Heidelberg University

Institute for Theoretical Physics

Quantum Field Theory I

Lecture Notes from 2019-2020

Lecturers:

Jörg Jäckel
Jan M. Pawlowski

Editors:

Jörg Jäckel
Jan M. Pawlowski
Katja Schwarz

Last edited: February 3, 2020

Contents

1. Introduction

Quantum Field Theory (QFT) describes the fundamental interactions of matter. It can be understood as the many body limit of quantum mechanical systems such as (an)harmonic oscillators. This is depicted in the lower horizontal map in Fig. 1.1. It can also be obtained from a quantisation of a classical field theory, depicted in the right vertical map in Fig. 1.1, historically called *second* quantisation. Evidently this simply is the quantisation introduced in quantum mechanics, as can be seen from Fig. 1.1.

$$
\begin{tikzcd}
x, p & \phi(\mathbf{x}),\ \Pi(\mathbf{y}) \\
\text{classical mechanics} \arrow[r] \arrow[d, "{[\hat{x}\,,\,\hat{p}]=\mathrm{i}\hbar}"'] & \text{classical field theory} \arrow[d, "{[\hat{\phi}(\mathbf{x})\,,\,\hat{\pi}(\mathbf{y})]=\mathrm{i}\hbar\delta(\mathbf{x}-\mathbf{y})}"] \\
\text{quantum mechanics} \arrow[r] & \text{quantum field theory} \\
\hat{x},\ \hat{p} & \hat{\phi}(\mathbf{x}),\ \hat{\pi}(\mathbf{y})
\end{tikzcd}
$$

Figure 1.1.: Different paths from classical mechanics to quantum field theory.

The applications of quantum field theory are manifold, and encompass all quantum systems from small to large scales. Amongst its variousq physical applications is the Standard Model & Beyond Standard Model physics including potentially even quantum gravity, condensed matter systems and ultracold gases. In particular modern theoretical particle physics is a great success story of quantum field theory. In the Standard Model the Higgs-boson corresponds to a scalar field (spin 0), leptons and quarks are described by (spin 1/2) fermion fields, and (spin 1) vector (gauge) fields are used for photons, $W^{\pm}$ and Z bosons, and gluons.

Due to its pivotal importance for the descriptions of general quantum systems, quantum field theoretical methods has been continuous and rapid advances from its early beginnings in the 30ties and 40ties of the 20th century. From the very beginning one of the key methods used in QFT applications is *perturbation theory*, that is the expansion of the physics at hand about a non-interacting case in order (number) of interactions (scatterings). This is a Taylor expansion in the interaction strength of the theory, in QFT this series is an *asymptotic* one (to be explained later). This will be the main method used in the applications in the current lecture course, and its success and limits will be discussed in detail. As a side remark we mention that, despite its matureness, even in perturbation theory there have been recently exciting new developments that go under the name of *resurgence*.

Despite its success the limits of perturbation theory are obvious, for strongly coupled or correlated systems an expansion in the number of scatterings may not be sufficient. A simple but relevant example is an observable $O(\lambda)$ that has the following dependence on the coupling,

$$
O(\lambda) = \mathrm{Pol}_0(\lambda) + \mathrm{Pol}_1(\lambda)\, \exp\left\{-\frac{\text{cont.}}{\lambda^2}\right\}\,, \tag{1.1}
$$

where $\mathrm{Pol}_i(\lambda)$ with $i = 1, 2$ are polynomials or converging series in λ. Evidently, an expansion of (1.1) about $\lambda = 0$ will only catch part of the series. Terms such as (1.1) arise typically from topological effects in QFT and quantum mechanics, and it is also here where the recent developments in resurgence have their merits.

In any case in general, the description of the physics of strongly correlated systems calls for non-perturbative methods. Prominent and important examples are the *lattice* approach to QFT, where the QFT is put on a space or space-time grid, and (functional) renormalisation group approaches, where the scale-dependence of the theory is resolved successively. Both approaches will be discussed briefly in the current lecture course, while more details are given for the latter, the renormalisation group, as it is also essential for perturbation theory. A full account of these approaches has to be subject of dedicated advanced QFT courses.

Let us now come back to the limits in Fig. 1.1, which we want to elucidate with two examples, the *oscillating masses/string* and *electrodynamics*.

Example 1-1: Oscillating masses / string. We consider a chain of oscillating point masses with a fixed position in x on a string (harmonic forces between next neighbours). This situation is depicted in Fig. 1.2.

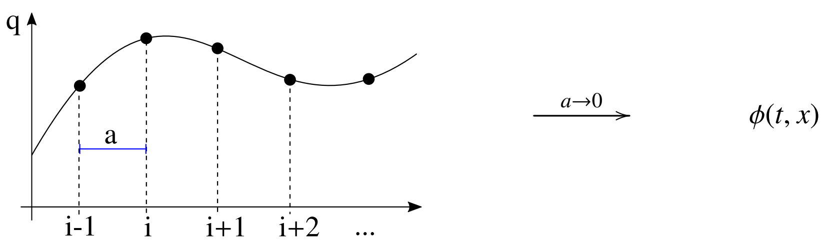


Figure 1.2.: Oscillating masses at a distance a on a string.

Now we take the limit of an infinitesimal *lattice* spacing a between the neighbouring point masses, thus storing more and more oscillating masses on a given intervall with length L on the string. The forces on a given point mass q_i are described by the equation of motion,

$$\partial_t^2 q_i = -c^2 \frac{(q_i - q_{i-1} + q_i - q_{i+1})}{a^2} \,, \tag{1.2}$$

triggered by the harmonic forces proportional to the distance with spring constant c^2/a^2, where we have dropped the mass. In the current one-dimensional example lattice spacing is seemingly a misnomer, but the example readily extends to d dimensions, where one typically considers a rectangular lattice of point masses. In $d = 2$ this leads to a two-dimensional rectangluar lattice, describing a membrane, in $d = 3$ we have a cubic lattice.

Coming back to our one-dimensional example, the *continuum* limit $a \to 0$ leads us to

$$\frac{q_i - q_{i-1}}{a} \to \partial_x \phi \,, \tag{1.3}$$

the differences turn into derivatives. We now also make this limit manifest in the position variable q_i. Instead of the single position variable we introduce the density of the mass points with

$$q_i \simeq \phi(ai) \overset{a\to 0}{\longrightarrow} \phi(x) \,, \quad \text{with} \quad x = i\, a \in \mathbb{R} \,. \tag{1.4}$$

with a 'density' field $\phi(x)$. Collecting all theses definitions and limits we are led to

$$\begin{aligned}\partial_t^2 q_i &= -c^2 \frac{(q_i - q_{i-1} + q_i - q_{i+1})}{a^2} \\ &\downarrow \\ \partial_t^2 \phi(t,x) &= -c^2 \partial_x^2 \phi(t,x)\,. \end{aligned} \tag{1.5}$$

Eq. (1.5) entails that in the limit $a \to 0$ the difference interactions turn into derivative (kinetic) terms. As the basic object in quantum field theory we will make use of the action of the theory at hand. In the current example the action of this system of oscillating masses is given by

$$S[q] = \int \mathrm{d}t\ \mathcal{L}(q(t), \dot{q}(t), t) = a \int \mathrm{d}t\ \sum_i \left[(\partial_t q_i)^2 - c^2 \frac{(q_{i+1} - q_i)^2}{a^2} \right] . \tag{1.6}$$

Perfoming the continuum limit $a \to 0$ in (1.6) the action turns into that of a classical scalar field theory

$$S[\phi] = \int \mathrm{d}t \int \mathrm{d}x \left((\partial_t \phi)^2 - c^2 (\partial_x \phi)^2 \right). \tag{1.7}$$

In general dimensions the action of a scalar field ϕ can be written as

$$S[\phi] \;=\; \int \mathrm{d}^d x \left(\frac{1}{2} (\partial_t \phi)^2 - (\nabla \phi)^2 - V(\phi) \right),$$

where V denotes the potential, and we have used the common QFT notation, setting $c = 1$. We close this discussion with two remarks:

(i) The problem can be simply described by a bunch of (coupled) harmonic oscillators.

(ii) The action $S[\phi]$ has Poincaré invariance (to be discussed later).

The second example is used to briefly recapitulate some basic facts of the most important example of a classical field theory, electrodynamics. This is also used for establishing some notation. If you feel that you are not familiar with some of the parts in this example, please recapitulate these parts. *Quantum* electrodynamics will serve as the first example for the quantisation of gauge theories in the second part of the lecture course, and some familiarity with it may come handy.

Example 1-2: Electrodynamics. Electrodynamics with only photons is a free fields theory. It is formulated in terms of the gauge (vector) field A_μ, in quantum electrodynamics this wil ldescribe the *photon*. Its classical action is given by

$$S[A_\mu] = \int \mathrm{d}^4 x\ \mathcal{L}(A_\mu(x), \partial_\mu A_\nu(x))\,, \tag{1.8}$$

where $x^0 = t$ and $\left(x^i\right) = \mathbf{x}$ for $i = 1, 2, 3$, and the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{1.9}$$

with the electromagnetic fieldstrength $F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu\,, \qquad \text{with} \qquad F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} F_{\rho\sigma}\,, \tag{1.10}$$

where the flat Minkowski metric $\eta_{\mu\nu}$ is used for lowering and raising indices. The diagonal metric η_μ has $\det \eta = -1$ and $\eta_{00} = -\eta_{ii}$ (so sum over i). In this lecture course we use the notation commonly used in QFT,

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \text{and} \qquad \eta^{\nu\rho}\eta_{\mu\rho} = \eta_\mu{}^\nu = \delta_\mu{}^\nu \,, \tag{1.11}$$

where the latter relation in (1.11) is the orthogonality relation. Again we close with a few remarks:

(i) The representation of the Minkowski metric with $\eta_{00} = 1$ is commonly used in general relativity and quantum gravity.

(ii) For general curved space-times with metrics $g_{\mu\nu}$ the orthogonality relation in (1.11) still holds

$$g^{\nu\rho} g_{\mu\rho} = g_\mu{}^\nu = \delta_\mu{}^\nu \,, \tag{1.12}$$

In both examples the fields will be quantised by inheriting the quantum mechanical quantisation from the underlying disrete systems, this can be decpited as

$$\begin{array}{ccc} q & \to & \phi \;,\; A_\mu \\ p & \to & \dot\phi \;,\; \dot A_\mu \\ & & \| \quad\;\; \| \\ & & \pi_\phi \;,\; \pi_{A_\mu} \end{array} \quad \xrightarrow{\text{Quantisation}} \quad \begin{array}{ccc} q & \to & \hat\phi \;,\; \hat A_\mu \\ p & \to & \hat\pi_\phi \;,\; \hat\pi_{A_\mu} \end{array} \;, \tag{1.13}$$

where the operators on the right hand side of (1.13) describe the annihilation and creation of particles. Again we remark:

(i) The Hilbertspace construction rests on the operators, e.g., $\hat\phi, \hat\pi_\phi$, the vacuum state (vector) $|\omega\rangle$. The creation operator $a^\dagger$ and annihilation operators a that are part of $\hat\phi, \hat\pi_\phi$ are used to create all states from the vacuum, for example the one particle state $|1\rangle \propto a^\dagger|\omega\rangle$.

(ii) Modern particle physics is described by *renormalisable* quantum field theories:

scalar fields (spin 0): Higgs

fermion fields (spin 1/2): leptons; quarks

vector fields (spin 1): photon; $W^\pm, Z$; gluons

spin 2: graviton (*perturbatively non-renormalisable*)

In this lecture course we will discuss the quantisation of all the above fields (except the graviton), including their quantum phenomenology as well as the important aspect of quantum symmetries and renormalisation.

2. Free Scalar Field

In this chapter we will discuss the free scalar field. Starting from classical field theory (section I) we move on to symmetries and the Noether theorem (section II). Lastly, we advance from classical theory to quantum field theory through quantisation. This requires the construction of the Fock space (section III), which is basically a sum of a set of Hilbert spaces.

I. Classical Theory

At first we consider a real scalar field $\phi(x)$. The scalar property entails, that ϕ is invariant under *Poincaré transformations*, that are space-time translations, spatial rotations, and Lorentz boosts, the lattter two forming the *Lorentz transformations*. Moreover, Poincaré transformations also include spatial and temporal reflections, altogether forming the *Poincaré group* $\mathcal{P}$, a non-compact Lie group. The invariance of a scalar field ϕ reads

$$\phi(x) \mapsto P(\phi(x)) = \phi(x)\,, \qquad P \in \mathcal{P}\,. \tag{2.1}$$

Poincaré transformations are those that leave the scalar product of space-time differences $(x-y)^2 = (x_\mu - y_\mu)\,\eta^{\mu\nu}\,(x_\nu - y_\nu)$ invariant. Trivially, a translation of the coordinates with a constant a: $x, y \to x+a, y+a$ leaves the scalar product invariant as the shift drops out from the difference $x-y$. *Continuous* Lorentz transformations Λ are rotations and boosts that leave the Minkowski metric invariant:

$$\left(\Lambda^T \eta \Lambda\right) = \eta\,, \qquad \text{in components:} \quad \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\mu = \eta_{\mu\nu}\,. \tag{2.2}$$

$$P = (\Lambda, a): \quad x^\mu \mapsto P(x^\mu) = \Lambda^\mu{}_\nu x^\nu + a^\mu\,, \tag{2.3}$$

Hence, a general Poincaré transformation $P \in \mathcal{P}$ is defined by the pair of translations a and Lorentz transformations Λ: $P = (\Lambda, a)$. The composition of Poincaré transformations is given by

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)\,. \tag{2.4}$$

We will consider Lorentz-invariant actions, and we exemplify the invariance within the standard scalar field theory, the ϕ^4-theory.

$$S[\phi] = \int \mathrm{d}^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x))\,, \tag{2.5}$$

with the Lagragian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\,, \qquad V(\phi) = \frac{1}{2}m^2\phi^2 + O(\phi^3)\,. \tag{2.6}$$

For the kinetic term Lorentz invariance follows from

$$\partial_\mu\phi\,\partial^\mu\phi \mapsto \partial_\nu\phi\,\Lambda_\mu{}^\nu\Lambda^\mu{}_\rho\,\partial^\rho\phi = \partial_\nu\phi\left(\Lambda^T\eta\Lambda\right)^\nu{}_\rho\,\partial^\rho\phi \overset{Eq.~(2.2)}{=} \partial_\nu\phi\,\partial^\nu\phi\,, \tag{2.7}$$

while it trivially follows for the potential term due to the scalar property of the field ϕ,

$$V(\phi) \mapsto V(\phi)\,. \tag{2.8}$$

In summary, the Lagragian and hence also the action are Lorenz invariant,

$$\mathcal{L}(\phi, \partial_\mu\phi) \mapsto \mathcal{L}(\phi, \partial_\mu\phi)\,, \qquad \Longrightarrow \qquad S[\phi] \mapsto S[\phi]\,. \tag{2.9}$$

We proceed by deriving the general solution to the equation of motion of the free theory. This general solution will turn out to be a simple superposition of plane wave solutions with general coefficients, the latter reflecting the fact, that the free theory is nothing but the continuum limit of a d-dimensional version of the string we have started with. The coefficients of the general solution characterise the density of these harmonic oscillators. This representation serves as the starting point for the canonical quantisation of the theory, by using the quantum mechanical commutaion relations for the single harmonic oscillators. To begin with, we discuss the *equation of motion* (EoM). For the case above it holds

$$\begin{aligned}
0 \stackrel{!}{=} \delta S &= \delta \int \mathrm{d}^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right)\\
&= \int \mathrm{d}^4x \left(\partial_\mu\phi\,(\partial_\nu\delta\phi)\,\eta^{\mu\nu} - m^2\phi\,\delta\phi\right) \quad \text{as} \quad \delta(\partial\phi)^2 = \delta\,(\partial_\mu\phi\,\partial_\nu\phi)\,\eta^{\mu\nu}\\
&= -\int \mathrm{d}^4x \left(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi\right)\delta\phi \quad \text{(using partial integration)}\\
&= -\int \mathrm{d}^4x\,\delta\phi\,(\partial^2 + m^2)\,\phi\,.
\end{aligned} \tag{2.10}$$

We conlcude that the scalar field satisfies the *Klein-Gordon* equation,

Klein-Gordon equation

$$(\partial^2 + m^2)\,\phi(x) = 0\,. \tag{2.11}$$

The Klein-Gordon equation is the equation of motivation for a four dimensional scalar field. We obtain the desired expression for the general solution for a scalar field ϕ with a linear superposition of all solutions of (2.11). Let us start with the solution for 1+0 dimensional theory, and subsequently generalise it to d dimensions. The quantised 1+0 dimensional theory simply is quantum mechanics,

$$\phi(t, \mathbf{x})\Big|_{1+0\ \mathrm{dim}} = \phi(t) = q(t)\,, \qquad \mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}m^2q^2 - \frac{\lambda}{4}q^4\,. \tag{2.12}$$

The first two terms correspond to a harmonic oscillator and the last is an anharmonic term. The equation of motion is then the Euler-Lagrange equation

$$\partial_t \frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0\,. \tag{2.13}$$

It follows that

$$\ddot{q} + m^2 q + \lambda q^3 = 0\,. \tag{2.14}$$

For $\lambda = 0$ this is the differential equation of a harmonic oscillator, which is solved by a plane wave

$$q(t) = A_0\,\mathrm{e}^{ikt} \quad \text{with} \quad k^2 - m^2 = 0\,. \tag{2.15}$$

When extending to d dimensions (1 time + (d-1) spacial dimensions), ϕ describes a density of coupled harmonic oscillators with the general solution

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \Big(\alpha(\mathbf{k})\, \mathrm{e}^{-ikx} + \alpha^*(\mathbf{k})\, \mathrm{e}^{ikx} \Big) \quad \text{with} \quad \omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}\,. \tag{2.16}$$

Note, that ϕ is real and satisfies Eq. (2.11). Further note, that

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)\, \theta(k^0)\,, \tag{2.17}$$

i.e. that this is a Lorentz invariant measure. This can be derived by using

$$\delta(g(x) - g(a)) = \frac{1}{|g'(a)|}\, \delta(x - a)\,, \tag{2.18}$$

where $g(x)$ is any C^1 function. Then

$$\delta(k^2 - m^2) = \delta((k^0)^2 - \mathbf{k}^2 - m^2) = \delta(k_0^2 - \omega_{\mathbf{k}}^2) = \frac{1}{|2\omega_{\mathbf{k}}|}\, \delta(k_0 - \omega_{\mathbf{k}})\,. \tag{2.19}$$

It follows

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)\, \theta(k^0) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} (2\pi)\, \frac{1}{|2\omega_{\mathbf{k}}|}\, \delta(k_0 - \omega_{\mathbf{k}})\, \theta(k^0) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}}\,. \tag{2.20}$$

Lastly, let us consider the case of a *complex scalar field*

$$\phi(x) = \frac{1}{\sqrt{2}} \Big(\phi_1(x) + \mathrm{i}\, \phi_2(x) \Big)\,, \tag{2.21}$$

where ϕ_1 and ϕ_2 are both real scalar fields. The action is given by

$$S[\phi] = \int \mathrm{d}^4 x\, \mathcal{L}(\phi, \partial_\mu \phi)\,, \tag{2.22}$$

with

$$\mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \phi\, \partial^\mu \phi^* - m^2 \phi\, \phi^* = \frac{1}{2} \Big[(\partial \phi_1)^2 + (\partial \phi_2)^2 - m^2 (\phi_1^2 + \phi_2^2) \Big]\,. \tag{2.23}$$

Then the general solution of Eq. (2.11) is given by

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \Big(\alpha(\mathbf{k})\, \mathrm{e}^{-ikx} + \beta^*(\mathbf{k})\, \mathrm{e}^{ikx} \Big) \quad \text{with} \quad \omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}\,. \tag{2.24}$$

Remarks:

(i) The action (2.22) is invariant under multiplication of ϕ with a global phase $\mathrm{e}^{i\omega}$. This global $U(1)$ symmetry implies a conserved charge, as will be discussed in the subsequent section.

(ii) An action that is invariant under a local $U(1)$ rotation, $\phi(x) \to \mathrm{e}^{i\omega(x)} \phi(x)$ is *gauge invariant*. The action (2.22) is not gauge-invariant, this would require the introduction of a $U(1)$ gauge field, to be discussed later.

II. Noether Theorem

Symmetries play a pivotal rôle in the description of quantum field theories. Their applications range from the direct deduction of physics simply from symmetry arguments, i.e. the exlcusion of processes based on their lack of symmetry to general construction principles (and hence restrictions) of quantum field theories, in particular for Beyond Standard Model physics. Most of these powerful symmetry principles originate in Noether's theorem. Loosely speaking it states, that

Continuous symmetries of the action leads to a conserved current density and a conserved charge.

We discuss this theorem first with a continuous symmetry with one parameter.

Example 2-1: $U(1)$-symmetry in the action of a complex scalar field.
A nice example to keep in mind is that of the field theory with a complex scalar with the action (2.22). The action is invariant the symmetry $\phi \to e^{i\,\omega}\phi$ as discussed at the end of the last chapter. An infinitesimal transformation with $\omega = \epsilon$ and $\epsilon \to 0$ is described by

$$e^{i\,\epsilon}\phi = \phi + \delta_\epsilon\phi\,, \qquad \text{with} \qquad \delta_\epsilon\phi = i\epsilon\phi, \quad \Delta\phi = \left.\frac{\partial\delta_\epsilon\phi}{\partial\epsilon}\right|_{\epsilon=0} = i\phi\,. \tag{2.25}$$

In (2.25), $\delta_\epsilon\phi$ is the infinitesimal transformation, and $\Delta\phi$ is the *generator* of the transformation. Evidently the action (2.22) is invariant under such the *global* transformation with $\partial_\mu\epsilon = 0$. Let us now consider a space-time dependent $\epsilon(x)$. Then, the action is shifted with

$$S[\phi(x)] \to S[\phi(x) + i\epsilon(x)\phi(x)] = S[\phi(x)] - i\int d^4x\,\partial_\mu\epsilon(x)\,\left[\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi\right] + O(\epsilon^2)\,. \tag{2.26}$$

With a partial integration of the last term we arrive at the form

$$S[\phi(x) + i\epsilon(x)\phi(x)] \simeq S[\phi(x)] + \int d^4x\,\epsilon(x)\partial_\mu j^\mu\,, \qquad j^\mu = i\left[\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi\right]\,. \tag{2.27}$$

This is an important result. It entails that for *global* symmetries the *local* variation of the action can be written as an integral of a total derivative $\partial_\mu j^\mu$, multiplied by $\epsilon(x)$. In turn, without the global symmetry, the non-invariant term cannot be written as a total derivative.
Clearly, the action is not invariant under the space-time dependent $U(1)$-transformation. We also see, that for chosing constant ϵ, the ϵ-dependent terms reduce to a total derivative and vanishes upon integration, as it must. Now we use that $\delta_\epsilon\phi$ simply is a specific variation of the field in the direction of the symmetry. This entails already that the term proportional to ϵ in (2.26) has to vanish on the equations of motion, as the latter are the stationary points of the action under a general variation. This leads us to

$$\partial_\mu j^\mu\big|_{\text{EoM}} = 0\,, \tag{2.28}$$

the theory has a conserved current. The current j^μ can be easily derived from the Lagrangian as it is only generated from the terms dependent on $\partial_\mu\phi$. We have

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi\,, \qquad \text{or} \qquad \partial_\mu j^\mu(x) = \left.\frac{\delta S[\phi]}{\delta\epsilon(x)}\right|_{\epsilon=0}\,. \tag{2.29}$$

The latter definition is the far more convenient one but requires some knowledge about functional derivatives. While not required at the present state of the lecture course, we suggest to the reader to get acquainted with functional derivatives as soon as possible, they facilitate quite some derivations and computations.

We also see that an additional term $\int d^4x\, \epsilon(x)\partial_\mu J^\mu$ in (2.26) would not have changed the existence of a conserved current. It simply would have led to a subtraction of J^μ on the right hand side of the first definition in (2.29), and would not have altered the second one. The latter fact again emphasises the naturality of using functional derivatives.
We emphasise again that (2.28) only holds true in the presence of a global symmetry. In turn, with a global symmetry (or for general variations) the right hand side is non-vanishing. An additional term occurs, namely $(\partial\mathcal{L}/\partial_\mu\phi)\Delta\phi$.
The conserved current (2.28) leads to a *Noether charge* Q, which is conserved on the EoM. We define

$$Q = \int d^3x\, j^0(x) = i \int d^3x\, [\phi^*\partial_t\phi - (\partial_t\phi^*)\phi] \,. \tag{2.30}$$

Using $\partial_t j^0 = \nabla\vec{j}$ from (2.28) we are led to

$$\partial_t Q\big|_{\text{EoM}} = \int d^3x\, \partial_t j^0(x) \overset{\text{EoM}}{=} \int d^3x\, \vec{\nabla}\vec{j}(x) = 0\,, \tag{2.31}$$

which can be also proven directly with the EoM. The charge (2.30) is nothing but the electric charge (up to normalisation) of the complex scalar field.

We now proceed with the derivation of the general theorem. Each step can be mapped back to our simple example discussed above. We consider an infinitesimal *global* symmetry transformation δ_ϵ with

$$\phi(x) \mapsto \phi(x) + \delta_\epsilon\phi(x)\,. \tag{2.32}$$

As has been discussed in our example, the global symmetry is described by a constant infinitesimal parameter ϵ with $\partial_\mu\epsilon = 0$. The transformation (2.32) is a symmetry of the action for

$$S[\phi(x)] \mapsto S[\phi(x) + \delta_\epsilon\phi(x)] = S[\phi(x)]\,. \tag{2.33}$$

For the general case it is convenient to consider the transformation of the Lagrangian $\mathcal{L}$. Eq. (2.33) holds if the Lagrangian transforms with

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\, \partial_\mu J^\mu(\phi)\,, \tag{2.34}$$

as has been also discussed briefly in our introductory example below (2.29). The last term, $\partial_\mu J^\mu$, vanishes if inserted in the action, as it is a divergence. This necessitates the absence of surface terms, that is $J^\mu[\phi](\|x\| \to \infty) = 0$. Then it follows

$$S[\phi] = \int d^4x\, \mathcal{L} \mapsto \int d^4x\, \mathcal{L} + \epsilon \int d^4x\, \partial_\mu J^\mu(\phi)\,. \tag{2.35}$$

In our simple example we have $J^\mu = 0$. Symmetries with $J^\mu \neq 0$ are e.g. space-time symmetries that lead to a conserved energy momentum tensor, to be discussed later.
Let us now assume that the theory has a *global* symmetry leading to (2.34). We have already seen in our example, that this leads to a local current, that is conserved on the equations of motion. Accordingly, we consider the explicit *global* symmetry transformation of the Lagrangian $\mathcal{L}$,

$$\mathcal{L} \mapsto \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}\,\delta_\epsilon\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\delta_\epsilon\phi\,. \tag{2.36}$$

The last term on the right hand side of (2.36) depends on the derivative of the variation. We rewrite this term as a total derivative and a term, where the derivative hits the $\partial_\mu\phi$-variation of the Lagrangian. This term combines with the first term on the right hand side of (2.36) to the equation of motion, indicated in red:

$$\begin{aligned}\mathcal{L} \mapsto \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}\,\delta_\epsilon \phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta_\epsilon\phi\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta_\epsilon\phi \\ = \mathcal{L} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta_\epsilon\phi\right) + (\mathrm{EoM})\,\delta_\epsilon\phi \\ \overset{!}{=} \mathcal{L} + \epsilon\,\partial_\mu J^\mu\,, \end{aligned} \tag{2.37}$$

The steps in (2.37) simply are the same as for the derivation of the Euler-Lagrange equations for the field theory (with variations restricted to the global symmetry). It is the identification of the non-invariant terms with a divergence $\epsilon\,\partial_\mu J^\mu$ in the last line of (2.37), that only holds for global symmetries. In summary we are led to the relation

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta_\epsilon\phi - \epsilon\,J^\mu\right) = -(\mathrm{EoM})\,\delta_\epsilon\phi\,. \tag{2.38}$$

that holds for general field configurations ϕ. For fields that satisfy the EoM, the right hand side vanishes and we are left with a conserved current, that is defined by the linear order in ϵ of the right hand side of (2.38). Using

$$\left.\frac{\partial \delta_\epsilon \phi}{\partial \epsilon}\right|_{\epsilon=0} = \Delta\phi\,, \tag{2.39}$$

we arrive at the general definition of the *Noether current* for a one-parameter global symmetry,

Conserved current (for a one-parameter global symmetry of a single scalar field)

$$j^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\Delta\phi - J^\mu\,, \quad \text{with} \quad \partial_\mu j^\mu = 0\,. \tag{2.40}$$

The conservation law can also be expressed in terms of the

Noether charge

$$Q(t)\,, := \int \mathrm{d}^3x\; j^0(t,\mathbf{x}) \quad \text{with} \quad \dot{Q}(t) = 0\,. \tag{2.41}$$

The Noether theorem extends readily to field theories with more than one field, indeed our example is such a case with (2.21). Then, the first term in j^μ needs to be replaced by a sum of the variations of the different fields. Also, the generalisation to the case of symmetries with N parameters $r = 1, ..., N$ is done by extending $\delta_\epsilon \to \delta_{\epsilon_r}$. This leads us to the generators of the symmetry, $\Delta_r\phi_i$ with

$$\Delta_r\phi_i = \left.\frac{\partial \delta_\epsilon \phi_i}{\partial \epsilon_r}\right|_{\epsilon=0}\,, \tag{2.42}$$

Repeating the derivation above in this general case leads us to the general definition of N Noether currents,

Conserved current (general case)

$$j_r^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta_r \phi_i - J_r^\mu\,, \quad \text{with} \quad \partial_\mu j^\mu = 0\,, \qquad r = 1, ..., N\,. \tag{2.43}$$

The N conserved currents are related to N conserved Noether charges

Noether charge (general case)

$$Q_r(t) := \int \mathrm{d}^3x\, j^0(t, \mathbf{x})\,, \quad \text{with} \quad \dot{Q}_r(t) = 0\,, \qquad r = 1, ..., N\,. \tag{2.44}$$

We close this derivation with a remark on general symmetries that involve both, transformation of the field as well as of its argument, the space-time variable x. This suggestes to split $\delta_\epsilon\phi$ into an infinitesimal transformation of the field ϕ, $\delta_\epsilon|_x\,\phi$, and the symmetry variation of the space-time variable $\delta_\epsilon x_\mu$. We obtain

$$\delta_\epsilon\phi = \delta_\epsilon|_x\,\phi + \delta_\epsilon x_\mu\,\partial^\mu\phi\,, \qquad \text{with} \qquad \Delta_r^{(\phi)}\phi_i = \left.\frac{\partial\delta_\epsilon|_x\,\phi_i}{\partial\epsilon_r}\right|_{\epsilon=0}, \qquad \Delta_r x = \left.\frac{\partial\delta_\epsilon x}{\partial\epsilon_r}\right|_{\epsilon=0}. \tag{2.45}$$

This split of the transformation translates into an according one for the Noether currents j^μ: they have a part that stems from the symmetry variation of the field, $\Delta_r^{(\phi)}\phi$, and the one which stems from the symmetry variation of the space-time variable x. Inserting the split (2.45) into the definition of the Noether current, (2.43), leads us to

$$j_r^\mu = \Delta_r^{(\phi)}\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \Delta_r x^\nu\,\partial_\nu\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - J_r^\mu\,. \tag{2.46}$$

