, η_z are parameters, no operators.

2. Calculate $[\eta, H]$. Show that it contains a term $\sim \sigma_y(b - b^\dagger)$ which is not present in the Hamiltonian and which would be generated. Give a condition for η_y, η_z so that this term does not occur.

3. $[\eta, H]$ contains also a term that is $\propto \sigma_x$ and is quadratic in the bosonic operators. Show that if one introduces a normal ordering for the bosons (creation operators stand left of annihilation operators) a term $\sim \sigma_x$ without any additional operators appears.

4. Write down the flow equations for Δ and λ by neglecting the term that is $\propto \sigma_x \times$ (normal ordered quadratic term in b, $b^\dagger$). Choose $\eta_z = -\lambda\omega\frac{\omega-\Delta}{\omega+\Delta}$. They should be similar to those in the lecture for a bath of bosons. Show that $\lambda(\ell) \to 0$ for $\ell \to \infty$ except for $\omega = \Delta(\ell = 0)$.

5. Show that for $\Delta = 0$ the flow equations diagonalize the Hamiltonian exactly.

6. Assume $\omega > \Delta(\ell = 0)$. Show that Δ decays monotonically as a function of ℓ and that it is bounded from below by 0. This means that for small Δ the flow equation yield a good result, irrespective how large λ is. This is a clear advantage compared to perturbation theory. Remark: You may reproduce perturbation theory by applying a unitary transformation $\exp(S)$ to the Hamiltonian and make use of $\exp(S)H\exp(-S) = H + [S, H] + \frac{1}{2}[S, [S, H]] + \ldots$ by choosing S such that $[S, H_d] = -H_r$ and by taking into account all terms up to second order.

7. A second advantage of flow equations is that they often allow to calculate upper bounds for quantities. If you want, try to derive an upper bound for $\Delta(\ell = \infty)$.