As a first relevant application of the Noether theorem we discuss the conservation of the *energy-momentum* or *stress-energy tensor*. The related conservation laws entail momentum and energy conservation. The underlying global symmetries are that of translation invariance of physics under a spatial as well as temporal shift of the laboratory system: the fundamental law of physics do not change with time or space. Hence we consider an infinitesimal global ($\partial_\mu\epsilon = 0$) space-time translation, $x \to x + \epsilon$, of the field

$$\phi(x) \mapsto \phi(x+\epsilon) = \phi(x) + \epsilon^\mu\partial_\mu\phi(x) + O(\epsilon^2)\,, \qquad \text{with} \qquad \Delta_\mu\phi = \eta_\mu{}^\nu\partial_\nu\phi\,, \tag{2.47}$$

or $\Delta_\mu^{(\phi)}\phi = 0$ and $\Delta_\mu x^\nu = \eta_\mu{}^\nu$. Translations have four parameters, $r = \mu = 0, ..., 3$. Note also that in (2.47) the *scalar* property of the field has been used, it is invariant under Poincaré transformations. This is different for fermion (spin 1/2) and vector fields (spin 1) to be considered later. We proceed by discussing the current $J^\mu{}_r$. Applying the infinitesimal transformations (2.47) to the Lagrangian amounts to simply taking a space-time derivative of $\mathcal{L}$, as the only x-dependence of the Lagrangian resides in the fields and its derivatives. We get

$$\mathcal{L}(\phi(x), \partial_\mu\phi(x)) \mapsto \mathcal{L} + \epsilon^\mu\partial_\mu\mathcal{L} = \mathcal{L} + \epsilon^\nu\partial_\mu\eta^\mu{}_\nu\mathcal{L}\,, \tag{2.48}$$

leading us to

$$J^\mu{}_\nu = \eta^\mu{}_\nu\mathcal{L}\,. \tag{2.49}$$

Inserting (2.49) in the definition of the Noether current (2.43) or (2.46) leads us to the energy-momentum tensor,

Energy-momentum tensor (or stress-energy tensor)

$$T^{\mu}_{\ \nu} := j^{\mu}_{\ \nu} = \partial_\nu \phi \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \eta^{\mu}_{\ \nu}\mathcal{L} \quad \text{with} \quad \partial_\mu T^{\mu\nu} = 0 \, . \tag{2.50}$$

Consequently, we have four conserved Noether currents, i.e. Noether charges,

$$P^\mu = \int d^3x \, T^{0\mu} \, , \tag{2.51}$$

the 4-momentum. The energy-density for time translations is given by the zeroth component of the 4-momentum, namely

$$\begin{aligned} P^0 = \int d^3x \, T^{00} &= \int d^3x \left((\partial^0\phi) \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} - \mathcal{L} \right) = \int d^3x \left((\partial^0 \phi)\, \pi - \mathcal{L} \right) \\ &= \int d^3x \, \mathcal{H} = H \, , \end{aligned} \tag{2.52}$$

with the canonical momentum of the field,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} \, , \tag{2.53}$$

and the Hamiltonian density

$$\mathcal{H} = \pi \, \partial_0 \phi - \mathcal{L} \, , \tag{2.54}$$

of the Hamiltonian H. The result (2.52) was to be expected, as P^0 is the Noether charge coming from the invariance of the system under translations in time. The Hamiltonian of a theory generates the time evolution of the system. Note also that the Hamiltonian density and the Hamiltonian are positive definite functionals. For instance, the Hamiltonian density for the standard case of a scalar field theory with $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$ is

$$\mathcal{H} = \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \, . \tag{2.55}$$

We also remark that the covariance of P^μ is seemingly not apparent. However, $\frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}$ transforms as the 0-component of a contravariant vector. Hence $\int d^3x \, \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}$ has the transformation properties of the measure $d^3x \, dx^0$, which is Lorentz invariant. We proceed with the spatial components of the vector of the Noether charges: P^i. We already discussed, that P^0 generates generates time translations, and the P^i are the generators of spatial translations of the fields within the *Poisson brackets*. To see this we use the explicit form of the P^i,

$$P^i = \int d^3x \, T^{0i} = \int d^3x \, \pi \, \partial^i \phi = \left(\int d^3x \, \pi \, \nabla\phi \right)^i \, . \tag{2.56}$$

Inserting (2.56) into the Poisson brackets with the fields generates infinitesimal spatial translations:

Generator of spatial translations with Poisson brackets

$$\left\{ P^i(\mathbf{x}) \, , \, \phi(\mathbf{x}) \right\} = -\nabla\phi \, , \qquad \text{with} \quad \left\{ \phi(\mathbf{x}) \, , \, \pi(\mathbf{y}) \right\} = \delta(\mathbf{x} - \mathbf{y}) \quad \text{and} \quad \left\{ \phi(\mathbf{x}) \, , \, \phi(\mathbf{y}) \right\} = 0 \, . \tag{2.57}$$

The property (2.57) is sustained in the quantisation, which promotes the Poisson brackets to commutators of the field (and momentum) operators. We close the discussion of the energy-momentum tensor and the respetive Noether charges with two remarks.

(i)* In general the canonical energy-momentum tensor is not symmetric, i.e. $T^{\mu\nu} \neq T^{\nu\mu}$. This originates in $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$. However, $T^{\mu\nu}$ can always be symmetrised by adding a divergence to the canonical EMT. Its symmetry is an important property for the coupling to gravity. An alternative -symmetric- definition results from the variation of the action with respect to the metric $g^{\mu\nu}$:

$$T^{\mu\nu}_{sym} = \frac{2}{\sqrt{-\det g}} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g=\eta}\,, \tag{2.58}$$

with

$$\frac{\delta g^{\alpha\beta}(x)}{\delta g^{\mu\nu}(y)} = \frac{1}{2}\left(\delta^\alpha{}_\mu \delta^\beta{}_\nu + \delta^\alpha{}_\nu \delta^\beta{}_\mu\right)\delta(x-y)\,, \qquad \text{and} \qquad \frac{\delta\sqrt{-g(x)}}{\delta g^{\mu\nu}(y)} = -\frac{1}{2}\sqrt{-g(x)}g_{\mu\nu}\delta(x-y)\,. \tag{2.59}$$

The derivation of the EMT from the variation of the metric inherits the symmetry of the latter.

(ii)* We have already suggested the use of functional derivatives in the derivation of the Noether theorem instead of using variations. Functional derivatives are conveniently defined by

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)\,. \tag{2.60}$$

Note also that the scalar field is invariant under a combined transformation of field and spacetime variable, i.e. $\phi'(x') = \phi(x)$, and so far we have only used $\phi(x) \mapsto \phi(x')$. For the combined transformation we are led to

$$\begin{aligned} \Delta\phi &= 0 \\ \Delta_\rho x^\nu &= \eta_\rho^{\ \nu} \\ J^\mu_{\ \rho} &= 0 \quad (\mathcal{L}' = \mathcal{L})\,. \end{aligned} \tag{2.61}$$

(iii) Finally, we briefly revisite the charge of a complex scalar field, used as an example in the beginning. The (free) Lagrangian of a complex scalar field, (2.21), is given by

$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\,\phi^*\,, \tag{2.62}$$

and is invariant under global $U(1)$ rotations, see (2.25), leading to $J^\mu = 0$. With

$$\Delta\phi = i\phi\,, \qquad \Delta\phi^* = -i\phi^*\,. \tag{2.63}$$

we obtain the Noether current of the complex field:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^* = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\mathrm{i}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\mathrm{i}\phi^* = \mathrm{i}\Big((\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*\Big). \tag{2.64}$$

The Nother current is the spatial integral of j^0:

Noether charge of a complex scalar field

$$Q = \int \mathrm{d}^3x\; j^0 = i\int \mathrm{d}^3x\left(\phi^*\partial_t\,\phi - (\partial_t\phi^*)\,\phi\right). \tag{2.65}$$

We have already noted in our introductory example, that the (Noether) charge is conserved on the equation of motion. This is now checked explicitly,

$$
\begin{aligned}
\dot{Q}\Big|_{\mathrm{EoM}} &= \mathrm{i}\int \mathrm{d}^3x\left(\dot{\phi}^*\,\dot{\phi} - \dot{\phi}^*\,\dot{\phi} + \phi^*\,\partial_t^2\phi - (\partial_t^2\phi^*)\,\phi\right)\\
\text{Eq. (2.11)} \rightarrow \qquad &= \mathrm{i}\int \mathrm{d}^3x\left(\phi^*(\nabla^2 - m^2)\,\phi - (\nabla^2 - m^2)\,\phi^*\,\phi\right)\\
&= \mathrm{i}\int \mathrm{d}^3x\left(\phi^*\,\Delta\phi - \Delta\phi^*\,\phi\right) = 0\,,
\end{aligned} \tag{2.66}
$$

where we have performed twice a partial integration for the last identity as well as the absense of boundary terms. On the equation of motion we can also use (2.24) for rewriting the Noether charge (2.65) as a momentum space integral,

$$
Q = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(\alpha^*(\mathbf{p})\,\alpha(\mathbf{p}) - \beta^*(\mathbf{p})\beta(\mathbf{p})\Big). \tag{2.67}
$$

In the next section we discuss the quantisation of the scalar theory. As already mentioned, in the quantisation procedure the coefficients α, α^* and β, β^* are elevated to annihilation and creation operators for particles and anti-particles, respectively. Then, their combination in (2.67) is simply the quantum field theoretical analogue of the number operator in quantum mechanics.

III. Quantisation

In short, quantum field theory is the field-theoretical limit of quantum mechanics. Therefore, the canonical quantisation relations of the position operator and the momentum operator of a single quantum mechanical system are simply carried over to the field theory, see Fig. 2.1. This limit has been roughly described in the beginning of this lecture course, see Fig. 1.1, and is put to work in the present section.

Quantum Mechanics		**Quantum Field Theory**
$[\hat{q}\,,\,\hat{p}] = \mathrm{i}\hbar$		$\left[\hat{\phi}(\mathbf{x})\,,\,\hat{\pi}(\mathbf{y})\right] = \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y})$
	$\Longrightarrow$	
$[\hat{q}\,,\,\hat{q}] = 0 = [\hat{p}\,,\,\hat{p}]$		$\left[\hat{\phi}(\mathbf{x})\,,\,\hat{\phi}(\mathbf{y})\right] = 0 = [\hat{\pi}(\mathbf{x})\,,\,\hat{\pi}(\mathbf{y})]\,,$

Figure 2.1.: Quantum field theory from the many body limit of quantum mechanics. The canonical quantisation relations of QM are transported to the canonical quantisation relations of QFT.

where $\hbar = 1$ and $c = 1$ on the right hand side. Note, that the expectation value $\left\langle\hat{\phi}(\mathbf{x})\right\rangle$ needs to yield the classical field.

III.1. Canonical commutation relations

The above general picture entails that the canonical quantisation in qunatum field theory can be performed analoguously to quantum mechanics. For emphasising this analogy, let us briefly recapitulate quantum mechanics as 1+0 dimensional quantum field theory. The generalisation to general dimensions is straightforward and is done subsequently.

Example 2-2: Quantum mechanics as 1+0-dimensional quantum field theory.
Reducing the dimensions to $d = 1 + 0$, the spatial integration in the action is removed. This integration is an integration over the density of (quantum) mechanical systems. We will stick to our QFT convention $\hbar = 1$, as this facilitates the generalisation to quantum field theory. In summary we are left with the action of a harmonic oscillator.

$$S[q] = \int \mathrm{d}t\, \mathcal{L} = \int \mathrm{d}t \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2\right). \tag{2.68}$$

The corresponding Hamiltonian reads

$$H = p\dot{q} - \mathcal{L} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \qquad \text{with} \quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q} \tag{2.69}$$

The quantisation entails $p, q \to \hat{p}, \hat{q}$, with the canonical commutation relation $[\hat{q}\,,\,\hat{p}] \coloneqq \mathrm{i}$ (with $\hbar = 1$). We introduce creation operators $a^\dagger$ and annihilation operators a as

$$\hat{q} = \frac{1}{\sqrt{2\omega}}\,(a + a^\dagger)\,, \qquad\qquad \hat{p} = -\mathrm{i}\sqrt{\frac{\omega}{2}}\,(a - a^\dagger)\,. \tag{2.70}$$

The (canonical) commutation relations for the creation and annihilation operators follow from that of $\hat{q}$ and $\hat{\pi}$,

$$[a\,,\,a]^\dagger = 1\,, \qquad\qquad [a\,,\,a] = \left[a^\dagger\,,\,a^\dagger\right] = 0\,. \tag{2.71}$$

The Hamilton operator can then be written in terms of the creation and annihilation operators,

$$\hat{H} = \left(a^\dagger\, a + \frac{1}{2}\right)\omega\,, \tag{2.72}$$

where $(1/2)\,\omega$ corresponds to the vacuum energy. In the Heisenberg picture the operators evolve with time, whereas the states are stationary, i.e.

$$\mathrm{i}\frac{\partial}{\partial t}\hat{O}(t) = \left[\hat{O}(t)\,,\,\hat{H}\right]\,, \qquad \text{with} \qquad \hat{O}(t) = \mathrm{e}^{\mathrm{i}\hat{H}t}\,\hat{O}(0)\,\mathrm{e}^{-\mathrm{i}\hat{H}t}\,. \tag{2.73}$$

Now we use (3.1) that the commutation relation do not evolve in time, we always have the canonical commutation relations:

$$[\hat{q}(t)\,,\,\hat{p}(t)] = \mathrm{e}^{\mathrm{i}\hat{H}t}\,[\hat{q}\,,\,\hat{p}]\,\mathrm{e}^{-\mathrm{i}\hat{H}t} = \mathrm{i}\,. \tag{2.74}$$

A final remark concerns a first step into the direction of a quantum field theory, namely the extension of the present harmonic operator to the superposition of many harmonic oscillators as discussed in the beginning of the lecture course. The total Hamiltonian is then a sum of the single ones, in the most general case also with different frequencies ω_i, that is $\sum_i H(\hat{q}_i, \hat{p}_i; \omega_i) = \hat{H}(\hat{a}_i^\dagger, \hat{a}; \omega_i)$. The commutation relation of the operators stay the same, and operators stemming from the different subsystems commute. This leads us to

$$\left[\hat{q}_i\,,\,\hat{p}_j\right] \coloneqq \mathrm{i}\,\delta_{ij}\,, \qquad\qquad \left[a_i\,,\,a_j^\dagger\right] = \delta_{ij}\,. \tag{2.75}$$

With this we close the brief recapitulation of some basic properties of the harmonic oscillator.

The alert reader may have noticed that all derivations and properties discussed in the quantum mechanical example above carry over to quantum field theory: the only difference of a field operator $\hat{\phi}$ in QFT and the position operator $\hat{q}$ in QM is the integration over spatial momentum in the former. This is a linear operation and we expect that we simply have to change the Kronecker-δ's in (2.75) into δ-functions. With this introductory remark we proceed to the 1+3 dimensional theory. where it is understood that this case serves as the generic case. From now on we shall drop the hat marking the operators, it is understood implicitly. The Hamiltonian density for a *real scalar field* operator is

$$\mathcal{H} = \pi\,\partial_0\,\phi - \mathcal{L} = \frac{1}{2}\Big[\pi(t,\mathbf{x})^2 + \phi(t,\mathbf{x})\left(-\Delta + m^2\right)\phi(t,\mathbf{x})\Big]\,, \tag{2.76}$$

with the Laplacian $\Delta = \nabla^2$ and the field momentum operator π, and the Lagrangian

$$\pi(t,\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}(t,\mathbf{x}) = \partial^0\phi = \dot{\phi}\,, \qquad\qquad \mathcal{L} = \frac{1}{2}\Big(\partial_0\phi\,\partial^0\phi - (\nabla\phi)^2 - m^2\phi^2\Big). \tag{2.77}$$

the commutation relations of the field operator ϕ with the momentum operator π follows from that in the quantum mechanical case: for a system of harmonic oscillators we have, see (2.75),

$$[q_i\,,\,\pi_j] = i\delta_{ij}\,. \tag{2.78}$$

In the many-body limit discussed in the beginning the operators q_i and π_j turn into spatial densities of quantum mechanical operators. More precisely their products q^2, p^2, qp in the quantum mechanical Lagrangian or Hamiltonian turn into densities proportional to the inverse spatial volume element $1/a^{d-1}$. For example we have $p_i^2/a^{d-1} \to \pi^2$. Consequently, the quatum mechanical commutation relations (2.78) have to be multiplied with a^{d-1}, where the exponent simply is the (inverse) dimension of space-time. Thus, in the many-body limit the product $1/a^{d-1}\delta_{ij}$ turns into the spatial δ-function, $\delta(\mathbf{x}-\mathbf{y})$, and we obtain canonical commutation relations for the field operators:

Canonical commutation relations

$$\begin{aligned}
[\phi(t,\mathbf{x})\,,\,\pi(t,\mathbf{y})] &= \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y}) \\
[\phi(t,\mathbf{x})\,,\,\phi(t,\mathbf{y})] &= [\pi(t,\mathbf{x})\,,\,\pi(t,\mathbf{y})] = 0\,.
\end{aligned} \tag{2.79}$$

The field operators ϕ, π that satisfy the canonical commutation relations (2.79) operators define the free scalar quantum field theory. We add a few remarks:

(i) The field operator ϕ satisfies the EoM, as do its matrix elements $\langle\phi\rangle$ of time-independent states.

(ii) The free field theory describes a (coupled) set of harmonic oscillators due to the presence of $\phi\Delta\phi$ in the action. In Fourier space this term turns into $\phi(-p)\,\mathbf{p}^2\,\phi(p)$. Consequently, we can *diagonalise* $\mathcal{L}$ and $\mathcal{H}$ in momentum space. This is done analogously to the 1+0 dimensional theory.

The field and momentum operators can be written as spatial momentum integrals by elevating the classical solutions (2.16) to operators,

$$\begin{aligned}
\phi(\mathbf{x}) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(a(\mathbf{p})\,e^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\,e^{\mathrm{i}px}\Big) \\
\pi(\mathbf{x}) &= \partial^0\phi(x) = -\mathrm{i}\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big(a(\mathbf{p})\,e^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\,e^{\mathrm{i}px}\Big),
\end{aligned} \tag{2.80}$$

with the onshell frequency (2.16),

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{2.81}$$

In (2.80) the coefficients $a, a^\dagger$ are operators that inherit their commutation relations from (2.79). The Fourier transform is defined as

$$\begin{aligned}\tilde\phi(p) &:= \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\phi(x)\\ \phi(x) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}px}\tilde\phi(p)\,.\end{aligned} \tag{2.82}$$

With the spatial Fourier transform ($t = 0$) we get the representation of the

Field operator in momentum space

$$\begin{aligned}\tilde\phi(\mathbf{p}) &:= \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\mathbf{px}}\,\phi(\mathbf{x}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(a(\mathbf{p}) + a^\dagger(-\mathbf{p})\Big),\\ \tilde\pi(\mathbf{p}) &:= \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\mathbf{px}}\,\partial^0\phi(\mathbf{x}) = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\Big(a(\mathbf{p}) - a^\dagger(-\mathbf{p})\Big).\end{aligned} \tag{2.83}$$

The commutation relations of the operators $a, a^\dagger$ follow from that of the field and its canonical momentum in (2.79). To begin with, we insert (2.83) into (2.79), and derive the commutation relations of $\tilde\phi(\mathbf{p}), \tilde\pi(\mathbf{p})$,

$$\begin{aligned}\Big[\tilde\phi(\mathbf{p})\,,\,\tilde\pi(\mathbf{q})\Big] &= \int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{e}^{-\mathrm{i}(\mathbf{px}+\mathbf{qy})}\,[\phi(\mathbf{x})\,,\,\pi(\mathbf{y})]\\ &= \int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{e}^{-\mathrm{i}(\mathbf{px}+\mathbf{qy})}\,\mathrm{i}\,\delta(\mathbf{x}-\mathbf{y}) = \mathrm{i}\int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\mathbf{p}+\mathbf{q})\mathbf{x}}\\ &= \mathrm{i}\,(2\pi)^3\,\delta(\mathbf{p}+\mathbf{q})\,,\end{aligned} \tag{2.84}$$

and

$$\Big[\tilde\phi(\mathbf{p})\,,\,\tilde\phi(\mathbf{q})\Big] = 0 = [\tilde\pi(\mathbf{p})\,,\,\tilde\pi(\mathbf{q})]\,. \tag{2.85}$$

Eqs. (2.84) & (2.85) entail that $\tilde\pi(\mathbf{q})$ is conjugate to $\tilde\phi(-\mathbf{q})$. Now we use that the creation and annihilation operators are related to sums of the field and momentum operators similarly to quantum mechanics:

Creation and annihilation operator

$$\begin{aligned}a(\mathbf{p}) &= \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\,\tilde\phi(\mathbf{p}) + \mathrm{i}\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde\pi(\mathbf{p})\\ a^\dagger(-\mathbf{p}) &= \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\,\tilde\phi(\mathbf{p}) - \mathrm{i}\,\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde\pi(\mathbf{p})\,.\end{aligned} \tag{2.86}$$

From the commutation relations (2.84), (2.85) of ϕ, π we deduce that for the creation and annihilation operators:

$$\Big[a(\mathbf{p})\,,\,a^\dagger(\mathbf{q})\Big] = -\frac{\mathrm{i}}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big[\tilde\phi(\mathbf{p})\,,\,\tilde\pi(-\mathbf{q})\Big] - \frac{\mathrm{i}}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\Big[\tilde\phi(-\mathbf{q})\,,\,\tilde\pi(\mathbf{p})\Big] = (2\pi)^3\,\delta(\mathbf{p}-\mathbf{q})\,, \tag{2.87}$$

and

$$[a(\mathbf{p}) \,,\, a(\mathbf{q})] \;\; = 0 = \;\; \left[a^\dagger(\mathbf{p}) \,,\, a^\dagger(\mathbf{q})\right] . \tag{2.88}$$

This sets up the canonical quantisation in a free scalar field theory.

III.2. Hamiltonian of the free scalar field

In the 1+0-dimensional quantum mechanical example in the beginning of this section and in the classical scalar field theory we have seen, that the Hamiltonian density can be diagonalised. In terms of quantum mechanical annihilation and creation operators it could be rewritten in terms of the number operator. Here we follow this derivation in the scalar quantum field theory. To that end we diagonalise the Hamiltonian density in momentum space. We start with the kinetic term,

$$-\int d^3\phi(\mathbf{x})\, \Delta\phi(\mathbf{x}) = \quad \int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{x}(\mathbf{p}+\mathbf{q})}\, \tilde\phi(\mathbf{p})\, \mathbf{q}^2\, \tilde\phi(\mathbf{q}) = \int \frac{d^3p}{(2\pi)^3}\tilde\phi(\mathbf{p})\, \mathbf{p}^2\, \tilde\phi(-\mathbf{p})\,, \tag{2.89}$$

where we have used that $\Delta e^{-i\mathbf{q}\mathbf{x}} = -\mathbf{q}^2 e^{-i\mathbf{q}\mathbf{x}}$. Analogously we get

$$m^2 \int d^3x\, \phi^2(\mathbf{x}) = m^2 \int \frac{d^3p}{(2\pi)^3}\tilde\phi(\mathbf{p})\, \tilde\phi(-\mathbf{p})\,, \qquad\qquad \int d^3x\, \pi^2(\mathbf{x}) = \int d^3p\, \tilde\pi(\mathbf{p})\, \tilde\pi(-\mathbf{p})\,. \tag{2.90}$$

All terms are diagonal in momentum space. Note that this complete diagonalisation is specific for free theories. Local interactions such as the ϕ^4-term cannot be diagonalised in momentum space. In the free case we arrive at the diagonal Hamiltonian:

Diagonal Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\tilde\pi(\mathbf{p})\, \tilde\pi(-\mathbf{p}) + \omega_{\mathbf{p}}^2\, \tilde\phi(\mathbf{p})\, \tilde\phi(-\mathbf{p})\right] \qquad \text{with} \qquad \omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2\,. \tag{2.91}$$

The physics interpretation of H is best done in terms of the annihilation and creation operators a, $a^\dagger$. Hence, we use (2.83) to rewrite

$$\begin{aligned} H \;\; &= \;\; \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}} \left\{a^\dagger(\mathbf{p})\, a(\mathbf{p}) + \frac{1}{2}\Big(a^\dagger(\mathbf{p})\, a^\dagger(-\mathbf{p}) + a(\mathbf{p})\, a(-\mathbf{p})\Big)\right. \\ &\qquad \left. + a^\dagger(\mathbf{p})\, a(\mathbf{p}) - \frac{1}{2}\Big(a^\dagger(\mathbf{p})\, a^\dagger(-\mathbf{p}) + a(\mathbf{p})\, a(-\mathbf{p})\Big) + \left[a(\mathbf{p})\,,\, a^\dagger(\mathbf{p})\right]\right\} \\ &= \;\; \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, a(\mathbf{p}) + \frac{1}{2}V\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\,, \end{aligned} \tag{2.92}$$

where we have used (2.88) for the last equality. We also used

$$\left[a(\mathbf{p})\,,\, a^\dagger(\mathbf{p})\right] = (2\pi)^3\delta(\mathbf{0}) = \int d^3x\, e^{i\mathbf{p}\mathbf{x}}\Big|_{\mathbf{p}=0} = V\,, \tag{2.93}$$

where V is the volume of $\mathbb{R}^3$. The second term in the last line of (2.92) contains *two infinities*. As in physics energy differences are measured and not total energies this infinite constant can be conveniently dropped. However, it reflects one of the many divergencies we will encounter on the way and we add a few remarks:

(i) The volume of $\mathbb{R}^3$ is an *infrared* infinity related to considering arbitrarily large wavelengths. It occurs in the infinite volume limit and can be dealt with by putting the theory in a finite volume, i.e. a box $\mathcal{B}$, a sphere S^3 (good for topological considerations) or a torus T^3 (periodic box, no artificial curvature or boundary, used in lattice formulations of QFT). All these choices lead to finite volume factor, and all of them, and others, have been considered in QFT.

(ii) In turn, the second infinity is given by the vacuum energy density and occurs as the integral $\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}$ diverges. It is related to the limit of large momenta or small wave lengths and hence is an *ultraviolet* infinity. It can be dealt with by *regularising* the momentum integral, e.g. with the constraint $\mathbf{p}^2 \leq \Lambda^2$ with some finite ultraviolet cutoff Λ.

(iii) Although we ave argued that this constant term can be dropped as only energy *differences* can be measured, it has to be considered in general: it plays a role at finite temperature or more generally for QFT with boundary conditions. The latter has an interesting and measurable application with the Casimir effect in QED: it introduces an attractive force between conducting plates, more generally the strength of the Casimir force and even the sign depends on the considered geometry. Last but not least the constant term is important for QFT in curved space-time and/or coupled to gravity, specifically but not exclusive for the *cosmological constant problem*.

For now we continue with our derivatiion and simply drop the term. This leads us to the diagonal Hamiltonian as a spatial momentum integral:

Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, a(\mathbf{p}) \qquad \text{with} \qquad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{2.94}$$

H is the Hamiltonian of a momentum continuum of harmonic oscillators with frequencies $\omega_{\mathbf{p}}$. As a, $a^\dagger$ are annihilation and creation operators respectively, the combination $a^\dagger(\mathbf{p})\, a(\mathbf{p})$ is simply the momentum density of the number operator. Accordingly, H simply counts the number of particles with a given momentum $\mathbf{p}$ and integrates over their energies $\omega_{\mathbf{p}}$. Thus, in summary it gives the total energy of a given state.

III.3. Fock space of scalar quantum field theory

This leads us directly to the question of the Hilbert space of the quantum field theory which we have constructed in terms of its operators. It is given by the *Fock space*, which is basically a sum of a set of Hilbert spaces of the n-particle states. The Fock space is no systematically constructed from the vacuum state and the operator algebra given by $a, a^\dagger$.

(i) Vacuum & generic states: The vacuum is the state with the lowest energy, and we define

Vacuum state

$$H\,|0\rangle = 0 \quad \text{with} \quad a(\mathbf{p})\,|0\rangle = 0 \quad \text{and} \quad \langle 0|0\rangle = 1\,. \tag{2.95}$$

Note that it is the definition (2.95) that leads to the interpretation of a and $a^\dagger$ as annihilation and creation operators respectively. With (2.95) we now create all states in the Hilbert Space by applying a, $a^\dagger$ on the vacuum state $|0\rangle$. Indeed it is sufficient to only consider $a^\dagger$, as any a can be commuted through to the right where it finally hits the vacuum state. We also remark that a general state is given by applying a sum of products of creation operators to the vacuum,

$$|\mathbf{f}\rangle = \sum_{N=0}^{\infty} \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_1}}} \cdots \frac{d^3 p_N}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_N}}} f(\mathbf{p}_1, ..., \mathbf{p}_N)\, a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N)\, |0\rangle\,, \tag{2.96}$$

where $\mathbf{f} = (f_0, f_1 ...)$ is the inifinite-dimensional vector of the coefficient functions $f_n(\mathbf{p}_1, ..., \mathbf{p}_N)$.

(ii) General one-particle state: In order to construct such a general state and for discussing its properties, we first consider the one-particle state with momentum $\mathbf{p}$, which is proportional to $a^\dagger(\mathbf{p})|0\rangle$. The normalised state is given by

One-particle state

$$|\mathbf{p}\rangle = \sqrt{2\omega_{\mathbf{p}}}\, a^\dagger(\mathbf{p})\, |0\rangle\,, \qquad \text{with} \qquad H\,|\mathbf{p}\rangle = \omega_{\mathbf{p}}\, |\mathbf{p}\rangle\,. \tag{2.97}$$

The state in (2.97) is normalised, and is an eigenstate of the Hamiltonian with the energy $\omega_{\mathbf{p}}$. We are now proving both of these properties. Let us first consider the latter property. Applying the Hamiltonian (2.94) to the state (2.97) leads us to

$$\begin{aligned} H\,|\mathbf{p}\rangle &= \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, a(\mathbf{p}')\, \sqrt{2\omega_{\mathbf{p}}}\, a^\dagger(\mathbf{p})\, |0\rangle \\ &= \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, \sqrt{2\omega_{\mathbf{p}}} \Big(\big[a(\mathbf{p}')\,,\, a^\dagger(\mathbf{p})\big] + a^\dagger(\mathbf{p})\, a(\mathbf{p}') \Big) |0\rangle \\ &= \omega_{\mathbf{p}}\, a^\dagger(\mathbf{p})\, \sqrt{2\omega_{\mathbf{p}}}\, |0\rangle = \omega_{\mathbf{p}}\, |\mathbf{p}\rangle\,. \end{aligned} \tag{2.98}$$

In (2.98) we have used the canonical commutation relations (2.88) as well as the vacuum property $a|0\rangle = 0$, see (2.95). The states (2.97) are orthonormal: first of all the scalar product of two states with momenta $\mathbf{p}$ and $\mathbf{q}$ is proportional to the spatial momentum δ-function. Moreover, it is scalar and hence the factor has to involve $\omega_{\mathbf{p}}$. This leads us to the normalisation of the momentum states with $(2\pi)^3\omega_{\mathbf{p}}\delta(\mathbf{p} - \mathbf{q})$. Integrated over the Lorentz-invariant measure $\int d^3p/(2\pi)^3 1/(2\omega_{\mathbf{p}})$ the total normalisation is unity. With (2.97) we arrive at

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2\sqrt{\omega_{\mathbf{p}}\,\omega_{\mathbf{q}}}\, \langle 0|\, a(\mathbf{p})\, a^\dagger(\mathbf{q})\, |0\rangle = 2\sqrt{\omega_{\mathbf{p}}\,\omega_{\mathbf{q}}}\, \langle 0|\, \big[a(\mathbf{p})\,,\, a^\dagger(\mathbf{q})\big]\, |0\rangle = 2\omega_{\mathbf{p}}\, (2\pi)^3\, \delta(\mathbf{p} - \mathbf{q})\,. \tag{2.99}$$

and

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \langle \mathbf{p}|\mathbf{q}\rangle = 1\,. \tag{2.100}$$

Eq. (2.97) is also an Eigenstate for the momentum operator and the general one-partical state is given by a weighted momentum integral of $|\mathbf{p}\rangle$,

General one-particle state

$$|f\rangle = \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_1}}} \cdots \frac{d^3 p_N}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_N}}} f(\mathbf{p}_1, ..., \mathbf{p}_N)\, a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N)\, |0\rangle\,, \tag{2.101}$$

where f($\mathbf{p}$) denotes the distribution of momenta present in the state. The norm of the general one-particle state is given by

$$\langle f|f\rangle = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}} \, f^*(\mathbf{p})\, f(\mathbf{q}) \langle 0|\, a(\mathbf{p})\, a^\dagger(\mathbf{q})\, |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} f^*(\mathbf{p})\, f(\mathbf{p})\,. \tag{2.102}$$

If the state is normalised to unity, $f^* f$ (or rather $f^* f/(2\omega_{\mathbf{p}})$) is nothing but the propability distribution of a given one-particle state in momentum space. While the momentum eigenstate (2.97) is also an eigenstate of the Hamiltonian, the general one-particle is not due to the momentum integral involved. We find

$$H\,|f\rangle = \int \frac{d^3p'}{(2\pi)^3} \omega_{\mathbf{p}'}\, a^\dagger(\mathbf{p}')\, a(\mathbf{p}') \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p})\, a^\dagger(\mathbf{p})\, |0\rangle = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\, a^\dagger(\mathbf{p})\, f(\mathbf{p})\, |0\rangle\ , \tag{2.103}$$

the energy is distributed according to the momentum distribution of the state. If sandwiched with this state this becomes even more obvious,

$$\langle f|H\,|f\rangle = \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2\omega_{\mathbf{p}}} f^*(\mathbf{p})\, f(\mathbf{p})\right] \omega_{\mathbf{p}}\,, \tag{2.104}$$

for normalised states $|f\rangle$ this is nothing but the energy distribution following from the probability $f^*(\mathbf{p})\, f(\mathbf{p})$.

(iii) N-particle state: The analysis done above for the one-particle state extends straightforwardly to the n-particle state. First we note that particle momentum states are eigenstates of the Hamiltonian and energy-momentum is additive. Assume now that we have a state $|\beta\rangle$ with $H\,|\beta\rangle = E_\beta\,|\beta\rangle$. Then $a^\dagger(\mathbf{p})\,|\beta\rangle$ is a state with one additional particle with momentum $\mathbf{p}$ and the energy $E_\beta + \omega_{\mathbf{p}}$:

$$H\Big(a^\dagger(\mathbf{p})\,|\beta\rangle\Big) = a^\dagger(\mathbf{p})\, H\,|\beta\rangle + \Big[H\,,\, a^\dagger(\mathbf{p})\Big]\,|\beta\rangle = (H + \omega_{\mathbf{p}})\Big(a^\dagger(\mathbf{p})\,|\beta\rangle\Big). \tag{2.105}$$

Accordingly, we simply have to consider N creation operators acting on the vacuum. This leads us to N-particle momentum states:

N-particle state

$$|\mathbf{p}_1 \cdots \mathbf{p}_N\rangle = \prod_{i=1}^N \sqrt{2\omega_{\mathbf{p}_i}}\, a^\dagger(\mathbf{p}_i)\,|0\rangle\ , \qquad \text{with} \qquad H\,|\mathbf{p}_1 \cdots \mathbf{p}_N\rangle = \left(\sum_{i=1}^N \omega_{\mathbf{p}_i}\right) |\mathbf{p}_1 \cdots \mathbf{p}_N\rangle\ , \tag{2.106}$$

where the latter property follows by starting with the vacuum, recursively multiplying the creating operators and using (2.105). Note also, that the N-particle states have Bose symmetry, i.e.

$$|\mathbf{p}_1 \cdots \mathbf{p}_i\, \mathbf{p}_{i+1} \cdots \mathbf{p}_N\rangle = |\mathbf{p}_1 \cdots \mathbf{p}_{i+1}\, \mathbf{p}_i \cdots \mathbf{p}_N\rangle\ , \tag{2.107}$$

as $\Big[a^\dagger(\mathbf{p_i})\,,\, a^\dagger(\mathbf{p_{i+1}})\Big] = 0$. While the energy relation in (2.106) follows concisely from (2.105), we re-derive it with the explicit form of the state and the Hamiltonian operator for further exemplifying the use of the commutation relation. This leads us to

$$H\,|\mathbf{p_1} \cdots \mathbf{p_N}\rangle = \int \frac{d^3p'}{(2\pi)^3} \omega_{\mathbf{p}'} \prod_{i=1}^N \sqrt{2\omega_{\mathbf{p}_i}} \Big(a^\dagger(\mathbf{p}')\, a(\mathbf{p}')\, a^\dagger(\mathbf{p_1}) \cdots a^\dagger(\mathbf{p_N})\,|0\rangle\Big). \tag{2.108}$$

In the next step we shift all creation operators to the left and all annihilation operators to the right, which is called *normal ordering*. Then we use Eq. (2.95) and the canonical commutation relations for the creation and annihilation operators, (2.88), to simplify the equation,

$$\begin{aligned} a(\mathbf{p'})\left(a^\dagger(\mathbf{p_1})\cdots a^\dagger(\mathbf{p_N})\right)|0\rangle &= \left(\left[a(\mathbf{p'})\,,\,a^\dagger(\mathbf{p_1})\right]a^\dagger(\mathbf{p_2})\cdots a^\dagger(\mathbf{p_N})+...\right.\\ &\quad ...+a^\dagger(\mathbf{p_1})\left[a(\mathbf{p'})\,,\,a^\dagger(\mathbf{p_2})\right]a^\dagger(\mathbf{p_3})\cdots a^\dagger(\mathbf{p_N})+...+...\\ &\quad \left....+a^\dagger(\mathbf{p_1})\cdots a^\dagger(\mathbf{p_{N-1}})\left[a(\mathbf{p'})\,,\,a^\dagger(\mathbf{N})\right]\right)|0\rangle\;. \end{aligned} \tag{2.109}$$

With (2.109) it follows that $|\mathbf{p}_1\cdots\mathbf{p}_N\rangle$ is an energy eigenstate with the energy $\sum\omega_{\mathbf{p}_i}$, see (2.106). The odering of operators of the final expression, namely all annihilation operators to the right, is called *normal ordering*. We have already used it implicitly in the definition of the Hamiltonian (2.94), and it will be discussed in more detail later.

The general N particle state is then given by a momentum integral of a distribution $f(\mathbf{p}_1,...,\mathbf{p}_N)$ of momentum eigenstates similarly to (2.101):

N-particle state

$$|f_N\rangle=\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}_1}}}\cdots\frac{d^3p_N}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}_N}}}\,f(\mathbf{p}_1,...,\mathbf{p}_N)\,a^\dagger(\mathbf{p}_1)\cdots a^\dagger(\mathbf{p}_N)\,|0\rangle\,. \tag{2.110}$$

This leads us directly to the definition of generic state in the Fock space of the scalar quantum field theory as a sum of general aN-particle states for all N as defined in (2.96). We close the discussion of the N-particle state with a remark on the generation of an $(N-1)$-particle state from an N-particle state. This is naturally achieved by applying an annihilation operator to the N-particle state, which is illustrated with the example of $N=1$ below:

Example 2-3: Annihilation operator applied to general one particle state.

$$\begin{aligned} a(\mathbf{p})\,|f\rangle &= a(\mathbf{p})\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}'}}}\mathrm{f}(\mathbf{p}')\,a^\dagger(\mathbf{p}')\,|0\rangle\\ &= \int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}'}}}\mathrm{f}(\mathbf{p}')\left[a(\mathbf{p})\,,\,a^\dagger(\mathbf{p}')\right]|0\rangle\\ &= \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\mathrm{f}(\mathbf{p})\,|0\rangle \end{aligned} \tag{2.111}$$

(iv) Field operator ϕ and coherent states: Let us now discuss the interpretation of the field operator $\phi(\mathbf{x})$ in Eq. (2.80). We first remark that applying ϕ to the vacuum state yields a one particle state,

$$\phi(\mathbf{x})\,|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a(\mathbf{p})\,e^{-i\mathbf{px}}+a^\dagger(\mathbf{p})\,e^{i\mathbf{px}}\right)|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\,e^{i\mathbf{px}}\,|\mathbf{p}\rangle\;. \tag{2.112}$$

Eq. (2.112) is the Fourier transformation of a momentum one-particle state, hence a position state. Testing this state with a momentum one-particle state results in

$$\langle \mathbf{p} | \, \phi(\mathbf{x}) \, |0\rangle = \mathrm{e}^{\mathbf{ipx}} \,, \tag{2.113}$$

is a plane wave travelling at momentum $\mathbf{p}$ and reminiscent of non-relativistic QM, as $\langle p|x\rangle = \mathrm{e}^{ipx}$. Applying ϕ to the dual (bra-vector) vacuum vector $\langle 0|$ leads to a similar conclusion for the annihilation operator part of ϕ. We conclude that the field operator $\phi(\mathbf{x})$ creates and annihilates a particle at the spatial position $\mathbf{x}$. Moreover, states with defined particle number have a vanishing expectation value of ϕ. In particular this applies to the vacuum,

$$\langle 0|\phi(\mathbf{x})|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \, \langle 0|a(\mathbf{p}) \, \mathrm{e}^{-\mathbf{ipx}} + a^\dagger(\mathbf{p}) \, \mathrm{e}^{\mathbf{ipx}}|0\rangle = 0 \,, \tag{2.114}$$

with $\langle 0|a|0\rangle = 0 = \langle 0|a^\dagger|0\rangle$. Similarly it follows, that

$$\begin{aligned} \langle \mathbf{p}|\phi(\mathbf{x})|\mathbf{p}\rangle &= 0 \\ &\vdots \\ \langle \mathbf{p_1} \cdots \mathbf{p_N}|\phi(\mathbf{x})|\mathbf{p_1} \cdots \mathbf{p_N}\rangle &= 0 \,, \end{aligned} \tag{2.115}$$

by using

$$|\mathbf{p_1} \cdots \mathbf{p_N}\rangle \simeq a^\dagger(\mathbf{p_1}) \cdots a^\dagger(\mathbf{p_N}) \, |0\rangle \;. \tag{2.116}$$

Let us now concentrate on the annihilation part of the field. A potential eigenvector for this part is one for the annihilation operator,

$$a(\mathbf{p}) \, |\alpha\rangle = \alpha(\mathbf{p}) \, |\alpha\rangle \;, \qquad \text{with} \qquad \langle \alpha|\alpha\rangle = 1 \,. \tag{2.117}$$

the state is unchanged by the annihilation (detection) of a particle with momentum $\mathbf{p}$ and the eigenvalue is the amplitude $\alpha(\mathbf{p})$. Eq. (2.117) defines a *coherent state*, heuristically one with minimal uncertainty, hence a 'classical' state. If the field operator is sandwiched with $|\alpha\rangle$, this interpretation is very suggestive. We find

$$\langle \alpha|\phi(\mathbf{x})|\alpha\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(\mathrm{e}^{-\mathbf{ipx}} \, \alpha(\mathbf{p}) + \mathrm{e}^{\mathbf{ipx}} \, \alpha^*(\mathbf{p}) \right) = \phi_{\mathrm{cl}}(\mathbf{x}) \,. \tag{2.118}$$

We emphasise,that $\alpha(\mathbf{p})$, $\alpha^*(\mathbf{p})$ are no operators, and Eq. (2.118) is equivalent to Eq. (2.16), i.e. the classical real scalar field $\phi_{\mathrm{cl}}(\mathbf{x})$. It is left to explicit construct $|\alpha\rangle$. Its defining property (2.117) implies that it must be a sum of N-particle states $|\alpha_N\rangle$ that are mapped into $|\alpha_{N-1}\rangle$ if hit with a. This leads us to

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \sum_{N=0}^{\infty} |\alpha_N\rangle \;, \qquad \text{with} \qquad a(\mathbf{p}) \, |\alpha_N\rangle = \alpha(\mathbf{p}) \, |\alpha_{N-1}\rangle \;, \tag{2.119}$$

with $|\alpha_{-1}\rangle = 0$ and a normalisation $\mathcal{N}(\alpha)$. Such an N-particle state is given by

$$|\alpha_N\rangle = \frac{1}{N!} \prod_{i=1}^{N} \left(\int \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p_i}}}} \, \alpha(\mathbf{p_i}) \right) |\mathbf{p_1} \cdots \mathbf{p_n}\rangle \;, \tag{2.120}$$

and the property in (2.119) is shown with successively using

$$a(\mathbf{p}) \, |\mathbf{p_1} \cdots \mathbf{p_n}\rangle = \sum_{i=1}^{n} (2\pi)^3 \, \sqrt{2\omega_{\mathbf{p_i}}} \; |\mathbf{p_1} \cdots \mathbf{p_{i-1}} \, \mathbf{p_{i+1}} \cdots \mathbf{p_n}\rangle \; \delta(\mathbf{p} - \mathbf{p_i}) \,, \tag{2.121}$$

more details can be found in Appendix I. It is left to compute the normalisation $\mathcal{N}(\alpha)$ necessary for $\langle\alpha|\alpha\rangle = 1$ in (2.119). This computation is deferred to Appendix I. This concludes our construction of the coherent state. The sum in (2.119) with the N-particle states $|\alpha_N\rangle$ defined in (2.120) is the exponential series and we write conveniently

Coherent state

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)}\exp\left(\int \frac{d^3p}{(2\pi)^3}\,\alpha(\mathbf{p})\,a^\dagger(\mathbf{p})\right)|0\rangle\,, \qquad \text{with} \qquad \mathcal{N}(\alpha) = \exp\left(\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,|\alpha(\mathbf{p})|^2\right). \tag{2.122}$$

The coherent states are not orthogonal which can be deduced from their scalar product,

$$\langle\beta|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)\,\mathcal{N}(\beta)}\exp\left(\int \frac{d^3p}{(2\pi)^3}\beta^*\,\alpha(\mathbf{p})\right) = \exp\left(-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(|\alpha(\mathbf{p})|^2 + |\beta(\mathbf{p})|^2 - 2\beta^*\,\alpha(\mathbf{p})\right)\right), \tag{2.123}$$

where we used the Baker–Campbell–Hausdorff formula

$$\exp(A)\,\exp(B) = \exp(B)\exp(A)\exp([A\,,\,B]) \quad \text{for} \quad [A\,,\,[A\,,\,B]] = 0 = [B\,,\,[B\,,\,A]]\,. \tag{2.124}$$

Indeed the set of coherent states is overcomplete. For completeness it is remarked, that in quantum mechanics (1+0 dimensional theory) we can easily show that

$$\frac{1}{\pi}\int d^2\alpha\;|\alpha\rangle\langle\alpha| = \mathbb{1}\,. \tag{2.125}$$

This completes our discussion of coherent states and the interpretation of the field operator.

(v) Conserved energy-momentum tensor: Let us now discuss the fate of the classical conservation laws in the quantisation procedure. To begin with, as nothing in the derivation of the Noether theorem made use of the nature of the field (functions or operators), the conservation laws should also hold in the quantised theory. Indeed we have already discussed the Hamiltonian as the generator of time translations. This is the 0-component of the conserved current $P^\mu = \int d^3x\,T^{0\mu}$, see (2.51). Corresponding to Eq. (2.51) we now calculate the spatial momentum operator as

$$\begin{aligned}
\mathrm{P}^i &= \int d^3x\,T^{0i} = \int d^3x\,\pi\,\partial^i\phi \\
&= \int d^3x\left\{(-\mathrm{i})\int \frac{d^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}'}}{2}}\left(a(\mathbf{p}')\,e^{-\mathrm{i}\mathbf{p}'\mathbf{x}} - a^\dagger(\mathbf{p}')\,e^{\mathrm{i}\mathbf{p}'\mathbf{x}}\right)\right. \\
&\qquad\left.\times\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\left(a(\mathbf{q})\,(-\mathrm{i}q^i)\,e^{-\mathrm{i}\mathbf{q}\mathbf{x}} + a^\dagger(\mathbf{q})\,\mathrm{i}q^i\,e^{\mathrm{i}\mathbf{q}\mathbf{x}}\right)\right\} \\
&= -\frac{\mathrm{i}}{2}\int \frac{d^3p}{(2\pi)^3}\Big(a(\mathbf{p})\,a(-\mathbf{p})\,\mathrm{i}\mathbf{p} + a(\mathbf{p})\,a^\dagger(\mathbf{p})\,\mathrm{i}\mathbf{p} \\
&\qquad -a^\dagger(\mathbf{p})\,a(\mathbf{p})\,(-\mathrm{i}\mathbf{p}) - a^\dagger(\mathbf{p})\,a^\dagger(-\mathbf{p})\,(-\mathrm{i}\mathbf{p})\Big)\,.
\end{aligned} \tag{2.126}$$

where we have used the Fourier representation of the field and its canonical momentum, (2.80). The first and last term in the momentum integral proportional to $a(\mathbf{p})\,a(-\mathbf{p})\,\mathbf{p}$ and $a^\dagger(\mathbf{p})\,a^\dagger(-\mathbf{p})\,\mathbf{p}$ vanish:

$a(\mathbf{p})\,a(-\mathbf{p})$ are even under $\mathbf{p} \to -\mathbf{p}$ as the operators commute, this follows similarly for $a^\dagger(\mathbf{p})\,a^\dagger(-\mathbf{p})$. Hence, the total integrands are odd under $\mathbf{p} \to -\mathbf{p}$ and the intergals vanish, i.e. $\int d^3p\; a(\mathbf{p})\,a(-\mathbf{p})\,\mathbf{p} = 0$. We normal order the operators (annihilation operator to the right) and arrive at

$$\begin{aligned} \mathrm{P}^i &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\Big(2\,a^\dagger(\mathbf{p})\,a(\mathbf{p})\,\mathbf{p} + \big[a^\dagger(\mathbf{p})\,,\,a(\mathbf{p})\big]\;\mathbf{p}\Big) \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\Big(2\,a^\dagger(\mathbf{p})\,a(\mathbf{p})\,\mathbf{p} - (2\pi)^3\delta(\mathbf{0})\,\mathbf{p}\Big). \end{aligned} \tag{2.127}$$

Analogously to the Hamiltonian in (2.93), the second term formally diverges. As for the vacuum energy we drop this term: we only can measure the *differences* of momenta. This leads us to

4-momentum operator

$$\begin{aligned} \mathrm{P}^0 &= H \\ \text{(spatial momentum)}\quad \mathbf{P} &= \int d^3x\,\pi\,\nabla\phi \\ &\simeq \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\,a^\dagger(\mathbf{p})\,a(\mathbf{p}) \quad \text{with} \quad \mathbf{P}\,|\mathbf{p}\rangle = \mathbf{p}\,|\mathbf{p}\rangle\;. \end{aligned} \tag{2.128}$$

(vi) Lorentz symmetry: We close the construction of the Fock space with discussing *Lorentz symmetry*. Let $U(\Lambda)$ denote the unitary Fock space representation of a Lorentz transformation Λ. Then

$$\begin{aligned} U(\Lambda)\,|0\rangle &= |0\rangle \\ U(\Lambda)\,|\mathbf{p}\rangle &= |\Lambda\mathbf{p}\rangle\;. \end{aligned} \tag{2.129}$$

Note, that

$$\langle \mathbf{q}|\mathbf{p}\rangle = 2\omega_{\mathbf{p}}(2\pi)^3\,\delta(\mathbf{p}-\mathbf{q}) \tag{2.130}$$

is Lorentz invariant, as

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}} = \int \frac{d^4p}{(2\pi)^4}\,2\pi\,\delta(p^2-m^2)\,\theta(p^0) \tag{2.131}$$

is invariant under proper orthochronous Lorentz transformations $\left(\det\Lambda = 1,\, \Lambda^0{}_0 > 0\right)$, and

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\,2\omega_{\mathbf{p}}(2\pi)^3\,\delta(\mathbf{p}-\mathbf{q}) = 1\;. \tag{2.132}$$

With this, we have completed the Fock space construction. Recall, that $\phi(x)$ generates a superposition of one particle states from the vacuum (Eq. (2.113)). Further we remark, that causality is encoded in the propagator

$$D(x-y) = \langle 0|\phi(x)\,\phi(y)|0\rangle \tag{2.133}$$

and its variants. This is further discussed in chapter 3, section I. In summary we have constructed and discussed the Hilbert space for the free real scalar field, the Fock space, in the paragraphs **(i)** - **(vi)**. The quantisation as well as the construction of the Fock space readily carries over to complex scalar field and $O(N)$-theories, $\phi = (\phi_1, ..., \phi_N)$. We refrain from repeating the identical steps of the construction and simply quote some of the important result. The action, the field operator and its canonical momentum are given by

$$
\begin{aligned}
S[\phi] &= \int \mathrm{d}^4x \left(\partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\,\phi^*\right), \quad \text{with} \quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\,\phi_2)\,, \\
\phi(\mathbf{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}\mathbf{px}} + b^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{px}}\right), \\
\pi(\mathbf{x}) &= \partial^0\phi^*(x) = -\mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_\mathbf{p}}{2}}\left(b(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}\mathbf{px}} - a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{px}}\right),
\end{aligned}
\tag{2.134}
$$

with the commutation relations

$$
\begin{aligned}
[\phi(\mathbf{x})\,,\,\pi(\mathbf{y})] &= \delta(\mathbf{x}-\mathbf{y})\,, \\
\left[a(\mathbf{p})\,,\,a^\dagger(\mathbf{q})\right] &= (2\pi)^3\,\delta(\mathbf{p}-\mathbf{q})\,, \\
\left[b(\mathbf{p})\,,\,b^\dagger(\mathbf{q})\right] &= (2\pi)^3\,\delta(\mathbf{p}-\mathbf{q})\,.
\end{aligned}
\tag{2.135}
$$

and all other commutators vanish. Evidently, $a, a\dagger$ are the annihilation and creation operators for particles while $b, b\dagger$ are the annihilation and creation operators for anti-particles. This can be also deduced from the respective Noether charges (electric charge) discussed below. The Hamiltonian operator is given by

$$
H = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\tilde{\pi}(\mathbf{p})\,\tilde{\pi}^\dagger(\mathbf{p}) + \omega_\mathbf{p}^2\,\tilde{\phi}(\mathbf{p})\,\tilde{\phi}^\dagger(\mathbf{p})\right),
\tag{2.136}
$$

and reads in the diagonal momentum representation

$$
H = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_\mathbf{p}\left(a^\dagger(\mathbf{p})\,a(\mathbf{p}) + b^\dagger(\mathbf{p})\,b(\mathbf{p})\right),
\tag{2.137}
$$

where the integrand corresponds to the sum of the energy of particles and antiparticles. Finally, the Noether charge operator from Eq. (2.65) is given by

$$
\begin{aligned}
Q &= i\int \mathrm{d}^3x\left(\phi^*\partial_t\,\phi - (\partial_t\phi^*)\,\phi\right) \\
&= i\int \mathrm{d}^3x\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}\,2\omega_\mathbf{q}}} \\
&\quad \Big((a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{px}} + b(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}\mathbf{px}})\cdot(-\mathrm{i}\omega_\mathbf{q}\,a(\mathbf{q})\,\mathrm{e}^{-\mathrm{i}\mathbf{qx}} + \mathrm{i}\omega_\mathbf{q}\,b^\dagger(\mathbf{q})\,\mathrm{e}^{\mathrm{i}\mathbf{qx}}) \\
&\quad -(\mathrm{i}\omega_\mathbf{p}\,a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}\mathbf{px}} - \mathrm{i}\omega_\mathbf{p}\,b(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}\mathbf{px}})\cdot(a(\mathbf{q})\,\mathrm{e}^{-\mathrm{i}\mathbf{qx}} + b^\dagger(\mathbf{q})\,\mathrm{e}^{\mathrm{i}\mathbf{qx}}\Big) \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(a^\dagger(\mathbf{p})\,a(\mathbf{p}) - b^\dagger(\mathbf{p})\,b(\mathbf{p})\right).
\end{aligned}
\tag{2.138}
$$

Again one sees that particle and anti-particle nature of the opposite Noether charge carries by the $a^\dagger a$ and $b^\dagger b$ parts. The 'classical' charge in a coherent state is given by

$$
\langle\alpha|Q|\alpha\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\alpha^*\,\alpha(\mathbf{p}) - \beta^*\,\beta(\mathbf{p})\right),
\tag{2.139}
$$

which agrees with the classical Noether charge derived in (2.65).

3. Perturbation Theory

Perturbation theory is a standard method in quantum field theory. It considers **interaction as a perturbation of the free theory**. Thus, we assume $\lambda \ll 1$ and expand the observables, e.g. scattering amplitudes, in order of λ. We remark, that strictly speaking it is an amplitude expansion of the field amplitude as the coupling always comes with powers of the field. This differentiation is important for strong field physics, e.g. electrodynamics in strong fields. There, the coupling is small, the fine structure constant α is of the order 10^{-2}, but perturbation theory at least has to be resummed.

I. Interaction Picture

Perturbation theory in QFT is typically done in the *interaction picture* which is a mixture of the *Heisenberg* and *Schrödinger picture* known from quantum mechanics (1+0 dimensional QFT). In short, in the interaction picture the operators evolve with the Hamiltonian of the free QFT, while the states are evolved with the interaction Hamiltonian. For the construction we first briefly recapitulate the Heisenberg and Schrödinger picture.

In the previous chapter, the Fock space construction was performed in the Heisenberg picture. As mentioned above, in the Heisenberg picture the operators evolve in time, whereas the states are stationary,

$$\begin{aligned} \mathrm{i}\partial_t \, |f\rangle &= 0\,, \\ \mathrm{i}\partial_t \, O(t) &= [O(t)\,,\, H]\,. \end{aligned} \tag{3.1}$$

The time evolution equation for the operator $O(t)$ in (3.1) has the simple solution

$$O(t) = \mathrm{e}^{\mathrm{i}Ht}\, O\, \mathrm{e}^{-\mathrm{i}Ht}\,, \tag{3.2}$$

with the *unitary* time evolution operator $\exp(\mathrm{i}H\,t)$ that describes the time evolution over a time distance $t = t_1 - t_2$. An important example is the field operator $\phi(x)$: In the discussion of the free field we only discussed equal time commutation relations and operators at $t = 0$ such as $\phi(\mathbf{x})$. The respective argument was the time translation invariance of the free theory. With (3.2) the time-dependent field operator $\phi(x)$ follows with

$$\begin{aligned} \phi(x) &= \mathrm{e}^{\mathrm{i}Ht}\, \phi(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}Ht} \\ &= \mathrm{e}^{\mathrm{i}Ht} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(a(\mathbf{p})\, \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}} \Big) \mathrm{e}^{-\mathrm{i}Ht} \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} + a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}(\omega_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} \Big) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \Big(a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}px} \Big), \end{aligned} \tag{3.3}$$

with

$$\mathrm{e}^{\mathrm{i}Ht}\, a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}Ht} = a(\mathbf{p})\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{p}}t}\,, \qquad \text{and} \qquad \mathrm{e}^{\mathrm{i}Ht}\, a^\dagger(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}Ht} = a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}\omega_{\mathbf{p}}t}\,. \tag{3.4}$$

Eq. (3.3) is nothing but the quantised version of the time-dependent classical solution of the equation of motion, (2.16), and the field operator in (3.3) collapses to the time-dependent classical free field if sandwiched in a coherent state. For the derivation of (3.4) we write $\exp\{iHt\} = \sum(1/n!)\, t^n\, H^n$. let us first consider the commutator of the Hamiltonian with the annihilation operator. With $H\,a(\mathbf{p}) - a(\mathbf{p})\,(H - \omega_{\mathbf{p}})$ we find

$$[H\,,\,a(\mathbf{p})] = \int \frac{d^3q}{(2\pi)^3}\omega_{\mathbf{q}}\Big(a^\dagger(\mathbf{q})\,[a(\mathbf{q})\,,\,a(\mathbf{p})] + \big[a^\dagger(\mathbf{q})\,,\,a(\mathbf{p})\big]\,a(\mathbf{q})\Big) = -\omega_{\mathbf{p}}\,a(\mathbf{p})\,. \tag{3.5}$$

where we have used the canonical commutation relations (2.88). With (3.5) we arrive at

$$e^{iHt}\,a(\mathbf{p})\,e^{-iHt} = a(\mathbf{p})\,e^{i(H-\omega_{\mathbf{p}})t}\,e^{-iHt} = a(\mathbf{p})\,e^{-i\omega_{\mathbf{p}}t}\,. \tag{3.6}$$

The time evolution of the creation operator follows similarily and we are led to (3.4). This closes our brief recapitulation of the Heisenberg picture.

In the *Schrödinger picture* the states evolve in time and the operators are stationary,

$$\begin{aligned} i\partial_t\,|f(t)\rangle &= H\,|f\rangle \\ i\partial_t\,O &= 0\,. \end{aligned} \tag{3.7}$$

As for the time evolution of the operators in the Heisenberg picture, the time evolution of the states in the Schrödinger picture have a simple solution in terms of the unitary time evolution operator $\exp(-i\,H\,t)$,

$$|f(t)\rangle = e^{-iHt}\,|f\rangle\ . \tag{3.8}$$

Hence, the time evolution operator $U(t,t') := e^{-iH(t-t')}$ either acts on the operators (Heisenberg) *or* the states (Schrödinger); the expectation values are the same. This leads us to an important property, the *causality* of a *local* Poincaré-invariant QFT: Causality is a necessary requirement of any physics description, and has to be present in a QFT on the microscopic level. In a *local* QFT we have point-like interactions, a necessary requirement for the causality of the QFT. It should be also present in the field operator. In particular the field operator should not connect space-like regions. This leads to the requirement

$$[\phi(x)\,,\,\phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{3.9}$$

Eq. (3.9) entails that two measurements with a space-like distance have no impact on each other. For proving (3.9) we consider

$$\begin{aligned} [\phi(x)\,,\,\phi(y)] &= \int \frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}\,2\omega_{\mathbf{q}}}}\Big(\big[a(\mathbf{p})\,,\,a^\dagger(\mathbf{q})\big]\,e^{-i(p-q)x} + \big[a^\dagger(\mathbf{p})\,,\,a(\mathbf{q})\big]\,e^{i(p-q)y}\Big) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{ip(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^3}\,\delta(p^2-m^2)\,\theta(p^0)\,e^{-ip(x-y)} - \int \frac{d^4p}{(2\pi)^3}\,\delta(p^2-m^2)\,\theta(p^0)\,e^{ip(x-y)}\,, \end{aligned} \tag{3.10}$$

with $p_0 = \sqrt{\vec{k}^2 + m^2} \geq 0$, and using (2.17) in the last step. Now we utilise the manifest Lorentz-invariant form of the momentum measures that in Eq. (3.10): For space-like separations $((x-y)^2 < 0$ there is a Lorentz transformation with

$$\Lambda(x-y) = -(x-y)\,, \tag{3.11}$$

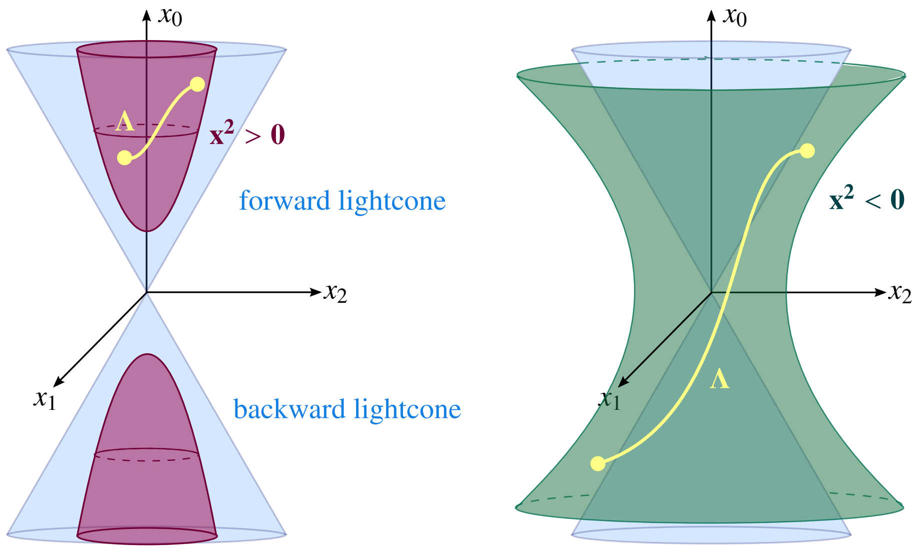


Figure 3.1.: Lorentz transformations connect all points on a x^2 = const. surface in the Minkowski diagram. Thus, for space-like separations we find a Lorentz transformation |*Lambda* with $\Lambda(x-y) = -(x-y)$, sketched in the figure on the right hand side. In turn, for time-like separations this is not possible, see the figure on the left hand side.

see figure 3.1. Hence, for $(x-y)^2 < 0$ we have

$$\int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\, \theta(p^0)\, e^{ip(x-y)} = \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\, \theta(p^0)\, e^{ip\,\Lambda(x-y)}$$
$$= \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\, \theta(p^0)\, e^{-ip(x-y)}\,. \tag{3.12}$$

With (3.12) the two terms in (3.10) cancel and we are led to the causality relation (3.9). Evidently, the derivation carries over directly to complex scalar fields (or more general N copies of scalar fields, $\phi = (\phi_1, ..., \phi_N)$), and we find

$$\left[\phi(x)\,,\, \phi^\dagger(y)\right] = 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{3.13}$$

In summary local interactions and the free field operators $\phi(x)$ lead to the necessary causality of the QFT on the microscopic level. We now proceed to the *interaction picture*. As already mentioned above, in the interaction picture we expand the theory in powers of interaction about the free theory, the latter being formulated in the Heisenberg picture. In turn, the states evolve with the interaction Hamiltonian. For this construction we decompose the Lagrangian density in a free and an interaction part

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_{\rm int}(\phi) = \frac{1}{2}\phi(x)\left(-\partial^2 - m^2\right)\phi(x) + \mathcal{L}_{\rm int}(\phi)\,, \tag{3.14}$$

where the interaction Lagrangian is given by the local pointlike interaction $-V(\phi)$,

$$\mathcal{L}_{\rm int}(\phi) = -V(\phi)\,, \tag{3.15}$$

is a polynomial in $\phi(x)$ at the same space-time point x. This decomposition carries over to the Hamiltonian density

$$\mathcal{H}(\pi,\phi) = \mathcal{H}_0(\pi,\phi) + \mathcal{H}_{\rm int}(\phi) = \frac{1}{2}\pi(x)^2 + \frac{1}{2}\phi(x)\left(-\Delta + m^2\right)\phi(x) + \mathcal{H}_{\rm int}(\phi)\,, \tag{3.16}$$

where the canonical momentum solely occurs in the free part and

$$\mathcal{H}_{\text{int}}(\phi) = V(\phi)\,. \tag{3.17}$$

For the time being we restrict ourselves to the ϕ^4-interaction:

interaction part of the Hamiltonian density

$$\mathcal{H}_{\text{int}}(\phi) = V(\phi) = \frac{\lambda}{4!}\,\phi(x)^4\,. \tag{3.18}$$

Note, that the normalisation factor of 4! varies in literature. Let us add a few remarks:

(i) higher order terms are excluded by lack of renormalisability (predictivity) in four space-tim dimensions.

(ii) ϕ^3 terms lack the discrete symmetry under $\phi \to -\phi$ and hence are not bounded from below classically.

(iii) as the simplest interacting QFT, the ϕ^4 theory is the 'workhorse' of QFT.

In the interaction picture, the operators evolve in time with the free Hamiltonian

$$\begin{aligned} \mathrm{i}\partial_t O &= [O\,,\,H_0] \\ \Rightarrow O(t) &= \mathrm{e}^{\mathrm{i}H_0 t}\,O\,\mathrm{e}^{-\mathrm{i}H_0 t}\,, \end{aligned} \tag{3.19}$$

with

$$H_0 = \int \mathrm{d}^3x\,\mathcal{H}_0\,. \tag{3.20}$$

On the other hand, the states evolve with the interaction Hamiltonian

$$\mathrm{i}\partial_t\,|f\rangle = H_{\text{int}}\,|f\rangle\,. \tag{3.21}$$

Note, that

$$\begin{aligned} [H_0\,,\,H_{\text{int}}] &\neq 0 \\ \Rightarrow \partial_t H_{\text{int}} &\neq 0 \quad \text{i.e.} \quad H_{\text{int}} = H_{\text{int}}(t)\,. \end{aligned} \tag{3.22}$$

Time evolution of a state can also be expressed as

$$|f(t)\rangle = U(t,t_0)\,|f(t_0)\rangle\,, \tag{3.23}$$

where $U(t,t_0)$ is the unitary time-evolution operator. With Eq. (3.19) we find the

time evolution of $\mathbf{U(t,t_0)}$

$$\mathrm{i}\partial_t U(t,t_0) = H_{\text{int}}(t)\,U(t,t_0)\,. \tag{3.24}$$

We remark, that the scattering matrix is defined as

S-matrix

$$S = \lim_{\substack{t_0\to-\infty \\ t\to+\infty}} U(t,t_0)\,. \tag{3.25}$$

Strictly speaking, this means λ is adiabatically switched on and off. Thus, initial state $|i\rangle$ and final state

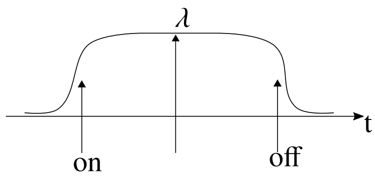


Figure 3.2.: Sketch of adiabatic switch on/off of λ, i.e. interaction.

$|f\rangle$ are given by:

$$\begin{aligned} |\text{state } t \to -\infty\rangle &= |i\rangle \\ |\text{state } t \to +\infty\rangle &= |f\rangle \ . \end{aligned} \tag{3.26}$$

Note, that for a proper treatment of the S-matrix the LSZ-formalism is used.
Next, we derive the explicit expression for $U(t, t_0)$. For this purpose, we take the infinitesimal form of Eq. (3.21) and use it to rewrite the state $|f(t)\rangle$ iteratively:

$$\begin{aligned} |f(t+\Delta t)\rangle &= |f(t)\rangle - \mathrm{i}\,\Delta t\, H_{\text{int}}(t)\ |f(t)\rangle \\ &= \Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t)\Big)\ |f(t)\rangle \\ &= \Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t)\Big)\Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t-\Delta t)\Big)\ |f(t-\Delta t)\rangle \\ &\ \vdots \\ &= \prod_{n=0}^{N}\Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t-n\,\Delta t)\Big)\ |f(t-N\,\Delta t)\rangle \ . \end{aligned} \tag{3.27}$$

Thus,

$$U(t+\Delta t, t-N\Delta t) = \prod_{n=0}^{N}\Big(1 - \mathrm{i}\,\Delta t\, H_{\text{int}}(t-n\,\Delta t)\Big). \tag{3.28}$$

We expand in powers of Δt:

$$U(t+\Delta t, t-N\Delta t) = 1 + (-\mathrm{i})\Delta t \sum_{n=0}^{N} H_{\text{int}}(t-n\,\Delta t) + (-\mathrm{i})^2(\Delta t)^2 \sum_{n<m} H_{\text{int}}(t-n\,\Delta t)\, H_{\text{int}}(t-m\,\Delta t) + \dots \,. \tag{3.29}$$

Note, that $n < m$ in the second sum corresponds to the time 'on the left' being larger than the time 'on the right' (time ordering). Now let $\Delta t \to 0$ with $N\,\Delta t = t - t_0$. Then Eq. (3.29) becomes

$$1 + (-\mathrm{i}) \int_{t_0}^{t} \mathrm{d}t'\ H_{\text{int}}(t') + (-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\ H_{\text{int}}(t')\, H_{\text{int}}(t'') + \dots \,, \tag{3.30}$$

where the first integral in the last line corresponds to the sum over n and the second integral to the sum over m. The integral limits give an equivalent ordering to $n < m$.
Finally, we obtain the

time-evolution operator

$$U(t, t_0) = T \exp\left(-\mathrm{i} \int_{t_0}^{t} \mathrm{d}t' \, H_{\mathrm{int}}(t') \right) \quad \text{for} \quad t > t_0 \,, \tag{3.31}$$

with the time ordering operator

$$T \, A(t) \, B(t') = A(t) \, B(t') \, \theta(t - t') + B(t') \, A(t) \, \theta(t' - t) \,. \tag{3.32}$$

Example 3-4: Time ordering for the second order term of $\mathbf{U(t, t_0)}$. This example shows, how the time ordering operator acts on the second order term in the expansion of Eq. (3.31), yielding the second order term of Eq. (3.30).

$$\begin{aligned}
&\frac{1}{2} T \int_{t_0}^{t} \mathrm{d}t' \, H_{\mathrm{int}}(t') \int_{t_0}^{t} \mathrm{d}t'' \, H_{\mathrm{int}}(t'') \\
= \; &\frac{1}{2} \left(\int_{t_0}^{t} \mathrm{d}t' \, H_{\mathrm{int}}(t') \int_{t_0}^{t'} \mathrm{d}t'' \, H_{\mathrm{int}}(t'') \qquad (t' < t) \right. \\
&\left. + \int_{t_0}^{t} \mathrm{d}t'' \, H_{\mathrm{int}}(t'') \int_{t_0}^{t''} \mathrm{d}t' \, H_{\mathrm{int}}(t') \right) \qquad (t'' > t') \\
= \; &\int_{t_0}^{t} \mathrm{d}t' \, H_{\mathrm{int}}(t') \int_{t_0}^{t'} \mathrm{d}t'' \, H_{\mathrm{int}}(t'') \,.
\end{aligned} \tag{3.33}$$

This works analogously for higher order terms, where $n!$ equal terms cancel with the $\frac{1}{n!}$ factor from the expansion.

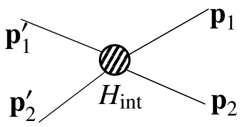
Note, that

$$H_{\mathrm{int}} = \int \mathrm{d}^3 x \, \phi^4(x) \sim a^2 \, (a^\dagger)^2 \,. \tag{3.34}$$

Hence, the interaction Hamiltonian creates two particles and annihilates them, leading to infinite vacuum processes $\langle 0 | H_{\mathrm{int}} | 0 \rangle$.

Example 3-5: 2 to 2 scattering.

$\left\langle \mathbf{p}_1' \, \mathbf{p}_2' \right| \sim$
at $t \to +\infty$



$\sim |\mathbf{p}_1 \, \mathbf{p}_2\rangle$
at $t \to -\infty$

The S-matrix is given by

$$S = \mathbb{1} + \mathrm{i}\,T\,, \tag{3.35}$$

where the unit matrix represents the part without scattering. For 2 to 2 scattering it is

$$\mathrm{i}\,T_{fi} \cong -\mathrm{i}\,\langle 0|a(\mathbf{p_1'})\,a(\mathbf{p_2'})\frac{\lambda}{4!}\int \mathrm{d}^4x\,\phi^4(x)\,a^\dagger(\mathbf{p_1})\,a^\dagger(\mathbf{p_2})|0\rangle\,, \tag{3.36}$$

where we have already dropped the infinite constants and used the first order interaction term, only. To explicitly compute $\mathrm{i}\,T_{fi}$, we use Eq. (2.88) and perform normal ordering (pull all creation operators in H_{int} to the left and all annihilation operators to the right). Using Eq. (2.95) all operators then vanish and only the terms with the commutators remain. Then we obtain

$$\mathrm{i}\,T_{fi} =: \mathrm{i}\,M(2\pi)^4\,\delta(p_1 + p_2 - p_1' - p_2') \quad \text{with} \quad \mathrm{i}\,M = \mathrm{i}\,\lambda\,, \tag{3.37}$$

where M denotes the matrix element.
In the following, normal ordered expressions will be marked by colons, e.g.

$$:a(\mathbf{p_1})\,a^\dagger(\mathbf{p_2}): = a^\dagger(\mathbf{p_2})\,a(\mathbf{p_1})\,. \tag{3.38}$$

For instance, normal ordering discards the infinite vacuum terms of the free Hamiltonian H_0 in Eq. (2.92), as

$$\begin{aligned} :H_0: \quad &= \quad \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} :\frac{1}{2}\,a^\dagger(\mathbf{p})\,a(\mathbf{p}) + \frac{1}{2}\,a(\mathbf{p})\,a^\dagger(\mathbf{p}): \\ &= \quad \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} a^\dagger(\mathbf{p})\,a(\mathbf{p})\,. \end{aligned} \tag{3.39}$$

Further, the normal ordered interaction Hamiltonian H_{int} already yields Eq. (3.37):

$$\begin{aligned} &\frac{\lambda}{4!}\int \mathrm{d}^4x\;:\phi(x)^4: \\ &\sim \frac{\lambda}{4!}:(a^\dagger)^2\,a^2 + a^\dagger\,a\,a^\dagger\,a + a\,a^\dagger\,a\,a^\dagger + a^\dagger\,a^2\,a^\dagger + a\,(a^\dagger)^2\,a + a^2\,(a^\dagger)^2: \\ &\sim \frac{\lambda}{4}(a^\dagger)^2\,a^2\,. \end{aligned} \tag{3.40}$$

Then,

$$\begin{aligned} &\frac{\lambda}{4}\left(\prod_i \int \frac{\mathrm{d}^3q_i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{q_i}}}\right)\mathrm{e}^{-\mathrm{i}x(q_3+q_4-q_1-q_2)}\sqrt{2\omega_{\mathbf{p_1'}}\,2\omega_{\mathbf{p_2'}}\,2\omega_{\mathbf{p_1}}\,2\omega_{\mathbf{p_2}}} \qquad (3.41) \\ &\qquad \cdot\langle 0|a(\mathbf{p_1'})\,a(\mathbf{p_2'})\Big(a^\dagger(\mathbf{q_1})\,a^\dagger(\mathbf{q_2})\,a(\mathbf{q_3})\,a(\mathbf{q_4})\Big)a^\dagger(\mathbf{p_1})\,a^\dagger(\mathbf{p_2})|0\rangle \\ &= \quad 4\cdot\frac{\lambda}{4}\left(\prod_i \int \frac{\mathrm{d}^3q_i}{(2\pi)^3}\sqrt{\frac{2\omega_{q_i}}{2\omega_{q_i}}}\right)\mathrm{e}^{-\mathrm{i}x(q_3+q_4-q_1-q_2)}\cdot\delta(\mathbf{p_1'}-\mathbf{q_1})\,\delta(\mathbf{p_2'}-\mathbf{q_2})\,\delta(\mathbf{p_1}-\mathbf{q_3})\,\delta(\mathbf{p_2}-\mathbf{q_4}) \\ &= \quad \lambda\,\mathrm{e}^{-\mathrm{i}x(p_1+p_2-p_1'-p_2')}\,, \end{aligned} \tag{3.42}$$

with e.g.

$$
\begin{aligned}
a(\mathbf{q_4})\, a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})\,|0\rangle &= \Big[a(\mathbf{q_4})\,,\, a^\dagger(\mathbf{p_1})\Big]\; a^\dagger(\mathbf{p_2})\,|0\rangle + a^\dagger(\mathbf{p_1})\, a(\mathbf{q_4})\, a^\dagger(\mathbf{p_2})\,|0\rangle \\
&= \Big((2\pi)^3\,\delta(\mathbf{q_4}-\mathbf{p_1})\, a^\dagger(\mathbf{p_2}) + a^\dagger(\mathbf{p_1})\;\Big[a(\mathbf{q_4})\,,\, a^\dagger(\mathbf{p_2})\Big] + a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})\, a(\mathbf{q_4})\Big)|0\rangle \\
&= (2\pi)^3\Big(\delta(\mathbf{q_4}-\mathbf{p_1})\, a^\dagger(\mathbf{p_2}) + \delta(\mathbf{q_4}-\mathbf{p_2})\, a^\dagger(\mathbf{p_1})\Big)\,. \qquad (3.43)
\end{aligned}
$$

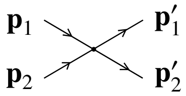
Lastly, using

$$
\frac{1}{(2\pi)^3}\int \mathrm{d}^4x\; \mathrm{e}^{-\mathrm{i}x(p_1+p_2-p_1'-p_2')} = \delta(p_1+p_2-p_1'-p_2')\,, \qquad (3.44)
$$

we obtain Eq. (3.37).
The difference between interaction Hamiltonian and normal ordered interaction Hamiltonian consequently gives the vacuum contributions:

$$
H_{\text{int}} = :H_{\text{int}}: + \frac{\lambda}{8}\int \mathrm{d}^4x\left(\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{p}}}\right)^2 + (a^\dagger\, a, a\, a^\dagger)\text{-terms}\,. \qquad (3.45)
$$

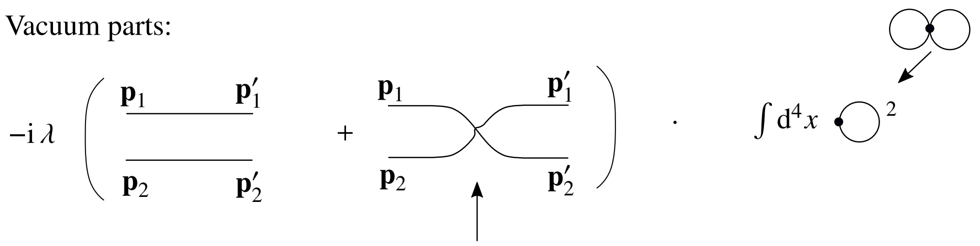
Let us now consider the interpretation of these terms:



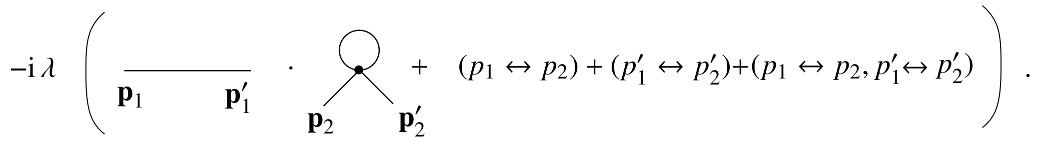
$: \quad -\mathrm{i}\,\lambda \quad \cdot \quad (2\pi)^4\,\delta(p_1+p_2-p_1'-p_2')$

interaction strength — 4-momentum conservation

Vacuum parts:



$\langle 0|\, a(\mathbf{p_1'})\, a(\mathbf{p_2'})\, a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})\,|0\rangle$



Note, that for the first term

$$
\begin{aligned}
&\langle 0|a(\mathbf{p_1'})\, a(\mathbf{p_2'})\, a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})|0\rangle\left(1 - \mathrm{i}\,\lambda\int \mathrm{d}^4x\; \mathrm{O}^2\right) \\
=\;& \langle 0|a(\mathbf{p_1'})\, a(\mathbf{p_2'})\, a^\dagger(\mathbf{p_1})\, a^\dagger(\mathbf{p_2})|0\rangle\left(\exp\left(-\mathrm{i}\,\lambda\int \mathrm{d}^4x\; \mathrm{O}^2\right) + O(\lambda^2)\right). \qquad (3.46)
\end{aligned}
$$

It can be shown, that the second order term is an infinite *phase*, that contains all vacuum processes. Nevertheless, as the phase/loops are infinite, the call for an appropriate treatment. Commonly one uses regularisation and renormalisation ("theory in a box", see chapter 7).

Next, we discuss a core ingredient of perturbation theory: the **propagator**.
Starting in the Heisenberg picture, we introduce the vacuum of the full theory $|\Omega\rangle$, with

$$\mathrm{i}\partial_t |\Omega\rangle = 0\,. \tag{3.47}$$

In the Heisenberg picture the operators evolve with the full Hamiltonian, i.e.

$$\mathrm{i}\partial_t \phi_H = [\phi_H\,,\,H]\,, \tag{3.48}$$

with

$$\phi_H = \mathrm{e}^{\mathrm{i}Ht}\,\phi(0,\mathbf{x})\,\mathrm{e}^{-\mathrm{i}Ht}\,. \tag{3.49}$$

We now link this to the interaction picture, where the states evolve with H_{int} and

$$\begin{aligned} |f(t)\rangle_I &= U(t,0)\,|f(0)\rangle_I \\ \mathrm{i}\partial_t \phi_I &= [\phi_I\,,\,H_0]\,. \end{aligned} \tag{3.50}$$

Hence,

$$\phi_I(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(a(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}px} + a^\dagger(\mathbf{p})\,\mathrm{e}^{\mathrm{i}px}\Big)_{p_0=\omega_{\mathbf{p}}}\,. \tag{3.51}$$

Using, that

$$U(t,0) = \mathrm{e}^{\mathrm{i}H_0 t}\,\mathrm{e}^{-\mathrm{i}Ht}\,, \tag{3.52}$$

it follows

$$\phi_H(x) = U(0,x^0)\,\phi_I(x)\,U(x^0,0)\,, \tag{3.53}$$

with

$$\begin{aligned} \phi_H(x)\,|f\rangle_H &= U(0,x^0)\,\phi_I(x)\,U(x^0,0)\,|f\rangle_H \\ \mathrm{i}\partial_t U(x^0,0)\,|f\rangle_H &= H_{\text{int}}\,U(x^0,0)\,|f\rangle_H\,. \end{aligned} \tag{3.54}$$

It is tempting to identify $U(x^0,0)\,|f\rangle_H$ with the interaction picture states $|f(t)\rangle_I$. At $t\to\pm\infty$, λ is switched off adiabatically, and $|f\rangle_I$ tend to free in/out states. Considering, that $U(0,\infty) = U(\infty,0)^{-1}$, we have

$$\begin{aligned} \langle\Omega|\,U(0,x^0) &= \langle\Omega|\,U(0,\infty)\,U(\infty,x^0) \\ &= \sum_n \langle\Omega|\,U(0,\infty)\,|n\rangle_I\ {}_I\langle n_I|\,U(\infty,x^0) \\ &= \langle\Omega|\,U(0,\infty)\,|0\rangle\,\langle 0|\,U(\infty,x^0)\,, \end{aligned} \tag{3.55}$$

where in the last step we used, that *adiabatically* indicates: $\big|\text{n-particles}\big\rangle_{\text{free}} \xrightarrow{U} \big|\text{n-particles}\big\rangle_{\text{full}}$. Also, it is

$$U(x^0,0)\,|\Omega\rangle = U(x^0,-\infty)\,|0\rangle\,\langle 0|\,U(-\infty,0)\,|\Omega\rangle\,. \tag{3.56}$$

Further note, that

$$\begin{aligned} \mathrm{i}\partial_t U(t,0) &= H_I(t)\,U(t,0)\,, \\ H_I(t) &= H_{\text{int}}(t) = \mathrm{e}^{\mathrm{i}H_0 t}\,H_{\text{int}}\,\mathrm{e}^{-\mathrm{i}H_0 t} \\ &= \frac{\lambda}{4!}\int \mathrm{d}^3 x\,\phi_I(x)^4\,, \\ \mathrm{i}\partial_t H_I &= [H_I\,,\,H_0]\,. \end{aligned} \tag{3.57}$$

Thus,

$$
\begin{aligned}
\mathrm{i}\partial_t \phi_H(x) &= U(0,t)\, H_{\text{int}}(x^0)\, \phi_I(t)\, U(t,0) - U(0,t)\, \phi_I\, H_{\text{int}}(x^0)\, U(t,0) + ... \\
&\quad ... + U(0,t)\ [\phi_I(x)\,,\ H_{\text{int}}(t)]\ U(t,0) \\
&= 0\,.
\end{aligned}
\tag{3.58}
$$

We now compute the propagator

$$
\begin{aligned}
&\langle\Omega|T\ \phi_H(x)\phi_H(y)|\Omega\rangle \\
=\ &\langle\Omega|\phi_H(x)\phi_H(y)|\Omega\rangle\ \theta(x^0-y^0) + \langle\Omega|\phi_H(y)\phi_H(x)|\Omega\rangle\ \theta(y^0-x^0)\,.
\end{aligned}
\tag{3.59}
$$

For $x^0 > 0 > y^0$:

$$\langle\Omega|T\ \phi_H(x)\phi_H(y)|\Omega\rangle$$

$$\text{using Eq. (3.53)} \quad \rightarrow \tag{3.60}$$

$$
\begin{aligned}
&= \langle\Omega|U(0,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,0)|\Omega\rangle \\
&= \langle 0|U(\infty,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-\infty)|0\rangle\cdot\frac{1}{\Big(\langle\Omega|\,U(0,\infty)\,|0\rangle\cdot\langle 0|\,U(-\infty,0)\,|\Omega\rangle\Big)^{-1}}\,,
\end{aligned}
\tag{3.61}
$$

where we used, that in general

$$U(x^0,z^0) = U(x^0,y^0)\,U(y^0,z^0) \quad \text{for} \quad x^0 > y^0 > z^0\,. \tag{3.62}$$

This follows straightforwardly from Eq. (3.31). Note, that the dominator in Eq. (3.61) is (a product of two) phases, i.e. $|\,\langle\Omega|\,U(0,\infty)\,|0\rangle\,| = 1$. This becomes evident, when considering:

$$|\,\langle\Omega|\,U(0,x^0)| = 1\,, \tag{3.63}$$

as from U being unitary it follows

$$|\,\langle\Omega|\,U(0,x^0)|^2 = \langle\Omega|\,U(0,x^0)\,U^\dagger(0,x^0)\,|\Omega\rangle = \langle\Omega|\Omega\rangle = 1\,. \tag{3.64}$$

And analogously

$$|\,\langle 0|\,U(-\infty,x^0)|^2 = 1\,. \tag{3.65}$$

Combining Eq. (3.64) and Eq. (3.65) yields

$$|\,\langle\Omega|\,U(0,\infty)\,|0\rangle\,| = 1\,. \tag{3.66}$$

Hence,

$$\langle\Omega|\,U(0,\infty)\,|0\rangle^{-1} = \langle\Omega|\,U(0,\infty)\,|0\rangle^* = \langle\Omega|\,U^\dagger(0,\infty)\,|0\rangle\,, \tag{3.67}$$

i.e. the normalisation factor in Eq. (3.61) is a phase. Likewise to Eq. (3.55), we use the adiabaticity and get

$$
\begin{aligned}
&\langle\Omega|\,U(0,\infty)\,|0\rangle^{-1}\,\langle 0|\,U(-\infty,0)\,|\Omega\rangle^{-1} \\
\text{using Eq. (3.67)} \quad \rightarrow \quad &= \langle 0|\,U(\infty,0)\,|\Omega\rangle\,\langle\Omega|\,U(0,-\infty)\,|0\rangle \\
&= \langle 0|U(\infty,0)\,U(0,-\infty)|0\rangle = \langle 0|U(-\infty,\infty)|0\rangle \\
&= \langle 0|S\,|0\rangle = \langle 0|T\,\exp\left(-\mathrm{i}\int \mathrm{d}t\,H_{\text{int}}(t)\right)|0\rangle\,.
\end{aligned}
\tag{3.68}
$$

We also have for the numerator of Eq. (3.61)

$$\begin{aligned} &\langle 0|U(\infty, x^0)\,\phi_I(x)\,U(x^0, y^0)\,\phi_I(y)\,U(y^0, -\infty)|0\rangle \\ = \quad &\langle 0|T\ \phi_I(x)\,\phi_I(y)\exp\left(-\mathrm{i}\int \mathrm{d}t\ H_{\mathrm{int}}(t)\right)|0\rangle\ . \end{aligned} \tag{3.69}$$

Finally, with $\phi_I = \phi$, and the analogous result for $y^0 > x^0$, we obtain for the

propagator (two-point function)

$$\langle\Omega|T\ \phi_H(x)\,\phi_H(y)|\Omega\rangle = \frac{\langle 0|T\ \phi(x)\,\phi(y)\exp\left(-\mathrm{i}\int \mathrm{d}t\ H_{\mathrm{int}}(t)\right)|0\rangle}{\langle 0|T\ \exp\left(-\mathrm{i}\int \mathrm{d}t\ H_{\mathrm{int}}(t)\right)|0\rangle}\ . \tag{3.70}$$

This is straightforwardly extended to the

propagator (n-point function)

$$\langle\Omega|T\ \phi_H(x_1)\cdots\phi_H(x_n)|\Omega\rangle = \frac{\langle 0|T\ \phi(x_1)\cdots\phi(x_n)\exp\left(-\mathrm{i}\int \mathrm{d}t\ H_{\mathrm{int}}(t)\right)|0\rangle}{\langle 0|T\ \exp\left(-\mathrm{i}\int \mathrm{d}t\ H_{\mathrm{int}}(t)\right)|0\rangle}\ . \tag{3.71}$$

Note, that the denominators in Eq. (3.70) and Eq. (3.71) are phases. For example, the linear term in λ is

$$-\mathrm{i}\,\langle 0|\int \mathrm{d}t\ H_{\mathrm{int}}|0\rangle = -\mathrm{i}\,\lambda\,\langle 0|\int \mathrm{d}^4x\ \phi(x)^4|0\rangle = -\frac{\mathrm{i}}{8}\int \mathrm{d}^4x\left(\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{q}}}\right)^2\ , \tag{3.72}$$

which cancels the vacuum term in Eq. (3.45). We remark, that both, phase factor (denominator) and the vacuum contributions in the nominator, are infinite and cancel.

II. Wick's Theorem

We have seen, that the computation of scattering amplitudes relates to the computation of time ordered n-point functions

$$\langle 0|T\ \phi(x_1)\cdots\phi(x_n)\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4y\ \mathcal{L}_{\mathrm{int}}(y)}|0\rangle\ , \tag{3.73}$$

where

$$-\int \mathrm{d}t\ H_{\mathrm{int}} = \int \mathrm{d}t\ L_{\mathrm{int}} = \int \mathrm{d}^4y\ \mathcal{L}_{\mathrm{int}}(y)\,. \tag{3.74}$$

Since the coupling is small, $\lambda \ll 1$ we can usually expand the exponential in powers of λ. For example in first order we then obtain,

$$\langle 0|T\ \phi(x_1)\cdots\phi(x_n)\prod_{i=1}^{m}\mathcal{L}_{\mathrm{int}}(y_i)|0\rangle = \frac{1}{(4!)^m}\langle 0|\phi(x_1)\cdots\phi(x_n)\,\phi(x_{n+1})\cdots\phi(x_{n+4m})|0\rangle\ , \tag{3.75}$$

with $x_{n+1},\ldots,x_{n+4} = y_1;\cdots;x_{n+4(m-1)},\ldots,x_{n+4m} = y_m$. The only building block in Eq. (3.73) is

$$\langle 0|T\ \phi(x_1)\cdots\phi(x_n)|0\rangle\ . \tag{3.76}$$

We stress that in this formula the important simplification is that: $\phi = \phi_I$ is free.

Using this feature, that $\phi = \phi_I$ is free, we can now approach calculating the time ordered n-point functions. Let us start with the example of the two-point function.
Indeed, for $x_1^0 > \cdots > x_n^0$ Eq. (3.76) reduces to $\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$, and we simply have to use the canonical commutation relations, Eq. (2.88). In the next step, we use normal ordering and the vanishing expectation value of the ordered parts. In particular we try to write,

$$\begin{aligned} T\phi(x_1)\cdots\phi(x_n) &= \; :\phi(x_1)\cdots\phi(x_n): + :(n-2)-\text{field operators}: \\ &\quad + :(n-4)-\text{field operators}: + \ldots + \text{rest without field operators.} \end{aligned} \tag{3.77}$$

The trick is now that all the normal ordered parts give vanishing expectation value and we have,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \text{rest without field operators.} \tag{3.78}$$

For the two-point function this works as follows:
Firstly, we rewrite

$$\phi(x) = \phi_+(x) + \phi_-(x)\,, \tag{3.79}$$

with

$$\begin{aligned} \phi_+(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\, a^\dagger(\mathbf{p})\, \mathrm{e}^{\mathrm{i}px} \\ \phi_-(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\, a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}px}\,. \end{aligned} \tag{3.80}$$

For $x^0 > y^0$ it is

$$\begin{aligned} T\,\phi(x)\,\phi(y) &= \phi_+(x)\,\phi_+(y) + \phi_+(x)\,\phi_-(y) + \phi_-(x)\,\phi_+(y) + \phi_-(x)\,\phi_-(y) \\ &= \phi_+(x)\,\phi_+(y) + \phi_+(x)\,\phi_-(y) + \Big(\phi_+(y)\,\phi_-(x) + [\phi_-(x)\,,\,\phi_+(y)]\Big) + \phi_-(x)\,\phi_-(y)\,. \end{aligned} \tag{3.81}$$

Thus,

$$T\,\phi(x)\,\phi(y)\Big|_{x_0>y_0} = \,:\phi(x)\,\phi(y): + [\phi_-(x)\,,\,\phi_+(y)]\,, \tag{3.82}$$

where

$$:\phi_-(x)\,\phi_+(y): = \phi_+(y)\,\phi_-(x) \quad \forall x\,, \tag{3.83}$$

from

$$:a(\mathbf{p})\,a^\dagger(\mathbf{q}): = a^\dagger(\mathbf{q})\,a(\mathbf{p})\,. \tag{3.84}$$

The procedure for the case $x^0 < y^0$ is completely analogous. The two parts can be written in one formula by using the θ-function.

Taking the vacuum expectation values, the normal ordered part vanishes. The time ordered propagator for the two-point function is called

Feynman-propagator

$$\begin{aligned} \mathcal{D}_F(x-y) &= \langle 0|T\,\phi(x)\phi(y)|0\rangle \\ &= [\phi_-(x)\,,\,\phi_+(y)]\,\theta(x^0-y^0) + [\phi_-(y)\,,\,\phi_+(x)]\,\theta(y^0-x^0)\,. \end{aligned} \tag{3.85}$$

The Feynman-propagator is the key-ingredient in (time ordered) perturbation theory. To explicitly calculate the Fenyman-propagator we consider

$$
\begin{aligned}
&[\phi_-(x)\,,\,\phi_+(y)]\;\theta(x^0-y^0)\\
=\;&\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}\,2\omega_{\mathbf{q}}}}\Big[a(\mathbf{p})\,,\,a^\dagger(\mathbf{q})\Big]\;\mathrm{e}^{-\mathrm{i}\,(px+qy)}\;\theta(x^0-y^0)\\
&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)}\;\theta(x^0-y^0)\\
\Rightarrow\quad &\mathcal{D}_F(x-y)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\Big(\mathrm{e}^{-\mathrm{i}\,p(x-y)}\;\theta(x^0-y^0)+\mathrm{e}^{\mathrm{i}\,p(x-y)}\;\theta(y^0-x^0)\Big)\,.
\end{aligned}
\tag{3.86}
$$

In the last step we have used that the case $x^0 < y^0$ can be treated analogously.
The Feynman propagator can be written as

Feynman-propagator (explicit)

$$
\mathcal{D}_F(x-y)\;=\;\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\,\epsilon}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,,\quad \text{as}\quad \epsilon\to 0\,. \tag{3.87}
$$

To see that this is true we use the

Residue theorem:
The contour integral of a function $f(z)$ around a closed, counterclockwise path encircling a domain where $f(z)$ has a finite number of isolated singularities (poles at $z = z_i$, $i = 1, 2, \ldots, n$) is

$$
\oint \mathrm{d}z\; f(z) = 2\pi\,\mathrm{i}\sum_{i=1}^{n}\mathrm{Res}(f, z_i)\,, \tag{3.88}
$$

where the residue of $f(z)$ at a simple pole z_i is $\mathrm{Res}(f, z_i) = \lim_{z\to z_i}(z - z_i)\, f(z)$.

The integrand in Eq. (3.87) has poles at $(p^0)^2 = \pm\sqrt{\mathbf{p}^2 + m^2 - \mathrm{i}\,\epsilon}$, as shown in figure 3.3. We now have to consider where we have to close the contour. For $x^0 - y^0 > 0$ the integrand grows exponentially as

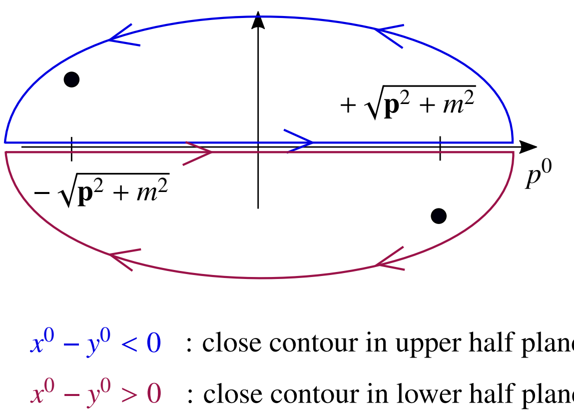


Figure 3.3.: Sketch of the poles of the integrand of Eq. (3.87) and the contour for the Residue theorem.

we go into the upper half plane. We therefore have to close the contour in the lower half plane where the integrand exponential is decreasing. The relevant pole is then at $p^0_- = \sqrt{\mathbf{p}^2 + m^2 - \mathrm{i}\,\epsilon} \to \omega_{\mathbf{p}}$, and thus

$$\begin{aligned} \mathcal{D}_F(x-y) &= -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{2\pi\,\mathrm{i}}{2\pi} \operatorname{res}_{p^0_-} \left(\frac{\mathrm{e}^{-\mathrm{i}\,p(x-y)}}{p^2 - m^2 + \mathrm{i}\,\epsilon} \right) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \mathrm{i}\, \frac{\mathrm{e}^{-\mathrm{i}\,p(x-y)}}{2\mathrm{i}\,\omega_{\mathbf{p}}} \bigg|_{p^0=\omega_{\mathbf{p}}} . \end{aligned} \tag{3.89}$$

This works similarly for $x^0 - y^0 < 0$. The difference is now just that we have to close the contour in the upper half plane. The distinction of these two cases can be accounted for by a θ-function. We therefore have the equivalence of Eq. (3.87) and Eq. (3.86).

We have parametrised the time ordered propagator in terms of commutators. On operator level we have

$$T\,\phi(x)\,\phi(y) \;=\; :\phi(x)\,\phi(y):\; + [\phi_-(x)\,,\,\phi_+(y)]\;\Theta(x^0-y^0) + [\phi_-(y)\,,\,\phi_+(x)]\;\Theta(y^0-x^0)\,. \tag{3.90}$$

This can be written as

$$T\,\phi(x)\,\phi(y) = :\phi(x)\,\phi(y):\; + \wick{\c\phi(x)\c\phi(y)}\,, \tag{3.91}$$

with the *contraction*

$$\begin{aligned} \wick{\c\phi(x)\,\c\phi(y)} &= [\phi_-(x)\,,\,\phi_+(y)]\;\Theta(x^0-y^0) + [\phi_-(y)\,,\,\phi_+(x)]\;\Theta(y^0-x^0) \\ &= \mathcal{D}_F(x-y)\,. \end{aligned} \tag{3.92}$$

Note, that $\mathcal{D}_F(x-y)$ is a c-number (and not an operator!). We use this, to generalise the time ordering to a product of n fields. This is

Wick's theorem

$$T\,\phi(x_1)\cdots\phi(x_n) = :\phi(x_1)\cdots\phi(x_n) + \text{all contractions}:\,, \tag{3.93}$$

where

$$\begin{aligned} & \wick{\phi(x_1)\cdots\c\phi(x_i)\cdots\c\phi(x_j)\cdots\phi(x_n)} \\ =\;& \phi(x_1)\cdots\phi(x_{i-1})\,\phi(x_{i+1})\cdots\phi(x_{j-1})\,\phi(x_{j+i})\cdots\phi(x_n)\,\wick{\c\phi(x_i)\,\c\phi(x_j)}. \end{aligned} \tag{3.94}$$

The content of this notation will become clearer by an example.

Example 3-6: 4-point correlation function.

$$\begin{aligned} T\,\phi(x_1)\cdots\phi(x_4) &= T\,\phi_1\,\phi_2\,\phi_3\,\phi_4 \\ &= \;:\phi_1\,\phi_2\,\phi_3\,\phi_4 + \wick{\c\phi_1\,\c\phi_2}\,\phi_3\,\phi_4 + \phi_1\,\phi_2\,\wick{\c\phi_3\,\c\phi_4} + \wick{\c\phi_1\,\phi_2\,\c\phi_3}\,\phi_4 \\ &\qquad + \wick{\c\phi_1\,\phi_2\,\phi_3\,\c\phi_4} + \phi_1\,\wick{\c\phi_2\,\c\phi_3}\,\phi_4 + \phi_1\,\wick{\c\phi_2\,\phi_3\,\c\phi_4} \\ &\qquad + \wick{\c1\phi_1\,\c1\phi_2\,\c2\phi_3\,\c2\phi_4} + \wick{\c1\phi_1\,\c2\phi_2\,\c1\phi_3\,\c2\phi_4} + \wick{\c1\phi_1\,\c2\phi_2\,\c2\phi_3\,\c1\phi_4}\;:\,, \end{aligned} \tag{3.95}$$

where e.g.

$$: \overbracket{\phi_1 \phi_2} \phi_3 \phi_4 : = : \phi_3 \phi_4 : \ \overbracket{\phi_1 \phi_2} \ = : \phi_3 \phi_4 : \ \mathcal{D}_F(x_1 - x_2)\,. \tag{3.96}$$

Importantly we have,

$$\langle 0| : \mathcal{O} : |0\rangle = 0\,. \tag{3.97}$$

With this it follows

$$\begin{aligned} \langle 0|T\ \phi(x_1)\cdots\phi(x_4)|0\rangle \ &= \ \mathcal{D}_F(x_1 - x_2)\,\mathcal{D}_F(x_2 - x_3) + \mathcal{D}_F(x_1 - x_3)\,\mathcal{D}_F(x_2 - x_4) \\ &\quad + \mathcal{D}_F(x_1 - x_4)\,\mathcal{D}_F(x_2 - x_3)\,, \end{aligned} \tag{3.98}$$

where each term corresponds to one of the terms with two contractions in Eq. (3.95).

It remains to prove Wick's theorem. We will do this by induction. First we show, that it holds for the one- and two-point function:

$$\begin{aligned} n = 1, 2: \qquad T\ \phi_1 \ &= \ : \phi_1 : \\ T\ \phi_1 \phi_2 \ &= \ : \phi_1 \phi_2 : + \overbracket{\phi_1 \phi_2}\,. \end{aligned} \tag{3.99}$$

Next, we assume that Wick's theorem applies to the n-point function, i.e. $T\ \phi_2 \cdots \phi_{n+1}$. Without loss of generality we can assume that $x_1^0 \geq x_i^0\ \forall i$. (If this is not the case we can just relabel the points and the time-ordering takes care of the rest.) Then

$$\begin{aligned} &T\ \phi_1 \cdots \phi_{n+1} && (3.100) \\ &\quad = \ \phi_1 T\ \phi_2 \cdots \phi_{n+1} \\ &\quad = \ \phi_1 \Big(: \phi_2 \cdots \phi_{n+1} + \text{all contractions} : \Big) \\ &\quad = \ \Big(\phi_{1_+} + \phi_{1_-}\Big)\Big(: \phi_2 \cdots \phi_{n+1} + \text{all contractions} : \Big) \\ &\quad = \ : \phi_1 \cdots \phi_{n+1} + [\phi_{1_-}, \phi_2]\ \phi_3 \cdots \phi_{n+1} + \phi_2\ [\phi_{1_-}, \phi_3]\ \phi_4 \cdots \phi_{n+1} + \cdots + \phi_2 \cdots [\phi_{1_-}, \phi_{n+1}] : \\ &\qquad + \Big(\phi_{1_+} + \phi_{1_-}\Big)\Big(: \text{all contractions} : \Big). && (3.101) \end{aligned}$$

Using

$$[\phi_{1_-}, \phi_i] = [\phi_{1_-}, \phi_{i_+}] = \overbracket{\phi_1 \phi_i}\,, \tag{3.102}$$

and similarly as in Eq. (3.101) for

$$\Big(\phi_{1_+} + \phi_{1_-}\Big)\Big(: \text{all contractions} : \Big), \tag{3.103}$$

we obtain

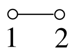
$$T\ \phi(x_1)\cdots\phi(x_{n+1}) = : \phi(x_1)\cdots\phi(x_{n+1}) + \text{all contractions} :, \tag{3.104}$$

which completes the induction.

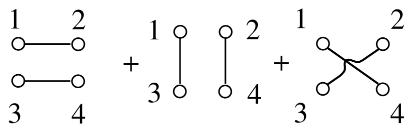
III. Feynman Rules

With Wick's theorem (Eq. (3.93)) we write every time ordered n-point function as product of Feynman propagators (Eq. (3.85)) plus the normal ordered terms. Indeed, since perturbation theory contains only vacuum expectation values of n-point functions, Wick's theorem reduces all those to products of Feynman propagators.

We introduce the diagrammatical notation

$$\mathcal{D}_F(x_1 - x_2) = \langle 0|T\ \phi_1\,\phi_2|0\rangle =$$  .

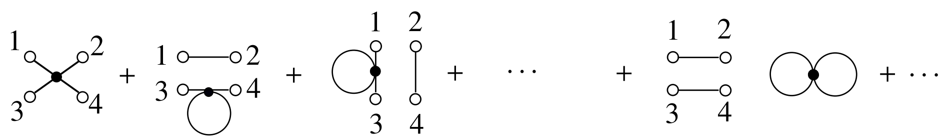
Now, let us again consider the 2-2 scattering, as it is a relevant example.
The zeroth order term in λ, i.e. the term without interaction is simply given by the expectation value of the 4-point function:

$$\mathcal{O}(\lambda^0): \qquad \langle 0|T\ \phi_1\,\phi_2\,\phi_3\,\phi_4|0\rangle =$$  .

The first order term $\mathcal{O}(\lambda^1)$ is

$$\begin{aligned}
&\frac{-\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x\ \langle 0|T\ \phi_1\,\phi_2\,\phi_3\,\phi_4\ \phi\,\phi\,\phi\,\phi|0\rangle \\
&\quad = \frac{-\mathrm{i}\lambda}{4!}\int \mathrm{d}^4x \Big[\phi_1\,\phi\,\phi_2\,\phi\,\phi_3\,\phi\,\phi_4\,\phi\ \cdot 4! + \phi\,\phi\Big(\phi_1\,\phi\,\phi_2\,\phi\,\phi_3\,\phi_4\Big)\cdot 12 + \text{ perm.} \qquad (3.105)\\
&\qquad\qquad + \phi\,\phi\,\phi\,\phi\Big(\phi_1\,\phi_2\,\phi_3\,\phi_4\Big)\cdot 3 + \textit{perm.}\Big].
\end{aligned}$$

Note, that the factor 4! accounts for all possibilities to contract ϕ^4 with $\phi_1\cdots\phi_4$, and the factors 12 and 3 account for permutations of the contractions, that give an identical expression. This will be discussed further below. Diagrammatically and without the symmetry factors this writes

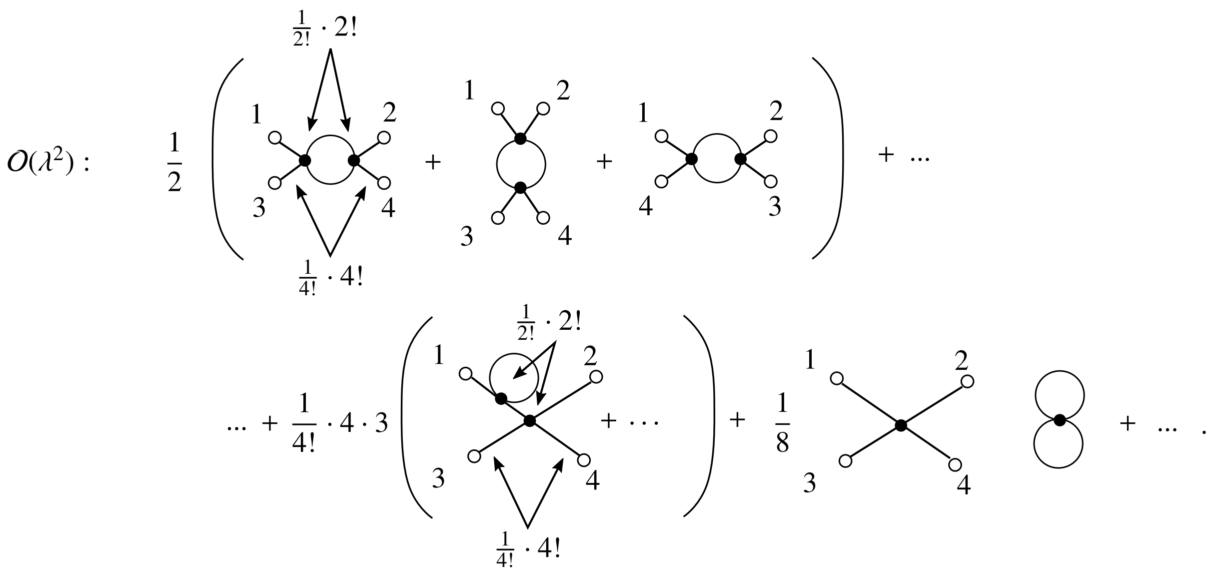
$\mathcal{O}(\lambda^1)$:  ,

where the vertices correspond to $\left(-\mathrm{i}\,\lambda\int \mathrm{d}^4x\right)$.

The second order term

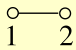
$$\frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{4!}\right)^2\int \mathrm{d}^4x\int \mathrm{d}^4z\ \langle 0|T\ \phi_1\,\phi_2\,\phi_3\,\phi_4\ \phi(x)^4\,\phi(z)^4|0\rangle$$

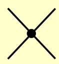
comprises



To determine the right prefactor for each diagram, we need to do some combinatorics: The permutations of how to contract $\phi\,\phi\,\phi\,\phi$ in H_{int} with the external fields gives a factor 4!, which cancels with the denominator in $\frac{-\mathrm{i}\lambda}{4!}$. This originally motivated the normalisation in Eq. (3.18). When loops are present, we further have to account for the symmetries that result from contracting the ϕ^4 amongst each others in H_{int}. For this purpose we introduce the **symmetry factor $\frac{1}{S}$**, where S corresponds to the number of interchanging components without changing the diagram.
Now we can write down the

Feynman rules (position space)

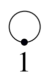
i)  $= \mathcal{D}_F(x_1 - x_2)$

ii)  $= (-\mathrm{i}\,\lambda) \int \mathrm{d}^4 x$

iii) multiplication with $\dfrac{1}{S}$.

(3.106)

However, there is one thing that we still have to take care of. From the first order onwards we encounter pieces that are completely disconnected from any of the points $x_1, \ldots, x_4$. They have the form of "vacuum bubbles". These terms contain infinities, but luckily we will see that these can be removed by a proper normaliaztion.
Nevertheless let us first have a look at vacuum bubbles and what kind of infinities they comprise. The simplest is actually just a closed loop of the propagator. This corresponds to the expression $\langle 0|T\ \phi_1\,\phi_1|0\rangle$. With Eq. (3.87), we find

$$\mathcal{D}_F(0) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\ \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\,\epsilon} =$$  .

This is a singularity, which will be removed by an appropriate adjustment of the computation (renormalisation). In particular, we note that the momentum dimension of $\mathcal{D}_F(0)$ is two. Therefore, we argue

$$\mathcal{D}_F(0) = M^2 + \text{infinite}\,. \tag{3.107}$$

However, the first such term in our perturbation theory is even worse. It is,

$$\sim (-\mathrm{i}\,\lambda) \int \mathrm{d}^4 x (\mathcal{D}_F(0))^2 s \sim V_4 \cdot (\infty)^2. \tag{3.108}$$

Luckily for now we do not have to deal with all those divergencies directly. The important step is to realize that so far we have only calculated the numerator of the products of Heisenberg operators that we are after. The full expression that we want is the vacuum expectation value Eq. (3.71)

$$\langle \Omega | T\ \phi_{H,1} \cdots \phi_{H,n} | \Omega \rangle := \frac{\langle 0 | T\ \phi_1 \cdots \phi_n \exp\left(\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}\right) | 0 \rangle}{\langle 0 | T\ \exp\left(\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}\right) | 0 \rangle}\,. \tag{3.109}$$

For the computation we note, that each term $\langle 0 | T\ \phi_1 \cdots \phi_n\ \frac{(\mathcal{L}_{\text{int}})^m}{m!} | 0 \rangle$ can be ordered in terms of contractions between the ϕ_i and the $\mathcal{L}_{\text{int}}$'s:

$$\begin{aligned} \langle 0 | T\ \phi_1 \cdots \phi_n\ \frac{(\mathcal{L}_{\text{int}})^m}{m!} | 0 \rangle \quad &= \quad \langle 0 | T\ \phi_1 \cdots \phi_n | 0 \rangle\ \frac{1}{m!}\ \langle 0 | T\ (\mathcal{L}_{\text{int}})^m | 0 \rangle \\ &+ \langle 0 | T\ \phi_1 \underbracket{\cdots \phi_n\ \mathcal{L}_{\text{int}}} | 0 \rangle\ \frac{1}{(m-1)!}\ \langle 0 | T\ (\mathcal{L}_{\text{int}})^{m-1} | 0 \rangle + \ldots\,, \end{aligned} \tag{3.110}$$

where "$\underbracket{\quad\quad}$" denotes all contractions, where internal fields from the interaction Hamiltonian are connected to *external* fields (and not amongst themselves).
We use that

$$\begin{aligned} &\frac{1}{m!}\ \langle 0 | T\ \phi_1 \cdots \phi_n\ (\mathcal{L}_{\text{int}})^m | 0 \rangle \Big|_{O(2-\mathcal{L}_{\text{int}}-\text{contr.})} \\ &\quad = \quad \frac{1}{m!}\ \langle 0 | T\ \phi_1 \underbracket{\cdots \phi_n\ (\mathcal{L}_{\text{int}}})^2 | 0 \rangle\ \ \langle 0 | T\ (\mathcal{L}_{\text{int}})^{m-2} | 0 \rangle \cdot \frac{m \cdot (m-1)}{2} \\ &\quad = \quad \frac{1}{2}\ \langle 0 | T\ \phi_1 \underbracket{\cdots \phi_n\ (\mathcal{L}_{\text{int}}})^2 | 0 \rangle\ \frac{1}{(m-2)!}\ \langle 0 | T\ (\mathcal{L}_{\text{int}})^{m-2} | 0 \rangle\,, \end{aligned} \tag{3.111}$$

and that in general the combinatorics factor for $l - \mathcal{L}_{\text{int}}$-contractions is

$$\frac{1}{m!}\binom{m}{l} = \frac{1}{m!}\frac{m!}{(m-l)!\,l!} = \frac{1}{(m-l)!\,l!}\,. \tag{3.112}$$

Then,

$$\begin{aligned} \langle 0 | T\ \phi_1 \cdots \phi_n \exp\left(\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}\right) | 0 \rangle &= \Big(\langle 0 | T\ \phi_1 \cdots \phi_n | 0 \rangle + \int \mathrm{d}^4 y_1\ \langle 0 | T\ \phi_1 \underbracket{\cdots \phi_n\ \mathcal{L}_{\text{int}}} | 0 \rangle \\ &+ \int \mathrm{d}^4 y_1 \mathrm{d}^4 y_2\ \langle 0 | T\ \phi_1 \underbracket{\cdots \phi_n\ (\mathcal{L}_{\text{int}}})^2/2 | 0 \rangle + \ldots \Big) \cdot\ \langle 0 | T\ \exp\left(\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}\right) | 0 \rangle\,. \end{aligned} \tag{3.113}$$

Consequently the denominator cancels all the vacuum terms. Therefore, the vacuum expectation value is given by

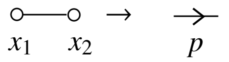
$$\frac{\langle 0 | T\ \phi_1 \cdots \phi_n \exp\left(\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}\right) | 0 \rangle}{\langle 0 | T\ \exp\left(\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}\right) | 0 \rangle} = \langle 0 | T\ \phi_1 \underbracket{\cdots \phi_n\ \mathrm{e}^{\mathrm{i} \int \mathrm{d}^4 x\ \mathcal{L}_{\text{int}}}} | 0 \rangle\,, \tag{3.114}$$

which corresponds to all diagrams without "vacuum bubbles".
As most computations are carried out in **momentum space**, we will conclude this section, by examining

the Fourier transforms.
The Feynman propagator becomes

$$\mathcal{D}_F(x_1 - x_2) \to \mathcal{D}_F(p) = \frac{\mathrm{i}}{p_1 - m^2 + \mathrm{i}\epsilon}(2\pi)^4\, \delta^4(p_1 - p_2)\,, \tag{3.115}$$

and



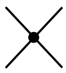
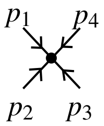
.

For the vertices we write

$$-\mathrm{i}\lambda \int \mathrm{d}^4x\, \phi(x)^4 = -\mathrm{i}\lambda \int \prod_{i=1}^{4} \frac{\mathrm{d}^4 p_i}{(2\pi)^4} \phi(p_i) \cdot (2\pi)^4\, \delta^4(p_1 + p_2 + p_3 + p_4)\,, \tag{3.116}$$

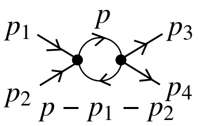
where the delta-function indicates momentum conservation. Hence,

$-\mathrm{i}\lambda \quad \to \quad -\mathrm{i}\lambda\,(2\pi)^4\, \delta^4(p_1 + p_2 + p_3 + p_4)$


$\to$


$p_4 = -(p_1 + p_2 + p_3)$.

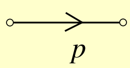
For example:



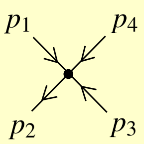
$$= (2\pi)^4\, \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \quad \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon}\, \frac{1}{(p - p_1 - p_2)^2 - \dot{m}^2 + \mathrm{i}\epsilon}$$

We now have the

Feynman rules (momentum space)

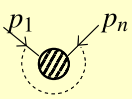
i) 

$= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}$

ii) 

$= -\mathrm{i}\lambda$ and $p_4 = -(p_1 + p_2 + p_3)$ (momentum conservation)

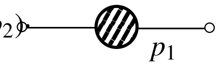
iii) $\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$ for each loop

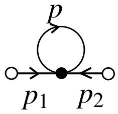
iv) $(2\pi)^4\, \delta^4(\sum_i p_i)$ for 

v) multiplication with $\dfrac{1}{S}$.

(3.117)

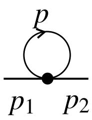
Example 3-7: two-point function in momentum space.

$\langle\Omega|T\ \phi_H(p_1)\,\phi_H(-p_2)|\Omega\rangle = (2\pi)^4\,\delta^4(p_1+p_2)$ 

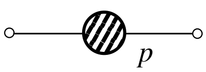
$= \dfrac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\epsilon}(2\pi)^4\,\delta^4(p_1+p_2) + \dfrac{1}{2}$  $+\,O(\lambda^2)\,.$

$\dfrac{1}{S}$

Without external propagators it is:

 $= -\mathrm{i}\lambda \int \dfrac{\mathrm{d}^4 p}{(2\pi)^4}\,\dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} = -\mathrm{i}\,\Pi + O(\lambda^2)\,.$

Heuristics:

 $= \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}(-\mathrm{i}\,\Pi)\,\dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} + O(\lambda^2)$

$$= \frac{\mathrm{i}}{p^2 - m^2 - \Pi + \mathrm{i}\epsilon} + O(\lambda^2) \quad .$$

It follows, that we have an **interacting mass** $m^2 - \Pi$, which is finite. In general (beyond 1-loop) it holds:

$$\Pi \to \Pi(p) \tag{3.118}$$

The proper treatment is again provided through renormalisation and the LSZ-formalism.

IV. Cross Section

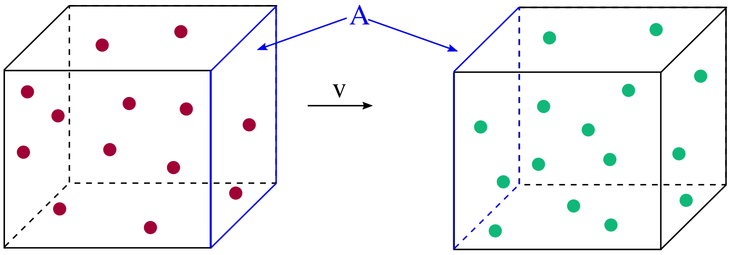
We start by considering an exemplary fixed target experiment. We first shoot a single point particle on a target with a number density ρ_A and “cross sectional” area σ of objects with radius r_A. For a thin target the A particles cover a fraction of the area,

$$\text{fraction}_{\text{covered}} = \rho_A l_A \sigma = \text{probability of scattering.} \tag{3.119}$$

When the target is thicker, multiple scattering becomes possible and the covered fraction gives us the average number of expected scatterings. We can therefore turn the equation around and obtain the cross section as,

$$\sigma = \frac{N_{\text{events}}}{\rho_A l_A} = \frac{N_{\text{events}}}{N_A/A}, \tag{3.120}$$

where N_A is the number of A particles and A is the area over which they are distributed.
This can be easily generalized to a situation where we have a whole bunch of particles B being scattered on the fixed target. Moreover, if both types of particle are not point-like the area to choose is the one where there is any overlap, i.e. scattering happening. (For billiard balls this would actually be $\pi(r_A + r_B)^2$.)



bunch of particles, type B
bunch length l_B
density ρ_B
velocity v

bunch of particles, type A
bunch length l_A
density ρ_A
velocity 0

The cross section is defined as

$$\sigma = \frac{N_{\text{events}}}{(N_B \cdot N_A)/A}\,, \tag{3.121}$$

where A is the scattering area (transverse). With a space-dependent density

$$(N_B \cdot N_A)/A = \int\limits_A \mathrm{d}^2x\, \rho_A(x)\,\rho_B(x)\, l_A\, l_B\,, \tag{3.122}$$

the cross section is

$$\sigma = \frac{N_{\text{events}}}{l_A\, l_B \int_A \mathrm{d}^2x\, \rho_A(x)\,\rho_B(x)}\,, \tag{3.123}$$

or for constant densities

$$\sigma \frac{N_{\text{events}}}{l_A\,\rho_A \cdot l_B\,\rho_B \cdot A}\,. \tag{3.124}$$

For the above example, we need to consider states, that are localised in space/momentum. Therefore, we consider the wave packet from Eq. (2.101):

$$\left|f_{\mathbf{p}}\right\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\mathrm{f}_{\mathbf{p}}(\mathbf{k})\,|\mathbf{k}\rangle\;, \tag{3.125}$$

with $\mathrm{f}_{\mathbf{p}}(\mathbf{k})$ being a packet at $\mathbf{p}$, e.g.

$$\mathrm{f}_{\mathbf{p}}(\mathbf{k}) \sim \mathrm{e}^{-(\mathbf{k}-\mathbf{p})^2/\mathcal{N}} \quad \text{Gaussian}\,. \tag{3.126}$$

Using the normalisation, it follows

$$\begin{aligned} 1 \quad &= \quad \left\langle f_{\mathbf{p}}\middle|f_{\mathbf{p}}\right\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int \frac{\mathrm{d}^3k'}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\,\frac{1}{2\omega_{\mathbf{k}'}}\,\mathrm{f}^*{}_{\mathbf{p}}(\mathbf{k}')\,\mathrm{f}_{\mathbf{p}}(\mathbf{k})\,\langle\mathbf{k}'|\mathbf{k}\rangle \\ &= \quad \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}|\mathrm{f}_{\mathbf{p}}(\mathbf{k})|^2 \quad \text{(see Eq. (2.102))}\,. \end{aligned} \tag{3.127}$$

The Gaussian is localised in $\mathbf{k}$ and $\mathbf{x}$. Recall, that a Fourier transform of a Gaussian remains a Gaussian. In our case, the initial state is given by

$$|i\rangle = \int \frac{\mathrm{d}^3k_A}{(2\pi)^3} \int \frac{\mathrm{d}^3k_B}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k_A}}\,2\omega_{\mathbf{k_B}}}\,\mathrm{f}_{\mathbf{p_A}}(\mathbf{k_A})\,\mathrm{f}_{\mathbf{p_B}}(\mathbf{k_B})\,|\mathbf{k_A}\,\mathbf{k_B}\rangle\;, \tag{3.128}$$

with

$$|\mathbf{k_A}\,\mathbf{k_B}\rangle \;=\; \sqrt{2\omega_{\mathbf{k_A}}\,2\omega_{\mathbf{k_B}}}\;a^\dagger(\mathbf{k_A})\,a^\dagger(\mathbf{k_B})\,|0\rangle\;. \tag{3.129}$$

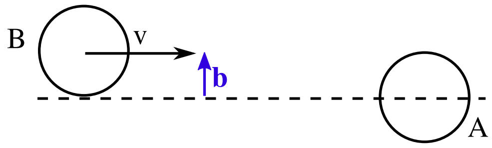


Figure 3.4.: Sketch of the impact parameter **b**.

Next, we introduce the impact parameter **b** (see figure 3.4). For this purpose, we recall, that the momentum operator **P** from Eq. (2.56) generates translations. Thus,

$$e^{-i\,\mathbf{P}\,\mathbf{b}}\,|\mathbf{k}\rangle \quad = \quad e^{-i\,\mathbf{k}\mathbf{b}}\,|\mathbf{k}\rangle \tag{3.130}$$

With this we rewrite the

initial state (with impact parameter **b**)

$$|i_{\mathbf{b}}\rangle = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k_A}}\, 2\omega_{\mathbf{k_B}}} \, f_{\mathbf{p_A}}(\mathbf{k_A})\, f_{\mathbf{p_B}}(\mathbf{k_B})\, e^{-i\mathbf{k_B}\mathbf{b}}\, |\mathbf{k_A}\,\mathbf{k_B}\rangle \ . \tag{3.131}$$

This shows, that the impact parameter only gives an additional phase shift.
We now want to ask for the probability for the initial state $|i_b\rangle$ to scatter into a given final state. For the moment we take the final state to be a 2-particle momentum eigenstate (this is a bit problmeatic with regards to the normalization, but we will clarify this below).
The situation we want to consider is that we are preparing our initial state such that the wave packets are very distant from each other in the far distant past $t_0 = -\infty$. We then need to time-evolve the situation, and finally we measure our scattering products much much later, i.e. at $t = +\infty$. Accordingly the transition amplitude is given by

$$\lim_{t_0\to\infty,\, t\to+\infty} \langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\, U(t,t_0)\,|i_{\mathbf{b}}\rangle = \langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\, S\,|i_{\mathbf{b}}\rangle \ , \tag{3.132}$$

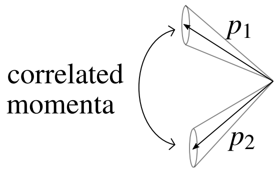
where we have used the time evolution operator $U(t, t_0)$ and the definition of the S-matrix from Eq. (3.25).
The probability is then given by $|\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\, S\,|i_{\mathbf{b}}\rangle|^2$.
In the following we will restrict ourselves to a collision of the bunch with a *single target*, i.e. $N_A = 1$. Let us now distribute N_B particles homogeneously over an area A. Then integration over the impact area A is equal to integration over the impact parameter **b** and the number of events in a dense beam is

$$N_{\text{events}} = \frac{N_B}{A} \int_A d^2b\, |\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\, S\,|i_{\mathbf{b}}\rangle|^2 \,. \tag{3.133}$$

With Eq. (3.121) it follows

$$\sigma(\mathbf{p_{f_1}}, \mathbf{p_{f_2}}) = \frac{N_{events}}{(N_B \cdot 1)/A} = \int_A d^2b\, |\langle \mathbf{p_{f_1}}\,\mathbf{p_{f_2}}\big|\, S\,|i_{\mathbf{b}}\rangle|^2 \,. \tag{3.134}$$

In this formula we have to be a bit careful. Since we have chosen our final state particles as momentum eigenstates, they are not properly normalizable (they can only be normalized to δ-functions. Hence, in this somewhat naive formula the cross section does not have the proper units of area. However, more realistic is a detection of a momentum region v_f, as detectors will in practice never be aligned with the beam. Hence, we will rather obtain something like:



It follows

$$\sigma(v_f) = \int \frac{\mathrm{d}^3 p_{f_1}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{f_1}}}} \int \frac{\mathrm{d}^3 p_{f_2}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{f_2}}}} \int \mathrm{d}^2 b \, |\langle \mathbf{p_{f_1}} \, \mathbf{p_{f_2}} | \, S \, | i_\mathbf{b} \rangle |^2 \,, \tag{3.135}$$

where

$$\int \frac{\mathrm{d}^3 p_{f_1}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{f_1}}}} \sim \int \frac{\mathrm{d}^4 p_{f_1}}{(2\pi)^4} (2\pi) \, \delta(p_{f_1}^2 - m^2) \tag{3.136}$$

implies that it is on-shell, i.e. that $p^2 = m^2$. Finally, we obtain the

differential cross section (for n particles)

$$\mathrm{d}\sigma = \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_{f_i}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{f_i}}}} \int \mathrm{d}^2 b \, |\langle \mathbf{p_{f_1}} \cdots \mathbf{p_{f_n}} | \, S \, | i_\mathbf{b} \rangle |^2 \,. \tag{3.137}$$

We assume now, that the $\mathbf{p_i}$ are not parallel to $\mathbf{p_B}$, so that there is no (trivial) forward scattering. Using

$$S_{fi} = \mathbb{1}_{fi} + \mathrm{i}\, T_{fi} \,, \quad \mathrm{i}\, T_{fi} = \mathrm{i}\, M_{fi} \, (2\pi)^4 \, \delta^4 \left(\sum p_{fi} - \sum k_i \right) , \tag{3.138}$$

we conclude

$$\mathrm{d}\sigma \;=\; \prod_i \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_i}}} \int \mathrm{d}^2 b \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k_A}}} \frac{\mathrm{d}^3 k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k_B}}} \tag{3.139}$$

$$\cdot \int \frac{\mathrm{d}^3 k_A'}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}_A'}} \frac{\mathrm{d}^3 k_B'}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}_B'}} \, \mathrm{f}_{\mathbf{p_A}}(\mathbf{k_A}) \, \mathrm{f}_{\mathbf{p_B}}(\mathbf{k_B}) \, \mathrm{f}^*_{\mathbf{p_A}}(\mathbf{k_A'}) \, \mathrm{f^*}_{\mathbf{p_B}}(\mathbf{k_B'}) \tag{3.140}$$

$$\cdot \mathrm{e}^{\mathbf{ib(k_B'-k_B)}} \, |M_{fi}|^2 \, (2\pi)^4 \, \delta^4 \left(\sum p_{fi} - \sum k_i \right) \cdot (2\pi)^4 \, \delta^4 \left(\sum p_{fi} - \sum k_i' \right) ,$$

with $k_1 = k_A$, $k_2 = k_B$.
In this formula the cross section now has the right units!
To explicitly compute this, we first consider the integral over the impact parameter, as solely the phase factor depends on $\mathbf{b}$. Therefore,

$$\int \mathrm{d}^2 b \; \mathrm{e}^{\mathbf{ib(k_B'-k_B)}} = (2\pi)^2 \, \delta^2(\mathbf{k}'_{\mathbf{B}_\perp} - \mathbf{k}_{\mathbf{B}_\perp}) \,. \tag{3.141}$$

Next we examine the integral over the primed momenta,

$$\int \mathrm{d}^3 k_A' \mathrm{d}^3 k_B' \, \delta^4 \left(\sum p_{fi} - \sum k_i' \right) \delta^2(\mathbf{k}'_{\mathbf{B}_\perp} - \mathbf{k}_{\mathbf{B}_\perp})$$

$$= \int \mathrm{d}\big(k_A^z\big)' \mathrm{d}\big(k_B^z\big)' \, \delta \left(\sum p_{fi}^z - \sum \big(k_i^z\big)' \right) \delta \left(\sum p_{fi}^0 - \sum \big(k_i^0\big)' \right)$$

$$= \int \mathrm{d}\,\big(k_A^z\big)' \, \delta \left(\sum p_{fi}^0 - \sum \big(k_i^0\big)' \right) , \tag{3.142}$$

with $k'_{B_\perp} = k_{B_\perp}$, $k'_{A_\perp} = k_{A_\perp}$, $\left(k_B^z\right)' = \sum p_{fi}^z - \left(k_A^z\right)'$. Here, the integral over the transverse δ-function for the $\mathbf{k'_{B_\perp}}$ is trivial. The equality $k'_{A_\perp} = k_{A_\perp}$ is due to the four dimensional δ-function. For example in the 1-direction it enforced,

$$p_{f,1}^1 + p_{f,2}^1 - k_A^1 - k_B^1 = p_{f,1}^1 + p_{f,2}^1 - k_A'^{,1} - k_B'^{,1}, \tag{3.143}$$

Using the equality of the transverse $k'_{B_\perp} = k_{B_\perp}$ we now obtain the desired equality for the A.
It follows

$$\begin{aligned}
&\int \mathrm{d}^3k'_A \mathrm{d}^3k'_B\, \delta^4\left(\sum p_{fi} - \sum k'_i\right)\, \delta^2(\mathbf{k'_{B_\perp}} - \mathbf{k_{B_\perp}}) \\
&= \int \mathrm{d}\left(k_A^z\right)'\, \delta\left(\sum p_{fi}^0 - \sqrt{(\mathbf{k_A}')^2 + m_A^2} - \sqrt{(\mathbf{k_B}\prime)^2 + \mathbf{m_B^2}}\right)\Bigg|_{\substack{k_{A/B\perp} = k'_{A/B_\perp} \\ \sum k_i^z = \sum (k_i^z)' \\ \sum k_i^0 = \sum (k_i^0)'}} \\
&= \frac{1}{\left|\frac{(k_A^z)'}{(k_A^0)'} - \frac{(k_B^z)'}{(k_B^0)'}\right|} \quad \overset{k_{A/B} = p_{A/B}}{\rightarrow} \quad \frac{1}{|v_A - v_B|}\,.
\end{aligned} \tag{3.144}$$

where we also used, that $(\mathbf{k'_A})^2 = k_{A_\perp}^2 + \left(\left(k_A^z\right)'\right)^2$ and $(\mathbf{k'_B})^2 = k_{B_\perp}^2 + \left(\sum p_i^z - \left(k_A^z\right)'\right)^2$.
As the wave packages $f_{\mathbf{p_{A/B}}}$ are located around $\mathbf{p_{A/B}}$, we can substitute $k'_{A/B} \to p_{A/B}$ in all prefactors. Then we obtain

$$\begin{aligned}
\mathrm{d}\sigma = &\prod_i \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p_{fi}}}} \frac{1}{4p_A^0\, p_B^0\, |v_A - v_B|} \\
&\cdot \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_B}{(2\pi)^3} \frac{1}{2k_A^0\, 2k_B^0} \left|\mathrm{f}_{\mathbf{p_A}}(\mathbf{k_A})\right|^2 \left|\mathrm{f}_{\mathbf{p_B}}(\mathbf{k_B})\right|^2 \\
&\cdot \left|M_{fi}\right|^2\, (2\pi)^4\, \delta^4\left(\sum p_{fi} - \sum k_i\right)\,.
\end{aligned} \tag{3.145}$$

Again, we use the localisation to replace $\sum k_i \to p_i = p_A + p_B$. Further, $\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega_\mathbf{k}} \left|f_\mathbf{p}(\mathbf{k})\right|^2 = 1$ and $\sum p_{fi} = \sum p_f - p_i$. Finally, we get the

differential cross section

$$\mathrm{d}\sigma = \frac{1}{4p_A^0\, p_B^0\, |v_A - v_B|} \left|M_{fi}\right|^2\, (2\pi)^4 \delta^4(p_f - p_i) \prod_{i=1}^n \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2p_{fi}^0}. \tag{3.146}$$

Note, that except for the first fraction all expressions are Lorentz invariant. The first fraction is invariant under boosts along the beam axis. Thus, $\mathrm{d}\sigma$ is a (differential) transverse area, i.e. **invariant under boosts along the beam axis**. We define the n-particle phase space factor as

$$\mathrm{d}\Pi_n := (2\pi)^4 \delta^4(p_f - p_i) \prod_{i=1}^n \frac{\mathrm{d}^3 p_{fi}}{(2\pi)^3} \frac{1}{2p_{fi}^0}\,. \tag{3.147}$$

Let us now consider the highly relativistic case. Then

$$\begin{aligned}
|s| &= (p_A + p_B)^2 = \left(p_A^0\right)^2 - \mathbf{p_A}^2 + \left(p_B^0\right)^2 - \mathbf{p_B}^2 + 2\, p_A^0\, p_B^0 - 2\, \mathbf{p_A}\, \mathbf{p_B} \\
&= m_A^2 + m_B^2 + 2\, p_A^0\, p_B^0 - 2\, \mathbf{p_A}\, \mathbf{p_B} \gg m_A^2 + m_B^2\,. \\
&\Rightarrow \quad 4p_A^0\, p_B^0\, |v_A - v_B| \to 2s\,,
\end{aligned} \tag{3.148}$$

and

$$d\sigma = \frac{1}{2s} \left|M_{fi}\right|^2 \, d\Pi_n \, . \tag{3.149}$$

Let us exemplarily discuss the 2-2 scattering in ϕ^4-theory in the highly relativistic case. Then, we have $n = 2$ in Eq. (3.149) and $p_{fi} = p_i$. It follows

$$\begin{aligned} \int d\Pi_2 &= \int (2\pi)^4 \, \delta^4\Big(p_1 + p_2 - (p_A + p_B)\Big) \cdot \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2p_1^0} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2p_2^0} \\ &\simeq \frac{1}{(2\pi)^2 \, 4 p_1^0 \, p_2^0} \int d^3 p_2 \, \delta\left(p_1^0 + p_2^0 - \sqrt{s}\right) \quad \text{for} \quad (p_A + p_B)^2 \gg m_A^2, m_B^2 \, . \end{aligned} \tag{3.150}$$

We compute this in the center of mass system (CMS). Therefore, we have $\mathbf{p_1} = -\mathbf{p_2} \Rightarrow p_1^0 = p_2^0$, i.e. equal masses. We also use

$$d^3 p_2 = d\Omega \, |\mathbf{p_2}|^2 d|\mathbf{p_2}| \, , \tag{3.151}$$

with the solid angle $d\Omega = d\varphi \sin\theta \, d\theta$.
It follows, $p_1^0 + p_2^0 - \sqrt{s} \simeq 2p_2^0 - \sqrt{s} = 2|\mathbf{p_2}|^2 - \sqrt{s}$, $p_i^0 = \sqrt{s}/2$

$$\int d\Pi_2 = \frac{1}{2} \frac{s/4}{(2\pi)^2 \, 4 p_1^0 \, p_2^0} \, d\Omega = \frac{1}{32\pi^2} d\Omega \, . \tag{3.152}$$

Now we use Eq. (3.37), i.e. that for classical scattering it is

$$\left|M_{fi}\right|^2 = \lambda^2 \, . \tag{3.153}$$

With this, we obtain the

differential cross section (2-2 scattering)

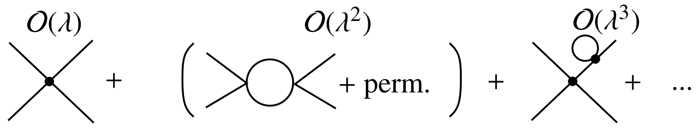
$$\frac{d\sigma}{d\Omega} = \frac{1}{2s} \left|M_{fi}\right|^2 \int_{d\Omega(p_2)\text{fixed}} d\Pi_2 = \frac{\lambda^2}{64\pi^2 \, s} \, . \tag{3.154}$$

Lastly, we discuss the **computation of the S-matrix elements**. In the 2-2 scattering example we used, that

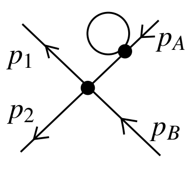
$$\left|M_{fi}\right|^2 = \lambda^2 + O(\lambda^3) \, . \tag{3.155}$$

We make an expansion in the Feynman diagrams:

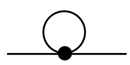
$\langle \mathbf{p_1} \, \mathbf{p_2} | \, i \, T \, | \mathbf{p_A} \, \mathbf{p_B} \rangle$ =



Computing

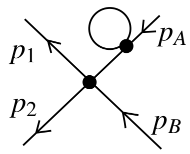


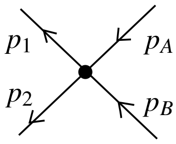
with

$\frac{1}{2}$  $= -i\,\Pi$

gives

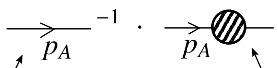
$$
\begin{aligned}
&\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\,(-\mathrm{i}\,\Pi)(-\mathrm{i}\lambda)\cdot(2\pi)^4\,\delta^4(p_A + p_B - p_1 - p_2) \\
&\qquad = \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)^{-1} \frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\,(-\mathrm{i}\,\Pi)\,\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\,(-\mathrm{i}\lambda)\cdot\delta^4(p_A + p_B - p_1 - p_2)
\end{aligned}
$$

$\Rightarrow \quad \frac{1}{2}$ 

$+$ 

$$
= \; -\mathrm{i}\lambda \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)^{-1} \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\,(-\mathrm{i}\,\Pi)\,\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)\cdot\delta^4(p_A + p_B - p_1 - p_2)
$$

$$
= \; -\mathrm{i}\lambda \left(\frac{\mathrm{i}}{p_A^2 - m^2 + \mathrm{i}\epsilon}\right)^{-1} \frac{\mathrm{i}}{p_A - (m^2 + \Pi) + \mathrm{i}\epsilon}\cdot\delta^4(p_A + p_B - p_1 - p_2) \tag{3.156}
$$

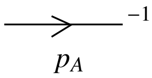
$= \left[-\mathrm{i}\lambda\,\delta^4(p_A + p_B - p_1 - p_2)\right]$ 

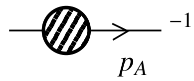
$+ \quad O(\lambda^3) \quad .$

(bare) free inverse propagator with p_A

full propagator with p_A

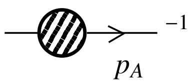
We remark, that the free inverse propagator is related to the fact, that the particle A in the initial state was prepared as a free state, which is only true for $t \to -\infty$. The correct state should relate to full (inverse) propagation, i.e.

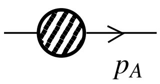


$\rightarrow$


.

This leads to

$1 \quad =$ 

$\cdot$ 

in the above equation. Thus, we conclude, that M_{fi} **is computed by computing amputated, connected scattering diagrams**. This will be discussed further in the subsequent section.

V. LSZ-Formalism

In the last section we have seen that the crucial QFT input in the calculation of scattering cross sections are the S-matrix elements. In the above example we have already intuited that the S-matrix elements are directly connected to Feynman diagrams. Now we aim to put this on a more solid footing. We will see that the S-matrix elements can be obtained with the LSZ-reduction formula, named after the three German physicists Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann.

V.1. The spectral function and the Källén-Lehmann representation of the propagator

In the previous section we have seen, that the naive preparation of our in-state lead to a product of the free inverse propagator with the full propagator in our scattering amplitudes (Eq. (3.156)). We have encountered a similar problem with vacuum bubbles before. In this section we shall see that

$$\phi_H(t \to \mp\infty) \to Z^{1/2}\, \phi_{\text{in/out}} \quad \text{(weak op. equivalence)}\,, \tag{3.157}$$

with $Z \leq 1$. So far, we have implicitly assumed $Z = 1$. In the following we determine Z by computing the two-point function and subsequent generalisation to the n-point function. We begin with the vacuum expectation value of the two-point function

$$\langle \phi_H(x)\, \phi_H(y)\rangle \quad = \quad \langle \Omega|\phi_H(x)\, \phi_H(y)|\Omega\rangle \tag{3.158}$$

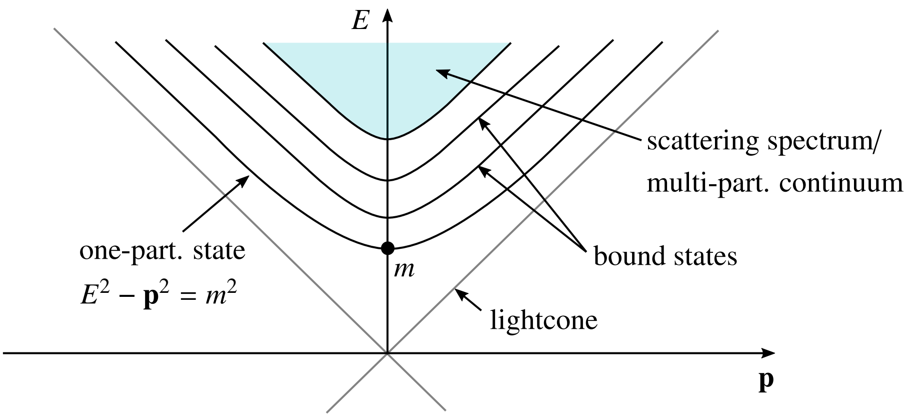
In a next step we want to insert a suitable factor of **1**. For this we choose $|\lambda, \mathbf{p}\rangle_H$ as eigenstates of H, i.e.

$$H\,|\lambda, \mathbf{p}\rangle = E_\lambda\; |\lambda, \mathbf{p}\rangle \tag{3.159}$$

and

$$\mathbf{P}\,|\lambda, \mathbf{p}\rangle = \mathbf{p}_\lambda\; |\lambda, \mathbf{p}\rangle\;, \tag{3.160}$$

with $E_\lambda^2 - \mathbf{p}_\lambda^2 = m_\lambda^2$ fixed (on-shell). The idea for the next steps is to integrate over all states (on-shell) with different masses to obtain the representation of the two-point function, which then will be off-shell. The fixed states $|\lambda, \mathbf{p}\rangle$ with fixed m_λ are connected by boosts.



Note, that for the vacuum state it is

$$E_\lambda = 0 \;:\; |\Omega\rangle\;, \tag{3.161}$$

and

$$\langle \Omega|\phi(x)|\Omega\rangle = 0\,. \tag{3.162}$$

As in quantum mechanics the sum over all energy eigenstates gives us the unit operator, i.e.

$$\mathbb{1} \quad = \quad |\Omega\rangle\,\langle\Omega| + \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{1}{2E_\lambda(\mathbf{p})}\,|\lambda, \mathbf{p}\rangle\,\langle\lambda, \mathbf{p}|\,. \tag{3.163}$$

We use, that with $\hat{P} = (H, \mathbf{P})$ we have

$$\phi_H(x) = \mathrm{e}^{\mathrm{i}\hat{P}x}\, \phi_H(0)\, \mathrm{e}^{-\mathrm{i}\hat{P}x}\,. \tag{3.164}$$

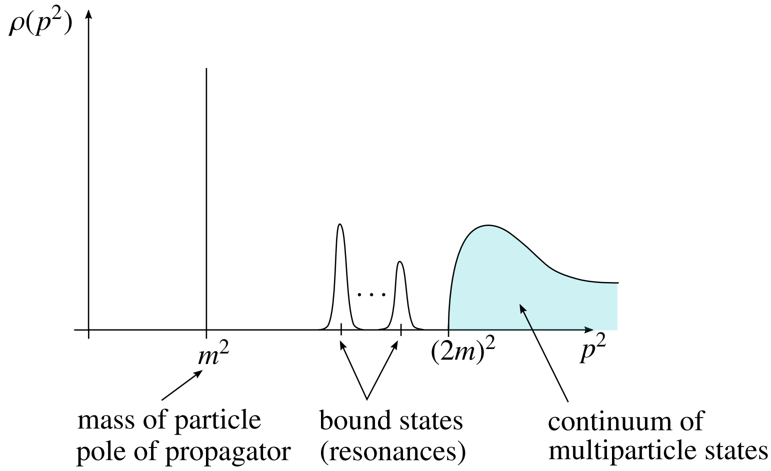


Figure 3.5.: Sketch of the spectral function.

Then we get, ($x^0 \geq y^0$)

$$\langle \phi_H(x)\, \phi_H(y) \rangle \quad = \quad \sum_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_\lambda(\mathbf{p})} \left| \langle \Omega |\, \phi_H(0)\, | \lambda, \mathbf{p} \rangle \right|^2 \cdot \mathrm{e}^{-\mathrm{i}\, p_\lambda (x-y)}. \tag{3.165}$$

We can now use the same trick of introducing an extra integration over p^0 that we used in our derivation of the Feynman propagator. Again the pole prescription that corresponds to the θ-functions that give us the time-ordering are included by an appropriate $i\epsilon$.

$$\langle \phi_H(x)\, \phi_H(y) \rangle \quad = \quad \sum_\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m_\lambda^2 + \mathrm{i}\epsilon}\, \mathrm{e}^{-\mathrm{i}\, p(x-y)} \cdot \left| \langle \Omega |\, \phi_H(0)\, | \lambda, \mathbf{p} \rangle \right|^2 . \tag{3.166}$$

Using the same steps for $x^0 \leq y^0$, we obtain in summary the

Källén-Lehmann spectral representation

$$\langle T\ \phi_H(x)\, \phi_H(y) \rangle = \int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\, \rho(M^2)\, \mathcal{D}_F(x-y; M^2)\,, \tag{3.167}$$

with the

spectral function

$$\rho(p^2) = \sum_\lambda (2\pi)\, \delta(p^2 - m_\lambda^2) \left| \langle \Omega |\, \phi_H(0)\, | \lambda \rangle \right|^2 . \tag{3.168}$$

The spectral function is depicted in figure 3.5 and has the representation

$$\rho(p^2) = Z \cdot 2\pi\, \delta(p^2 - m^2) + \theta(p^2 - m_1^2) + \cdots\,, \tag{3.169}$$

where m_1^2 denotes the mass in the first residue. Hence,

$$\langle T\ \phi(x)\, \phi(y) \rangle = Z\, \mathcal{D}_F(x-y; m^2) + \int_{m_1^2}^\infty \frac{\mathrm{d}M^2}{2\pi}\, \rho(M^2)\, \mathcal{D}_F(x-y; M^2)\,. \tag{3.170}$$

Note, that $\mathcal{D}_F(x-y; M^2)$ carries the one-particle pole of ϕ.
To relate this to ϕ_{in}, we consider one-particle states $|\lambda_1\rangle$ in $|\lambda\rangle\langle\lambda|$:

$$\rho \sim \sum_{\substack{\text{one-part.}\\ \text{states}\lambda_1}} \mathrm{e}^{-\mathrm{i}p_\lambda(x-y)} \left|\langle\Omega|\,\phi(0)\,|\lambda_1\rangle\right|^2 , \tag{3.171}$$

with, $U = U(-\infty, 0)$

$$\begin{aligned} \left|\langle\Omega|\,\phi(0)\,|\lambda_1\rangle\right|^2 &= \left|\langle\Omega|\,U^{-1}\,U\,\phi\,U^{-1}\,U\,|\lambda_1\rangle\right|^2 \\ &= \left|{}_I\langle 0|\,\phi_H(-\infty)\,|\lambda_1\rangle_I\right|^2 \\ &= \left|{}_I\langle 0|\,Z^{1/2}\,\phi_{\text{in}}\,|\lambda_1\rangle_I\right|^2 \\ &= Z\,. \end{aligned} \tag{3.172}$$

Let us now determine Z. For this purpose, we consider the *not time ordered* expectation value $\langle\phi(x)\,\phi(y)\rangle$. Then $\mathcal{D}_F$ in Eq. (3.170) is substituted by the not time ordered propagator D. Note also, that

$$\begin{aligned} \left[\frac{\partial}{\partial y^0}\,\langle[\phi(x)\,,\,\phi(y)]\rangle\right]_{x^0=y^0} &= \left\langle[\phi(x)\,,\,\Pi(y)]_{x^0=y^0}\right\rangle \\ &= \mathrm{i}\,\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{3.173}$$

and

$$\left[\frac{\partial}{\partial y^0}\Big(D(x-y) - D(y-x)\Big)\right]_{x^0=y^0} = \mathrm{i}\,\delta^3(\mathbf{x}-\mathbf{y})\,. \tag{3.174}$$

Next, we integrate over space, i.e. evaluate $\int \mathrm{d}^3x\,\left\langle[\phi(x)\,,\,\Pi(y)]_{x^0=y^0}\right\rangle$. With Eq. (3.170) and Eq. (3.173) we obtain

$$1 = Z + \int_{m_1^2}^{\infty} \frac{\mathrm{d}M^2}{2\pi}\,\rho(M^2) \tag{3.175}$$

and, as the integral term is larger than zero,

$$0 \le Z \le 1\,. \tag{3.176}$$

Note, that $Z = 1$ in free theory and $Z < 1$ in interacting theory. Also note, that $1 - Z$ accounts for the overlap of $\phi\,|\Omega\rangle$ with multi-particle states and that in the limit $t \to \mp\infty$:

$$\phi(x) \to Z^{1/2}\,\phi_{\text{in/out}} \quad \text{(weak op. equivalence)}\,. \tag{3.177}$$

In momentum space the spectral representation reads,

$$\begin{aligned} D_F(p^2) &= \int d^4x \exp(ipx)\langle\Omega|T\phi_H(x)\phi_H(0)|\Omega\rangle \\ &= \int_0^{\infty} \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon}. \end{aligned} \tag{3.178}$$

Close to the mass shell, i.e. $p^2 - m^2 = 0$, the propagator is dominated by the one particle state,

$$\mathcal{D}_F(p^2 \to m^2) \quad \simeq \quad \frac{\mathrm{i}\, Z}{p^2 - m^2 + \mathrm{i}\epsilon} \tag{3.179}$$

$$\left(\quad = \quad \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}px} \langle T\ \phi(x)\phi(0)\rangle \right).$$

This can be easily seen by splitting the integral in Eq. (3.167) into a part from 0 to m_1^2 and one from m_1^2 to ∞,

$$D_F(p^2) = \int_0^{m_1^2} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon} + \int_{m_1^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}. \tag{3.180}$$

The first part gives Eq. (3.179) whereas the second is finite in the limit $p^2 \to m^2$.
Indeed from this spectral representatiom we can obtain an interesting general result for the decay of the propagator at large space-like momenta (or even better do it in Euclidean space $p^2 \to -p_E^2$),

$$\begin{aligned} \left| \lim_{p_E^2 \to \infty} p_E^2 D_F \right| &= \left| \int_0^{\infty} \frac{dM^2}{2\pi} \frac{p_E^2}{p_E^2 + M^2} \rho(M^2) \right| \geq \int_0^{q_0^2} \frac{dM^2}{2\pi} \frac{p_E^2}{p_E^2 + M^2} \rho(M^2) \\ &\geq \int_0^{q_0^2} \frac{dM^2}{2\pi} \frac{p_E^2}{p_E^2 + q_0^2} \rho(M^2) \\ &\geq \int_0^{q_0^2} \frac{dM^2}{2\pi} \frac{p_E^2}{2q_0^2} \geq A \qquad \text{for} \quad p_E^2 \geq q_0^2. \end{aligned} \tag{3.181}$$

Here, we have used that $\rho(M^2) \geq 0$. Physically this tells us that the propagator cannot drop faster than $1/p_E^2$. This is to some degree a problem. We have already seen that loop integrals are often divergent. One way to make them more convergent would be to have a situation where the propagator drops faster than $1/p_E^2$, e.g. as $1/p_E^4$ or even exponentially. However, as we have just seen this is forbidden by Eq. (3.181). That said, there are some caveats that have to go along with this. The above naive manipulations are only guaranteed to be correct if the integral converges. This is far from a trivial requirement. For example we have already encountered divergences in quantum field theory, so some care is needed. Moreover, there could also be more complicated theories, where the spectral function grows, e.g. exponentially. For example string theory (or at least toy models of it) can exhibit such a feature.
Note, m^2 is *not* simply the mass parameter m_0^2 in the Lagrangian. It is really the physical mass/energy of the one-particle state

V.2. The LSZ reduction formula

Now we will derive the **LSZ-reduction formula**. For this purpose we extend the analysis of the two-point function to an n-point function. The latter will be related to the S-matrix elements. As in Eq. (3.179) we evaluate the Fourier transform

$$\int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}px} \langle T\ \phi(x)\, \phi(x_2) \cdots \phi(x_n)\rangle\ . \tag{3.182}$$

With $T_+ > x_2^0, ..., x_n^0$ and $T_- < x_2^0, ..., x_n^0$, we split

$$\int \mathrm{d}x^0\ \mathrm{e}^{\mathrm{i}p^0x^0} = \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+} + \int_{T_+}^{+\infty}\right)\mathrm{d}x^0\ \mathrm{e}^{\mathrm{i}p^0x^0}\,, \tag{3.183}$$

where the first and the third integral give poles and the second one is finite. It follows

$$\begin{aligned}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px}\,\langle T\ \phi(x)\,\phi(x_2)\cdots\phi(x_n)\rangle &= \int_{T_+}^{\infty} \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px}\,\langle \phi(x) T\ \phi(x_2)\cdots\phi(x_n)\rangle + \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+}\right)\cdots \qquad (3.184)\\ &= \int_{T_+}^{\infty} \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px} \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega_\mathbf{q}}\cdot\langle\phi(x)|\lambda,\mathbf{q}\rangle\,\langle\lambda,\mathbf{q}|T\ \phi_2\cdots\phi_n\rangle + \cdots\,.\end{aligned}$$

Using $\langle\phi(x)|\lambda,\mathbf{p}\rangle = \langle\Omega|\,\phi(0)\,|\lambda\rangle\ \mathrm{e}^{-\mathrm{i}qx}$:

$$\begin{aligned}&\int_{T_+}^{\infty} \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2\omega_\mathbf{q}}\mathrm{d}x^0\ \mathrm{e}^{\mathrm{i}(p^0-q^0+\mathrm{i}\epsilon)x^0}\ \langle\Omega|\,\phi(0)\,|\lambda\rangle\,\langle\lambda,\mathbf{q}|T\ \phi_2\cdots\phi_n\rangle\cdot(2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{q})\\ &= \sum\!\!\!\!\!\!\int \frac{1}{2\omega_\mathbf{p}}\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}(p^0-\omega_\mathbf{p}+\mathrm{i}\epsilon)T_+}}{p^0-\omega_\mathbf{p}+\mathrm{i}\epsilon}\ \langle\Omega|\,\phi(0)\,|\lambda\rangle\,\langle\lambda,\mathbf{p}|\phi_2\cdots\phi_n\rangle\,. \qquad (3.185)\end{aligned}$$

Focussing on the one particle pole. For $p^0\to\omega_\mathbf{p}$: (using Källén-Lehmann)

$$\lim_{p^0\to\omega_\mathbf{p}}\int_{T_+}^{+\infty}\mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px}\ \langle T\ \phi(x)\,\phi(x_2)\cdots\phi(x_n)\rangle = \frac{\mathrm{i}\,Z^{1/2}}{p^2-m^2+\mathrm{i}\epsilon}\,\langle\mathbf{p}|T\ \phi(x_2)\cdots\phi(x_n)\rangle + \text{finite}\,. \tag{3.186}$$

Analogously we find for the $\int_{-\infty}^{T_-}$-term:

$$\lim_{p^0\to-\omega_\mathbf{p}}\int_{-\infty}^{T_-}\mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px}\ \Big\langle\Big(T\ \phi(x_2)\cdots\phi(x_n)\Big)\phi(x)\Big\rangle = \frac{\mathrm{i}\,Z^{1/2}}{p^2-m^2+\mathrm{i}\epsilon}\,\langle T\ \phi(x_2)\cdots\phi(x_n)|{-\mathbf{p}}\rangle + \text{finite}\,. \tag{3.187}$$

As mentioned before, the last term $\int_{T_-}^{T_+}\cdots$ is finite as the integration interval has a finite length (compact). We remark, that the above analysis can be repeated iteratively for all $\phi(x_i)$. Strictly speaking, one should separate the fields spacially: $\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}px} \to \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}px}\,\mathrm{f}_\mathbf{p}(\mathbf{k})$.
Importantly, states $|\mathbf{p}\rangle$ are at time $t\to-\infty$ and states $\langle\mathbf{p}|$ are at time $t\to+\infty$, and that after iteration we have

$$_{-\infty}\langle\mathbf{p_1}\cdots\mathbf{p_n}|\mathbf{k_1}\cdots\mathbf{k_m}\rangle_{+\infty} = \langle\mathbf{p_1}\cdots\mathbf{p_n}|\,S\,|\mathbf{k_1}\cdots\mathbf{k_m}\rangle\,. \tag{3.188}$$

With this we obtain the

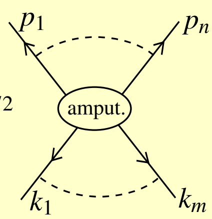
LSZ-reduction formula

$$\langle \mathbf{p_1} \cdots \mathbf{p_n} | \, S \, | \mathbf{k_1} \cdots \mathbf{k_m} \rangle \Big|_{\text{on-shell}}$$
$$= \int \prod_{i=1}^{n} \mathrm{d}^4 x_i \, \mathrm{e}^{\mathrm{i} p_i x_i} \prod_{j=1}^{m} \mathrm{d}^4 y_i \, \mathrm{e}^{-\mathrm{i} k_j y_j} \prod_{i=1}^{n} \left(\partial^2_{x_i} + m^2 \right) \prod_{j=1}^{m} \left(\partial^2_{y_i} + m^2 \right)$$
$$\cdot Z^{(n+m)/2} \langle T \; \phi(x_1) \cdots \phi(x_n) \, \phi(y_1) \cdots \phi(y_n) \rangle \; , \tag{3.189}$$

where on-shell means $p^2 = m^2$ being the physical mass pole and not the mass parameter m_0^2 in the Lagrangian.

This entails for the S-matrix elements with Eq. (3.190)

$$\langle \mathbf{p_1} \cdots \mathbf{p_n} | \, S \, | \mathbf{k_1} \cdots \mathbf{k_m} \rangle \Big|_{\text{on-shell}} = Z^{(n+m)/2}$$



(3.190)

Concluding, we remark: Z is called wave function (or field strength) renormalisation, as it multiplies the field. Note that

$$\left\langle T \; Z^{-1/2} \, \phi(x) \, Z^{-1/2} \, \phi(y) \right\rangle_{p^2 \to m^2} = \mathcal{D}_F(x-y; m^2) \tag{3.191}$$

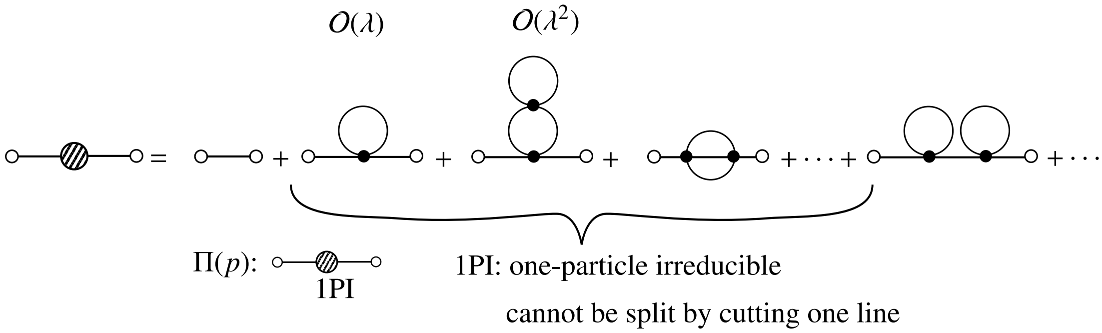
i.e. Z renormalises the field. With this we also see, that

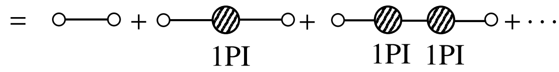
$$Z^{(n+m)/2} \; \langle T \; \phi(p_1) \cdots \phi(k_m) \rangle_{amput.}$$
$$\cong \; Z^{(n+m)/2} \prod_i \frac{p_i^2 + m^2}{Z^{1/2}} \prod_j \frac{k_j^2 + m^2}{Z^{1/2}} \; \langle T \; \phi(p_1) \cdots \phi(k_m) \rangle$$
$$= \; \prod_i (p_i^2 + m^2) \prod_j (k_j^2 + m^2) \left\langle T \; Z^{-1/2} \, \phi(p_1) \cdots Z^{-1/2} \, \phi(k_m) \right\rangle \; , \tag{3.192}$$

where $\left\langle T \; Z^{-1/2} \, \phi(p_1) \cdots Z^{-1/2} \, \phi(k_m) \right\rangle$ is just the expectation value of the renormalised fields.

Let us now consider the structure of $\langle T \; \phi_1 \cdots \phi_n \rangle$.

Example 3-8: n=2.





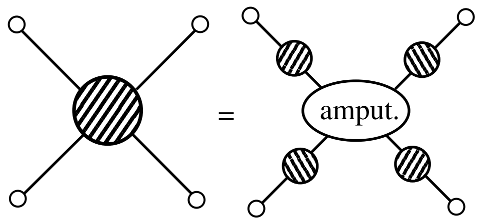
$$= \frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon}\Big(- \mathrm{i}\,\Pi(p)\Big)\frac{\mathrm{i}}{p^2 - m_0^2 + \mathrm{i}\epsilon} + \cdots$$

$$= \frac{\mathrm{i}}{p^2 - \left(m_0^2 + \Pi(p)\right) + \mathrm{i}\epsilon} \tag{3.193}$$

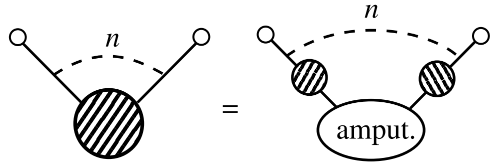
with

$$\frac{\mathrm{i}}{p^2 - \left(m_0^2 + \Pi(p)\right) + \mathrm{i}\epsilon} \overset{p^2\to m^2}{\longrightarrow} \frac{\mathrm{i}\,Z}{p^2 - m^2 + \mathrm{i}\epsilon} \tag{3.194}$$

Example 3-9: n=4.



And in general:



4. Fermions

I. Fields and Lorentz Invariance

So far we have discussed the quantisation of a scalar field, i.e. particles with spin zero (Higgs boson). The scalar field is invariant under Lorentz transformations:

$$\phi(x) \xrightarrow{\Lambda} \phi'(x') = \phi(x)\,, \tag{4.1}$$

with

$$x^\mu \quad \to \quad (x')^\mu = \Lambda^\mu{}_\nu\, x^\nu \tag{4.2}$$

and $\phi'(x) = \phi(\Lambda^{-1}\, x)$. However, for vector fields we have

$$\begin{aligned} A^\mu(x) \quad &\to \quad \Lambda^\mu{}_\nu\, A^\nu(x) \\ (A')^\mu(x) \quad &= \quad \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}\, x)\,, \end{aligned} \tag{4.3}$$

and for Tensor fields (e.g. the fieldstrength in QED, QCD, week):

$$F^{\mu\nu}(x) \quad \to \quad \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, F^{\rho\sigma}(x)\,. \tag{4.4}$$

Note, that the graviton is an example for a particle with spin two.
In the following we will present some mathematical background on group theory, as it is important to understand the properties of fermions. In general we can write

$$\phi^i(x) \quad \to \quad R(\Lambda)^i{}_j\, \phi^j(x)\,, \tag{4.5}$$

with the general index i, e.g. $i = \{\}, \mu, \mu\nu, \ldots$ and the representation R. The representation is chosen accordingly to the field, i.e.

$$\begin{aligned} \text{scalar:} \quad R(\Lambda) \quad &= \quad 1 \quad \textit{trivial representation} \\ \text{vector:} \quad R(\Lambda) \quad &= \quad \Lambda \quad \textit{fundamental representation} \\ \text{(2nd rank) tensor:} \quad R(\Lambda) \quad &= \quad \left(\Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\right) \quad \textit{tensor representation}\,. \end{aligned} \tag{4.6}$$

Let G denote a group. Then, the representation $R:\ G \to R(G)$ has the properties:

$$\begin{aligned} R(\mathbb{1}) \quad &= \quad \mathbb{1}\,, \\ R(g\cdot h) \quad &= \quad R(g)\cdot R(h)\,. \end{aligned} \tag{4.7}$$

For instance, for rotations in $\mathbb{R}^3$, i.e. the SO(3) group we have

$$\begin{aligned} \text{trivial rep:} \quad R(\Lambda) \quad &= \quad 1 \quad \Lambda \in \mathrm{SO}(3) \\ \text{fundamental rep:} \quad R(\Lambda) \quad &= \quad \Lambda \quad \text{Lie group}\,. \end{aligned} \tag{4.8}$$

A useful property of Lie groups is, that we can write every element in terms of an exponential

$$\Lambda = \mathrm{e}^{\mathrm{i}\omega\mathbf{J}}, \tag{4.9}$$

where ω is a vector and $\mathbf{J}$ is a Lie-algebra with the generators J^i. Let us consider for example the case of the fundamental, i.e. matrix representation. In this case we have three generators,

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad J^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.10}$$

Let us now consider the direction

$$\omega = (0, 0, \alpha). \tag{4.11}$$

Inserting this we have

$$\begin{aligned} \exp(i\omega J) &= \exp\left(i\alpha J^3\right) \qquad (4.12) \\ &= \exp\left(\alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\alpha^2}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 + \ldots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\alpha^2}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 + \ldots \\ &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

This corresponds to a rotation around the three-axis. An important property of the generators J are the commutation relations. In the case of our SO(3) example they can be checked to be,

$$[J^i, J^j] = i\epsilon^{ijk} J_k. \tag{4.13}$$

However, there are also other representations. For example if we consider functions on the three coordinates rotations around the axis x^i are generated by,

$$\begin{aligned} J^i &= -\mathrm{i}\epsilon^{ijk} x^j \partial^k \\ &= -\frac{1}{2}\epsilon^{ijk} J^{jk}, \quad J^{jk} = -\mathrm{i}\left(x^j\partial^k - x^k\partial^j\right). \qquad (4.14) \end{aligned}$$

You probably have checked already in quantum mechanics that these, too fulfill the commutation relations Eq. (4.13). Note, that this is for instance used in quantum mechanics in the n-dimensional representation of spins with $n = 2s + 1$.

As this chapter is about fermions, we now consider the situation with spin 1/2. Then, we have the generators $\frac{\sigma^i}{2}$, with

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = \mathrm{i}\epsilon^{ijk}\frac{\sigma^k}{2}, \tag{4.15}$$

where σ^i are the Pauli matrices (spinor representation)

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.16}$$

We remark, that the Lie algebra provides *local* information about the Lie group (tangential space). We will now present two important examples.

Example 4-10: SO(3) and SU(2)$\simeq$ S^3. Lie algebra:

$$\left[t^a, t^b\right] = \mathrm{i}\epsilon^{abc}\, t^c . \tag{4.17}$$

As the Lie group is a differentiable manifold, the SU(2) is the double covering of SO(3)$\simeq$ RP^3, which is visualised in figure 4.1

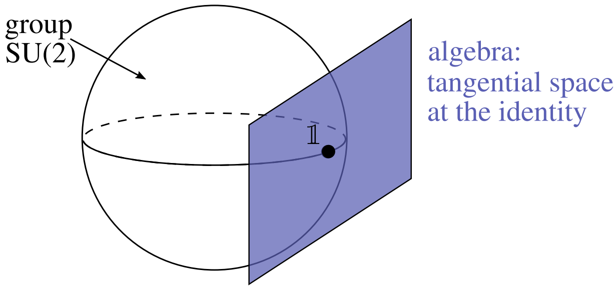


Figure 4.1.: Schematic representation of the SU(2) and the Lie algebra.

Example 4-11: SO(1,3) and SL (2,$\mathbb{C}$). Consider the infinitesimal Lorentz transformation $\Lambda \in$ SO(1,3):

$$\Lambda_\mu^{\ \nu} = \delta_\mu^{\ \nu} + \mathrm{i}\, T_\mu^{\ \nu} . \tag{4.18}$$

From

$$\Lambda_\mu^{\ \nu}\, \Lambda_\rho^{\ \sigma}\, \eta_{\nu\sigma} = \eta_{\mu\rho} \tag{4.19}$$

follows:

$$\begin{aligned} \left(\delta_\mu^{\ \nu} + \mathrm{i}\, T_\mu^{\ \nu}\right)\left(\delta_\rho^{\ \sigma} + \mathrm{i}\, T_\rho^{\ \sigma}\right)\eta_{\nu\sigma} &= \eta_{\mu\rho} + O(T^2) \\ \Rightarrow\ T_{\mu\rho} + T_{\rho\mu} &= 0 . \end{aligned} \tag{4.20}$$

We conclude, that T has $\frac{16-4}{2} = 6$ free parts, of which three are given by boosts and the other three by the

generators M

$$T_\mu^{\ \nu} = \frac{\omega^{\rho\sigma}}{2}\left(M_{\rho\sigma}\right)_\mu^{\ \nu} , \tag{4.21}$$

with

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\,(\eta^{\nu\rho}\, M^{\mu\sigma} - \eta^{\mu\rho}\, M^{\nu\sigma} - \eta^{\nu\sigma}\, M^{\mu\rho} + \eta^{\mu\sigma}\, M^{\nu\rho}) . \tag{4.22}$$

Eq. (4.22) is the Lie algebra of SO(1,3) rotations. To see this, we extend the SO(3)-generators of rotations, J^{ij} in Eq. (4.14) to boosts (J^{0i}) and find

$$J^{\mu\nu} = \mathrm{i}\,(x^\mu \partial^\nu - x^\nu \partial^\mu)\ , \tag{4.23}$$

which satisfy Eq. (4.22). To find general representations, we also look for M, that satisfy Eq. (4.22). For the fundamental representation we obtain for example

$$(M^{\mu\nu})_{\rho\sigma} = \mathrm{i}\left(\delta^\mu{}_\rho\, \delta^\nu{}_\sigma - \delta^\mu{}_\sigma\, \delta^\nu{}_\rho\right)\ . \tag{4.24}$$

Thus, boosts and rotations are given by

$$\begin{aligned} J_i &= \frac{1}{2}\epsilon_{ijk}\, M_{jk} \quad \textit{rotations} \\ K_i &= M_{0i} \quad \textit{boosts}\,. \end{aligned} \tag{4.25}$$

Example 4-12: Boosts along $\mathbf{x_1}$-axis.

$$\begin{aligned} \Lambda^\mu{}_\nu &= \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}} \\ &= \left(\mathrm{e}^{iw\,K_1}\right)^\mu{}_\nu\ , \end{aligned} \tag{4.26}$$

with the rapidity $w = \mathrm{artanh}\frac{v}{2}$ and the generator

$$i(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.27}$$

The generators in Eq. (4.25) make the structure of the Lorentz group apparent, as we can now formulate the Lie-algebra in terms of **J** and **K**

$$\begin{aligned} \left[J^i\,,\, J^j\right] &= \mathrm{i}\epsilon^{ijk}\, J^k \\ \left[K^i\,,\, K^j\right] &= -\mathrm{i}\epsilon^{ijk}\, J^k \\ \left[J^i\,,\, K^j\right] &= \mathrm{i}\epsilon^{ijk}\, K^k\,. \end{aligned} \tag{4.28}$$

We remark, that SU(2) and SL(2,$\mathbb{C}$) with the generators ($J^i + \mathrm{i}K^i, J^i - \mathrm{i}K^i$) are the universal covering groups of SO(3) and SO(1,3), respectively. Note: For a universal covering group $\tilde{G}$ of G it holds:

$$'\text{simply connected group } \tilde{G} \supseteq G'\,. \tag{4.29}$$

II. Spinor Fields

In section I we have discussed the mathematical structure of the Lorentz group. Let us now use these concepts to describe spinor fields. For this purpose we combine boosts K^i and rotations J^i into

$$\begin{aligned} N_+^i &= \frac{1}{2}\left(J^i + \mathrm{i}\, K^i\right) \\ N_-^i &= \frac{1}{2}\left(J^i - \mathrm{i}\, K^i\right). \end{aligned} \tag{4.30}$$

The N_+'s and N_-'s have the SO(3) i.e. SU(2) Lie-algebra:

$$\left[N_\pm^i\,,\, N_\pm^j\right] = \mathrm{i}\,\epsilon_{ijk}\, N_\pm^k \qquad [N_+^i, N_-^j] = 0. \tag{4.31}$$

Hence, we can formulate a two-dimensional spin 1/2 representation:

$$\begin{aligned} \text{left-handed:}\quad \Lambda_L &= \exp\left(\frac{\mathrm{i}}{2}\,\sigma^i\,(w_i - \mathrm{i}\,v_i)\right) \\ \text{right-handed:}\quad \Lambda_R &= \exp\left\{\frac{\mathrm{i}}{2}\,\sigma^i\,(w_i + \mathrm{i}\,v_i)\right\}, \end{aligned} \tag{4.32}$$

where w_i and v_i denote rotations and boosts, respectively and Λ_L, Λ_R $\in$ SL(2,$\mathbb{C}$)[1]. Note, that under parity transformations

$$\begin{aligned} \left(x^0, \mathbf{x}\right) &\xrightarrow{P} \left(x^0, -\mathbf{x}\right) \\ \Rightarrow \mathbf{J} &\xrightarrow{P} \mathbf{J} \quad \textit{pseudo-vector} \\ \Rightarrow \mathbf{K} &\xrightarrow{P} -\mathbf{K} \quad \textit{vector}\,. \end{aligned} \tag{4.33}$$

To determine, how $\Lambda_{L/R}$ act on coordinates, we define

$$\hat{x} = x_\mu \sigma^\mu\,, \tag{4.34}$$

with

$$(\sigma^\mu) = \left(\sigma^0, \sigma^1, \sigma^2, \sigma^3\right), \quad \sigma^0 = \mathbb{1}_{2\times 2}\,, \tag{4.35}$$

and the Pauli matrices $\sigma^{1,2,3}$. Then:

$$\hat{x} = \begin{pmatrix} x_0 - x_3 & x_1 + \mathrm{i}\, x_2 \\ x_1 - \mathrm{i}\, x_2 & x_0 + x_3 \end{pmatrix}, \tag{4.36}$$

and

$$\det \hat{x} = x_\mu x^\mu\,. \tag{4.37}$$

Note, that Lorentz transformations leave the determinant unchanged

$$\hat{x}' = \Lambda_L\, \hat{x}\, \Lambda_L^\dagger \quad \text{with} \quad \det \hat{x}' = \det \hat{x}\,, \tag{4.38}$$

as $\det \Lambda_L^{(\dagger)} = 1$.

[1] universal covering group of the Lorentz group

Example 4-13: Boosts along 3-direction. Let us consider the transformation where only v_3 is non-vanishing.

$$\begin{aligned}
\Lambda_L &= \exp\left[\frac{i}{2}\sigma^3(-iv_3)\right] \qquad (4.39)\\
&= \exp\left[\frac{v_3\sigma^3}{2}\right]\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{v_3}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}(\left(\frac{v_3}{2}\right)^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots\\
&= \begin{pmatrix} \exp(v_3/2) & 0 \\ 0 & \exp(-v_3/2) \end{pmatrix}\\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}
\end{aligned}$$

We can now calculate the transformed field,

$$\begin{aligned}
\hat{x}' &= \Lambda_L \hat{x} \Lambda_L^\dagger \qquad (4.40)\\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} a(x_0 - x_3) & a^{-1}(x_1 + ix_2) \\ a(x_1 + ix_2) & a^{-1}(x_0 + x_3) \end{pmatrix}\\
&= \begin{pmatrix} a^2(x_0 - x_3) & x_1 + ix_2 \\ x_1 - ix_2 & a^{-2}(x_0 + x_3) \end{pmatrix}.
\end{aligned}$$

At this point it is clear that the coordinates x_1 and x_2 remain unchanged under this transformation as befits a boost along the x_3-axis. Moreover we can extract,

$$x_0' = \frac{1}{2}(a^2 + a^{-2})x_0 - \frac{1}{2}(a^2 - a^{-2})x_3 \qquad (4.41)$$

and

$$x_3' = \frac{1}{2}(a^2 + a^{-2})x_3 - \frac{1}{2}(a^2 - a^{-2})x_0 \qquad (4.42)$$

Identifying

$$\gamma = \frac{1}{2}(a^2 + a^{-2}), \qquad \gamma v = \frac{1}{2}(a^2 - a^{-2}) \qquad (4.43)$$

we have the standard form for a Lorentz boost along the x_3-axis.

We remark, that Λ_L and $-\Lambda_L$ give the same $\hat{x}'$ (double covering). Further

$$\Lambda_{L/R}^\dagger = \Lambda_{R/L}^{-1}, \qquad (4.44)$$

and σ maps $L \to R$. Also note, that σ^μ transforms as a vector.

We can now try to formulate the field equations for a two-component spinor that have the right transformation properties under Lorentz transformations,

$$\mathcal{D}_L \Psi_L = 0, \qquad (4.45)$$

where Ψ_L is the left-handed *Weyl spinor*. Under Lorentz transformation, it holds

$$\begin{aligned}
\Psi_L(x) &\xrightarrow{\Lambda} \Lambda_L\Psi_L(x) \\
\mathcal{D}_L\Psi_L(x) &\xrightarrow{\Lambda} \mathcal{D}_L'\Lambda_L\Psi_L(x) = \Lambda_R\mathcal{D}_L\Psi_L(x) \\
\Rightarrow \mathcal{D}_L' &= \Lambda_R\mathcal{D}_L\Lambda_R^\dagger\,, \quad \text{as} \quad \Lambda_R^\dagger = \Lambda_L^{-1}\,,
\end{aligned} \tag{4.46}$$

with $\mathcal{D}_L = \mathrm{i}\,\bar{\sigma}^\mu\partial_\mu$ and $\bar{\sigma} = \left(\sigma^0, -\vec{\sigma}\right)$. Analogously, this holds for the right-handed Weyl spinor, with $\mathcal{D}_R = \mathrm{i}\,\sigma^\mu\partial_\mu$, which yields the

Weyl equations

$$\begin{aligned}
\mathrm{i}\,\bar{\sigma}^\mu\partial_\mu\,\Psi_L &= 0 \\
\mathrm{i}\,\sigma^\mu\partial_\mu\,\Psi_R &= 0\,,
\end{aligned} \tag{4.47}$$

which form the equations of motion of two-component spinors. Note, that the Weyl equations (4.47) do not have parity invariance.

Let us now connect Eq. (4.47) to the Klein-Gordon equation (2.11):

$$\begin{aligned}
& \sigma^\mu\partial_\mu\,(\bar{\sigma}^\nu\partial_\nu\,\Psi_L = 0) \\
= \;& \frac{1}{2}\left[\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right]\partial_\mu\partial_\nu\,\Psi_L = \eta^{\mu\nu}\,\partial_\mu\partial_\nu\,\Psi_L \\
\Rightarrow \;& \partial_\mu\partial^\mu\,\Psi_L = \partial^2\,\Psi_L = 0\,,
\end{aligned} \tag{4.48}$$

where we used that

$$\begin{aligned}
&\frac{1}{2}[\sigma^0\sigma^0 + \sigma^0\sigma^0] = \frac{1}{2}[1\cdot 1 + 1\cdot 1] = 1 \qquad (4.49)\\
&\frac{1}{2}[\sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0] = \frac{1}{2}[1(-\sigma^i) + \sigma^i 1] = 0 \\
&\frac{1}{2}[\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i] = \frac{1}{2}[\sigma^i(-\sigma^j) + \sigma^j(-\sigma^i)] = -\frac{1}{2}[\sigma^i\sigma^j + \sigma^j\sigma^i] = -1.
\end{aligned}$$

Similarly, one shows $\partial^2\,\Psi_R = 0$, which implies, that the Weyl spinors also satisfy the *m*assless Klein-Gorden equation.

When writing down the derivative term we have seen that $\mathcal{D}_L\Psi_L$ transforms as a right handed spinor. We therefore cannot simply add a term $\sim \Psi_L$, which of course transforms as a left-handed field, to the equation of motion in order to obtain a mass term. Moreover, we may want to demand parity invariance (this is not always true in the Standard Model, but it holds in QED).

To achieve this we have to combine left- and right-handed spinors. This gives the **Dirac spinor**

$$\Psi_D = \begin{pmatrix}\Psi_L \\ \Psi_R\end{pmatrix}, \tag{4.50}$$

which basically partitions space into the space of left- and right-handed spinors. Then, $\mathcal{D}_L$ maps left- to right-handed spinors and $\mathcal{D}_R$ maps right- to left-handed spinors. Now we combine the Weyl-operators $\mathcal{D}_{L/R}$

$$\begin{pmatrix}0 & \mathcal{D}_R \\ \mathcal{D}_L & 0\end{pmatrix}\begin{pmatrix}\Psi_L \\ \Psi_R\end{pmatrix} = \mathrm{i}\,\gamma^\mu\partial_\mu\,\Psi_D\,, \tag{4.51}$$

with the matrix γ^μ (chiral representation)

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{4.52}$$

The γ matrices are Lorentz invariant and have the desired parity invariance. Further they satisfy the

Clifford algebra

$$\{\gamma^\mu , \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{4.53}$$

To perform Lorentz transformations, we now define the four-dimensional spin $1/2$ representation of Λ:

$$\Lambda_{1/2} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} , \tag{4.54}$$

with

$$\begin{aligned} \Psi_D &\xrightarrow{\Lambda} \Lambda_{1/2}\, \Psi_D = \begin{pmatrix} \Lambda_L \Psi_L \\ \Lambda_R \Psi_R \end{pmatrix} \\ \mathrm{i}\,\gamma^\mu \partial_\mu \Psi_D &\xrightarrow{\Lambda} \Lambda_{1/2}\,\mathrm{i}\,\gamma^\mu \partial_\mu\, \Lambda_{1/2}^{-1}\, \Lambda_{1/2}\, \Psi_D = \Lambda_{1/2}\,\mathrm{i}\,\gamma^\mu \partial_\mu\, \Psi_D . \end{aligned} \tag{4.55}$$

With $\Psi = \Psi_D$, we can formulate the

Dirac equation

$$(\mathrm{i}\,\partial\!\!\!/ - m)\, \Psi = 0 , \tag{4.56}$$

with the short notation

$$w\!\!\!/ := \gamma^\mu w_\mu . \tag{4.57}$$

Note, that the Dirac spinor satisfies the *massive* Klein-Gordon-equation (2.11):

$$\begin{aligned} & \left(-\mathrm{i}\,\gamma^\mu \partial_\mu - m\right) (\mathrm{i}\,\gamma^\nu \partial_\nu - m)\ \Psi = \left(\gamma^\mu\, \gamma^\nu \partial_\mu \partial_\nu + m^2\right)\, \Psi \\ = \ & \left(\frac{1}{2}\, \{\gamma^\mu , \gamma^\nu\}\, \partial_\mu \partial_\nu + m^2\right) \Psi = \left(\frac{1}{2}\, (2\eta^{\mu\nu})\, \partial_\mu \partial_\nu + m^2\right) \Psi \\ \Rightarrow \ & \left(\partial_\mu \partial^\mu + m^2\right)\, \Psi = 0 . \end{aligned} \tag{4.58}$$

In the following, we consider **the Lagrangian and the Hamiltonian of the spinor field**. The Lagrangian transforms as a Lorentz scalar $\sim (\mathrm{i}\partial\!\!\!/ - m)\ \Psi$

$$\begin{aligned} \mathcal{L} &= \bar{\Psi}\, (\mathrm{i}\partial\!\!\!/ - m)\ \Psi \\ &\xrightarrow{\Lambda} \bar{\Psi}' \Lambda_{1/2}\, (\mathrm{i}\partial\!\!\!/ - m)\ \Psi \\ & . \end{aligned} \tag{4.59}$$

Therefore we have to have,

$$\bar{\Psi}' = \bar{\Psi} \Lambda_{1/2}^{-1} , \tag{4.60}$$

as the Lagrangian is Lorentz invariant. We will show below that for this to be the case we can use,

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \,, \tag{4.61}$$

which is called *Dirac conjugate*, with

$$\begin{aligned} \bar{\Psi}' &= \Psi^\dagger \Lambda_{1/2}^\dagger \gamma^0 = \Psi^\dagger \gamma^0 \gamma^0 \Lambda_{1/2}^\dagger \gamma^0 \\ \text{Eq. (4.76)} \quad \rightarrow \quad &= \bar{\Psi} \Lambda_{1/2}^{-1} \,. \end{aligned} \tag{4.62}$$

The equation of motion is given by the Dirac equation (4.56)

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} &= 0 = (\mathrm{i}\partial\!\!\!/ - m)\, \Psi \\ \frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} &= 0 = \bar{\Psi} \left(\mathrm{i}\overleftarrow{\partial\!\!\!/} - m \right) , \end{aligned} \tag{4.63}$$

with the short notation $f\,\overleftarrow{\partial_\mu} = -\partial_\mu f$. Then the Hamiltonian density is given by

$$\begin{aligned} \mathcal{H} &= \Pi_\Psi \dot{\Psi} - \mathcal{L} = \mathrm{i}\, \bar{\Psi} \gamma^0 \dot{\Psi} - \mathcal{L} \\ &= \Psi^\dagger \gamma^0 \left(\mathrm{i}\, \vec{\gamma}\, \vec{\partial} + m \right) \Psi \,, \end{aligned} \tag{4.64}$$

where $\vec{\gamma}\,\vec{\partial} = \gamma^i \partial_i = \gamma^i \frac{\partial}{\partial x^i}$.

We still have to demonstrate,

$$\bar{\Psi}' = \bar{\Psi} \Lambda_{1/2}^{-1}. \tag{4.65}$$

In principle we could directly verify this by using our formula for the γ-matrices as well as the Lorentz transformation, Eq. (4.54). However, let us do it in a way that tells us a few more details on the spinoir representation of the Lorentz group.
We can check by direct computation that the spin-representations of the generators M (Eq. (4.21)) is given by,

$$S^{\mu\nu} = \frac{\mathrm{i}}{4} \left[\gamma^\mu \,, \gamma^\nu \right] , \tag{4.66}$$

with

$$\begin{aligned} \left[\gamma^\mu \,, \gamma^\nu \right] &= \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} \\ \sigma\bar{\sigma} &: \quad L \rightarrow L \\ \bar{\sigma}\sigma &: \quad R \rightarrow R \end{aligned} \tag{4.67}$$

Accordingly the block-diagonal structure that we have between L and R is explicit.
Note, that (see Eq. (4.25))

$$\begin{aligned} K_{i_L} &= S_{0 i_L} = -\mathrm{i}\, \frac{\sigma_i}{2} = \mathrm{i}\, \frac{\bar{\sigma}_i}{2} \\ J_{i_L} &= \frac{1}{2} \epsilon_{ijk} S_{jk} = -\frac{\mathrm{i}}{2} \epsilon_{ijk} \left[\frac{\sigma_j}{2} \,, \frac{\sigma_k}{2} \right] = -\frac{\mathrm{i}}{2} \epsilon_{ijk} \left(\mathrm{i}\, \epsilon_{jkl} \frac{\sigma_l}{2} \right) = \frac{\sigma_i}{2} \,. \end{aligned} \tag{4.68}$$

Analogously, it follows

$$\begin{aligned} K_{i_R} &= \mathrm{i}\,\frac{\sigma_i}{2} \\ J_{i_R} &= \frac{\sigma_i}{2}\,, \end{aligned} \tag{4.69}$$

and hence

$$\Lambda_{1/2} = \mathrm{e}^{\mathrm{i}\,\frac{w_{\mu\nu}}{2}\,S^{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \tag{4.70}$$

with (see Eq. (4.32))

$$\begin{aligned} \Lambda_L &= \exp\left(\mathrm{i}\,\frac{\sigma^i}{2}\,(w_i - \mathrm{i}\,v_i)\right) \\ \Lambda_R &= \exp\left(\mathrm{i}\,\frac{\sigma^i}{2}\,(w_i + \mathrm{i}\,v_i)\right), \end{aligned} \tag{4.71}$$

and $w_{0i} = v_i$, $w_{ij} = \epsilon_{ijk}\,w_k$. This fully agrees with our earlier result on the L and R representation of the Lorentz group.

Next, let us find the inverse of $\Lambda_{1/2}$. γ^0 is hermitian, i.e.

$$\left(\gamma^0\right)^2 = \mathbb{1}_{4\times 4}, \quad \left(\gamma^0\right)^\dagger = \gamma^0\,. \tag{4.72}$$

On the other hand, γ^i is anti-hermitian:

$$\left(\gamma^i\right)^2 = -\mathbb{1}_{4\times 4}, \quad \left(\gamma^i\right)^\dagger = -\gamma^i\,. \tag{4.73}$$

Note, that these properties are representation independent. From Eq. (4.52) (i.e. choosing chiral representation) it also follows

$$\boxed{\gamma^0\left(\gamma^i\right)^\dagger\gamma^0 = \gamma^i\,,} \tag{4.74}$$

and we conclude

$$\gamma^0\,(S^{\mu\nu})^\dagger\,\gamma^0 = S^{\mu\nu}\,. \tag{4.75}$$

Thus, the inverse Lorentz transformation for the four-dimensional spin 1/2 representation is given by

$$\boxed{\gamma^0\Lambda^\dagger_{1/2}\gamma^0 = \Lambda^{-1}_{1/2}\,.} \tag{4.76}$$

Let us now discuss some **invariants and general properties**. The derivations above made use of a specific representation of our spinors in left- and right-handed Weyl spinors. In particular for massive Dirac fermions, this is not the best adapted representation. The γ's and Ψ's can be rotated with unitary transformations U, without changing the Lagrangian in Eq. (4.60). Thus a different representation can be obtained by

$$\gamma \to U^\dagger\,\gamma\,U\,. \tag{4.77}$$

This leaves the Clifford algebra unchanged. When transforming the generators $S_{\mu\nu} \to U^\dagger S_{\mu\nu} U$ we have to find a way, to project on the left- and right-handed eigenspaces. For this purpose we define

$$\gamma_5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{4.78}$$

with properties

$$\begin{aligned}\gamma_5^2 &= \mathbb{1} \quad \to \text{eigenvalues } \pm 1\\ \{\gamma_5\,,\,\gamma^\mu\} &= 0\\ \left[S_{\mu\nu}\,,\,\gamma_5\right] &= 0 \quad \to S_{\mu\nu},\ y_5 \text{ can be diagonalised at the same time}\,.\end{aligned} \tag{4.79}$$

Note, that in chiral representation $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently, we find the

projection operators on L/R spaces

$$P_{L/R} = \frac{\mathbb{1} \mp \gamma_5}{2}\,, \tag{4.80}$$

with $P_{L/R}^2 = P_{L/R}$ and $P_L + P_R = \mathbb{1}$. Hence,

$$P_{L/R}\,\Psi = \Psi_{L/R}\,. \tag{4.81}$$

Next, we discuss **Dirac matrices and Dirac field bilinears**. So far we found that $\bar\Psi$ is a Lorentz scalar. One easily finds, that $\bar\Psi\,\gamma^\mu\,\Psi$ is a 4-vector. Using $\gamma^\mu \to \Lambda^\mu{}_\nu\gamma^\nu$, we find a basis of sixteen 4×4 matrices, defined as antisymmetric combinations of γ-matrices:

$\mathbb{1}$	scalar	1 of these
γ^μ	vector	4 of these
$[\gamma^\mu\,,\,\gamma^\nu] := \gamma^{[\mu}\gamma^{\nu]}$	tensor	6 of these
$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$	pseudo-vector	4 of these
γ_5	pseudo-scalar	1 of these
		16 total

The prefix *pseudo* indicates, that these quantities transform usual under continuous Lorentz transformations, but with an additional sign change under parity transformations.
We can now look for symmetries of the Lagrangian.

$$\begin{aligned}\Psi \to \mathrm{e}^{\mathrm{i}\alpha}\Psi \quad &\Rightarrow \quad \bar\Psi \to \bar\Psi\mathrm{e}^{-\mathrm{i}\alpha}\\ \Psi \to \mathrm{e}^{+\mathrm{i}\gamma_5\alpha}\Psi \quad &\Rightarrow \quad \bar\Psi \to \bar\Psi\mathrm{e}^{+\mathrm{i}\gamma_5\alpha} \quad \text{for} \quad m=0.\end{aligned} \tag{4.82}$$

While the first is a symmetry for the full Dirac Lagrangian, the second is only a symmetry if the mass is vanishing.
From this we obtain two conserved currents out of Dirac field bilinears, namely

$$j^\mu = \bar\Psi\,\gamma^\mu\,\Psi \quad \text{and} \quad j_5^\mu = \bar\Psi\,\gamma^\mu\gamma_5\Psi\,. \tag{4.83}$$

These are conserved as

$$\begin{aligned}\partial_\mu \left(\bar\Psi \gamma^\mu \Psi\right)\Big|_{\mathrm{EOM}} &= \mathrm{i}\, m \bar\Psi \Psi - \mathrm{i}\, m \bar\Psi \Psi = 0 \\ \partial_\mu \left(\bar\Psi \gamma^\mu \gamma_5 \Psi\right) &= 2\mathrm{i}\, m \bar\Psi \gamma_5 \Psi\Big|_{m=0} = 0\,. \end{aligned} \tag{4.84}$$

Note again, that the axial current j_5^μ is only conserved for $m = 0$ (chiral symmetry).
Importantly note that,

$$j^0 = \bar\Psi\gamma^0\Psi = \Psi^\dagger\Psi. \tag{4.85}$$

We will now find **solutions of the Dirac equation**. As $\Psi(x)$ satisfies the Klein-Gorden equation (2.11). In our quantisation procedure these will then be the functions multiplying our creation and annihilation operators.
We write

$$\Psi(x) = u(p)\, \mathrm{e}^{-\mathrm{i}px}\,, \tag{4.86}$$

with $u(p)$ being a vector and $p^2 = m^2$. Thus,

$$\mathrm{e}^{-\mathrm{i}px}\, (\mathrm{i}\partial\!\!\!/ - m)\, \Psi(x) = \left(p\!\!\!/ - m\right) u(p) = 0\,. \tag{4.87}$$

Similarly, with

$$\Psi(x) = v(p)\mathrm{e}^{\mathrm{i}px} \tag{4.88}$$

we find

$$\left(p\!\!\!/ + m\right) v(p) = 0\,, \tag{4.89}$$

for $p^2 = m^2$.
Let us now simplify the equations of motion by going to the rest frame coordinate system. the Dirac equation then simplifies,

$$\begin{aligned} p &= (p_0, 0) \\ m\left(\gamma^0 - \mathbb{1}\right) u(p) &= 0\,. \end{aligned} \tag{4.90}$$

With the chiral representation, we have

$$\left(\gamma^0 - \mathbb{1}\right) = \begin{pmatrix} -\mathbb{1}_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & -\mathbb{1}_{2\times2} \end{pmatrix}. \tag{4.91}$$

However, for the purpose of finding solutions and in particular discussing the non-relativistic limit a different representation of the Dirac matrices is useful. As already transforming matrices with a unitary matrix as

$$\gamma' = U^\dagger \gamma U \tag{4.92}$$

the γ' fulfil the same anti-commutation relations as the original γ in such they are also a suitable representation of the Dirac algebra.
When investigating the solutions to the equations of motion a different, the so-called Dirac representation, is handy. This can be obtained by using,

$$U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{4.93}$$

We then have,

$$\begin{aligned}\gamma^0 &= \begin{pmatrix}\mathbb{1}_{2\times 2} & 0\\ 0 & -\mathbb{1}_{2\times 2}\end{pmatrix},\ \gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\\ \Rightarrow & \left(\gamma^0 - \mathbb{1}\right) = 2\begin{pmatrix}0 & 0\\ 0 & -\mathbb{1}_{2\times 2}\end{pmatrix}.\end{aligned} \tag{4.94}$$

With this, we find

$$\begin{aligned}u_s(p^0) &= \sqrt{2m}\begin{pmatrix}\chi_s\\ 0\end{pmatrix}\\ v_s(p^0) &= \sqrt{2m}\begin{pmatrix}0\\ \epsilon\chi_s\end{pmatrix},\end{aligned} \tag{4.95}$$

with $s = 1/2, -1/2$, $\chi_{1/2} = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_{-1/2} = \begin{pmatrix}0\\1\end{pmatrix}$ and the metric in spinor space $\epsilon = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Note, that $\epsilon^{-1}\sigma\epsilon = \bar{\sigma}$.

We can now simply do a Lorentz boost. However, we need to do this in the Dirac basis,

$$\Lambda_D = U^\dagger \Lambda_{\text{chiral}} U \frac{1}{2}\begin{pmatrix}\Lambda_L + \Lambda_R & \Lambda_R - \Lambda_L\\ \Lambda_R - \Lambda_L & \Lambda_L + \Lambda_R\end{pmatrix} \tag{4.96}$$

With this we obtain,

general solutions of the Dirac equation

$$\begin{aligned}u_s(p) &= \frac{1}{\sqrt{2m}}\frac{\not{p} + m}{\sqrt{p^0 + m}}u_s(p^0) = \sqrt{p^0 + m}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\,\vec{p}}{p^0+m}\chi_s\end{pmatrix}\\ v_s(p) &= \frac{1}{\sqrt{2m}}\frac{\not{p} - m}{\sqrt{p^0 + m}}v_s(p^0) = -\sqrt{p^0 + m}\begin{pmatrix}\frac{\vec{\sigma}\,\vec{p}}{p^0+m}\epsilon\chi_s\\ \epsilon\chi_s\end{pmatrix},.\end{aligned} \tag{4.97}$$

We will now introduce some relations between the solutions. We have

$$\begin{aligned}\bar{u}_r(p)\,u_s(p) &= 2m\delta_{rs}\\ \bar{v}_r(p)\,v_s(p) &= -2m\delta_{rs}\end{aligned} \tag{4.98}$$

$$\bar{u}_r(p)\,v_s(p) = 0 = \bar{v}_r(p)\,u_s(p) \tag{4.99}$$

$$\begin{aligned}\sum_s u_s(p)_\xi\,\bar{u}_s(p)_{\bar{\xi}} &= (\not{p} + m)_{\xi\bar{\xi}}\\ \sum_s v_s(p)_\xi\,\bar{v}_s(p)_{\bar{\xi}} &= (\not{p} - m)_{\xi\bar{\xi}},.\end{aligned} \tag{4.100}$$

The calculation to Eq. (4.98) goes as follows:

$$\begin{aligned}
\bar{u}_r(p)\,u_s(p) &&=&\; u_r^\dagger(p^0)\,\frac{(\not{p}+m)\,\gamma^0\,(\not{p}+m)}{p^0+m}\,u_s(p^0) \\
\gamma^0\gamma^\dagger\gamma^0=\gamma &\quad\rightarrow &=&\; u_r^\dagger(p^0)\,\gamma^0\,\frac{(\not{p}+m)(\not{p}+m)}{p^0+m}\,u_s(p^0) \\
{\scriptstyle u_r^\dagger(p^0)=u_r(p^0),\; u_r(p^0)\gamma^0\gamma^i\,u_s(p^0)=0} &\quad\rightarrow &=&\; u_r(p^0)\,\gamma^0\,\frac{p^2+m^2+2p^0m\gamma^0}{p^0+m}\,u_s(p^0) \\
\gamma^0\,u_s(p^0)=u_s(p^0) &\quad\rightarrow &=&\; 2u_r(p^0)\,\gamma^0\,\frac{m(p^0+m)}{p^0+m}\,u_s(p^0) \\
&&=&\; 2m\delta_{rs}\,, \qquad (4.101)
\end{aligned}$$

and analogously for $\bar{v}_r\,v_s$. Eq. (4.99) follows from $(\not{p}-m)(\not{p}+m)=0$ and Eq. (4.100) is proven by showing it at the basis $u_s(p), v_s(p)$, i.e.

$$\begin{aligned}
\sum_s u_s(p)\,\bar{u}_s(p)\,u_r(p) &= \sum_s u_s(p)\,2m\delta_{rs} \\
&= 2m\,u_r(p) = \frac{2m(\not{p}+m)}{\sqrt{p^0+m}}\,u_r(p^0) = \frac{(\not{p}+m)^2}{\sqrt{p^0+m}}\,u_r(p^0) \\
&= (\not{p}+m)\,u_r(p) \qquad (4.102) \\
\sum_s u_s(p)\,\bar{u}_s(p)\,v_r(p) &= 0 = (\not{p}+m)\,v_r(p)\,, \qquad (4.103)
\end{aligned}$$

and similarly for $\sum_s v_s(p)\,\bar{v}_s(p)$.

III. Quantisation

First, we try to quantise fermions similarly to scalars (bosons), as performed in section III. In analogy to Eq. (2.134), we have the

general solution to the Dirac equation

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2p^0}}\sum_s\left[e^{-ipx}\,a_s(\mathbf{p})\,u_s(\mathbf{p}) + e^{+ipx}\,b_s^\dagger(\mathbf{p})\,v_s(\mathbf{p})\right], \quad \text{with} \quad p^0 = \sqrt{\mathbf{p}^2+m^2}\,. \qquad (4.104)$$

The Hamiltonian follows from Eq. (4.64)

$$\begin{aligned}
H &= \int d^3x\,\mathcal{H} = \int d^3x\,\Psi^\dagger(\mathbf{x})\,\gamma^0\left(i\,\vec{\gamma}\,\vec{\partial}+m\right)\Psi(\mathbf{x}) \\
&= \int \frac{d^3p}{(2\pi)^3}\,\frac{2p^0}{2p^0}\,p^0\sum_s\left[a_s^\dagger(\mathbf{p})\,a_s(\mathbf{p}) - b_s(\mathbf{p})\,b_s^\dagger(\mathbf{p})\right]. \qquad (4.105)
\end{aligned}$$

Note the "−" sign instead of the "+" in Eq. (2.137) in the last line! Here, we have used Eq. (4.98) and

$$
\begin{aligned}
(\vec{\gamma}\,\mathbf{p} + m)\; u(p) &= \Big(-(\not{p} - m) + \gamma^0\, p^0 \Big) u(p) \\
&= \gamma^0\, p^0\, u(p) \\
(-\vec{\gamma}\,\mathbf{p} + m)\; v(p) &= -\gamma^0\, p^0\, v(p)\,,
\end{aligned}
\tag{4.106}
$$

where the "−" in the last line corresponds to the "−" in Eq. (4.105). If we now suggest commuting operators, e.g.

$$
b_s\, b_s^\dagger = b_s^\dagger\, b_s + \text{c-number}\,,
$$

this would imply that

$$
H \simeq \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p^0 \sum_s \Big[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) - b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p}) \Big]\,,
$$

.

The minus sign is a problem, because it would create a Hamiltonian that is unbounded from below.

Therefore, we suggest

$$
b_s\, b_s^\dagger = -b_s^\dagger\, b_s + \text{c-number}\,.
\tag{4.107}
$$

Further, demanding

$$
\Big[\Psi(\mathbf{x})\,,\, \mathrm{i}\,\Psi^\dagger(\mathbf{y}) \Big] = \mathrm{i}\,\delta(\mathbf{x} - \mathbf{y})\,,
\tag{4.108}
$$

implies

$$
\begin{aligned}
\Big[a_s(\mathbf{p})\,,\, a_r^\dagger(\mathbf{q}) \Big] &= (2\pi)^3\, \delta(\mathbf{p} - \mathbf{q})\, \delta_{sr} \\
&= -\Big[b_s(\mathbf{p})\,,\, b_r^\dagger(\mathbf{q}) \Big]\,.
\end{aligned}
\tag{4.109}
$$

Again, note the additional "−" int the last line, which rescues causality, but does not cure the issue with the minus sign in the Hamiltonian! Hence, we define the

anti-commutation relations of creation and annihilation operator (for fermions)

$$
\begin{aligned}
\Big\{ a_s(\mathbf{p})\,,\, a_r^\dagger(\mathbf{q}) \Big\} &= (2\pi)^3\, \delta_{sr}\, \delta(\mathbf{p} - \mathbf{q}) \\
\Big\{ b_s(\mathbf{p})\,,\, b_r^\dagger(\mathbf{q}) \Big\} &= (2\pi)^3\, \delta_{sr}\, \delta(\mathbf{p} - \mathbf{q})\,.
\end{aligned}
\tag{4.110}
$$

Note, that the anti-commutators of a-a, b-b, b-$a^{(\dagger)}$ vanish and in particular $a_s(\mathbf{p})\, a_s(\mathbf{p}) = a_s^2 = 0$ (Grassmann variables). It follow the

anti-commutation relations of field operators (for fermions)

$$
\begin{aligned}
\Big\{ \Psi_\xi(\mathbf{x})\,,\, \Psi_{\xi'}^\dagger(\mathbf{y}) \Big\} &= \delta_{\xi\xi'}\, \delta(\mathbf{x} - \mathbf{y}) \\
\Big\{ \Psi_\xi(\mathbf{x})\,,\, \Psi_{\xi'}(\mathbf{y}) \Big\} = 0 &= \Big\{ \Psi_\xi^\dagger(\mathbf{x})\,,\, \Psi_{\xi'}^\dagger(\mathbf{y}) \Big\}\,.
\end{aligned}
\tag{4.111}
$$

Similarly to section III, we will now construct the **Fock space**. Again, we define a vacuum state $|0\rangle$ (compare to Eq. (2.95)), with

$$
\sqrt{2 w_{\mathbf{p}}}\, a_s(\mathbf{p})\, |0\rangle = 0 = \sqrt{2 w_{\mathbf{p}}}\, b_s(\mathbf{p})\, |0\rangle\,.
\tag{4.112}
$$

The one-particle states are given by

$$|\mathbf{p}, s\rangle = \sqrt{2w_\mathbf{p}}\, a_s^\dagger(\mathbf{p})\, |0\rangle \ , \tag{4.113}$$

and $\sqrt{2w_\mathbf{p}}\, b_s^\dagger(\mathbf{p})\, |0\rangle$ for anti-particles. The states are normalised to

$$\langle \mathbf{q}, r|\mathbf{p}, s\rangle = (2\pi)^3\, 2p^0\, \delta_{rs}\, \delta(\mathbf{p} - \mathbf{q})\,. \tag{4.114}$$

Note, that the states are *antisymmetric*, as e.g. for two-particle states:

$$\sim a_s^\dagger(\mathbf{p})\, a_r^\dagger(\mathbf{q})\, |0\rangle = -\, a_r^\dagger(\mathbf{q})\, a_s^\dagger(\mathbf{p})\, |0\rangle \ . \tag{4.115}$$

In particular, it is

$$a_r^\dagger(\mathbf{p})\, a_r^\dagger(\mathbf{p})\, |0\rangle = 0\,, \tag{4.116}$$

which mirrors the Fermi-exclusion principle. Next, we consider **continuous symmetries**. Again, we define the 4-momentum operator (compare to Eq. (2.128)):

$$\begin{aligned} P^0 &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p^0 \sum_s \left[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right], \quad p^0 = E > 0 \\ &= H \\ P^i &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p^i \sum_s \left[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right] . \end{aligned} \tag{4.117}$$

Ψ is a complex field and the Lagrangian is invariant under $\Psi \to \mathrm{e}^{ie\alpha}\,\Psi$, $\bar\Psi \to \bar\Psi\,\mathrm{e}^{-ie\alpha}$, as shown in Eq. (4.82). This leads to conserved currents and similarly to Eq. (2.65), we can formulate the

Noether charge (for fermions)

$$\begin{aligned} Q &= \int \mathrm{d}^3x\, j^0 = e \int \mathrm{d}^3x\, \Psi^\dagger(x)\, \Psi(x) = e \int \mathrm{d}^3x\, \bar\Psi(x)\, \gamma^0\, \Psi(x) \\ &\simeq e \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s \left[a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p}) - b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right] , \end{aligned} \tag{4.118}$$

where e is the elementary charge, and $\left(a_s^\dagger(\mathbf{p})\, a_s(\mathbf{p})\right)$, $\left(b_s^\dagger(\mathbf{p})\, b_s(\mathbf{p})\right)$ correspond to a fermion with charge e and an anti-fermion with charge $-e$, respectively.
This actually also solves a small oddity that we ignored earlier on. In terms of ordinary fields the Noether charge density $\bar\Psi\gamma^0\Psi = \Psi^\dagger\Psi \geq 0$. This makes sense if we have only one type of particle and this carries only one type of charge. We can then only have either total positive or total negative charge (one can multiply the Noether current which is particle number by the positive or negative charge). If we now allow ourselves antiparticles we can naturally have both positive and negative charges for one type of field which describes both of them in a unified way.

Interlude: Grassmann numbers
In the case of the scalar field the classical field theory could be obtained by setting the commutators to zero (as in quantum mechanics). However, for the fermion field we now have decided to use anti-commutators. Setting those to zero tells us that the spinors still anti-commute. Therefore they are **not** classical functions with ordinary complex values. Instead, they are so-called anti-commuting or Grassmann numbers.

Let us introduce them by giving rules for calculating with them. If we have two Grassmann numbers θ and η we have,

$$\theta\eta = -\eta\theta. \tag{4.119}$$

This implements their anti-commuting nature.
Grassmann numbers can be multiplied (and added to) ordinary complex numbers (this makes it an algebra),

$$A + B\theta + C\eta \tag{4.120}$$

Two such numbers can be multiplied together, e.g.

$$(A_1+B_1\theta+C_1\eta)(A_2+B_2\theta+C_2\eta) = A_1A_2+(B_1A_2+A_1B_2)\theta+(C_1A_2+A_1C_2)\eta+(B_1C_2-C_1B_2)\theta\eta. \tag{4.121}$$

Here, we have used that due to the anti-commuting nature we have,

$$\theta^2 = \eta^2 = 0. \tag{4.122}$$

In general we can also have functions of Grassmann numbers. Due to the vanishing of the second powers they are at most linear in the Grassmann variable,

$$f(\theta) = A + B\theta. \tag{4.123}$$

We can also take derivatives with respect to Grassmann variables. Because of the linearity of the functions there is the obvious definition,

$$\frac{d}{d\theta} f(\theta) = B. \tag{4.124}$$

Similarly requiring that the integral is shift invariant under $\theta \to \theta + \eta$ suggests the definition,

$$\int d\theta f(\theta) = \int d\theta (A + B\theta) = \int d\theta ((A + B\eta) + B\theta) = B. \tag{4.125}$$

Integration does the same as differentiation!
For multiple Grassmann variables this can be easily generalized, e.g.

$$\int d\theta \int d\eta\, \eta\theta = 1. \tag{4.126}$$

Note, that one has to be very careful of the order of integration because every swap in the order corresponds to adding a minus sign.
We can also introduce complex Grassmann variables (to deal with something like the complex Dirac spinor fields). This essentially adds a Hermitean conjugation, where again we have to be slightly careful about the order and the corresponding sign.

$$(\theta\eta)^\star = \eta^\star\theta^\star = -\theta^\star\eta^\star. \tag{4.127}$$

Just as in the case of ordinary complex numbers we can separate the complex Grassmann numbers into two real ones,

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}, \qquad \theta^\star = \frac{\theta_1 - i\theta_2}{\sqrt{2}}. \tag{4.128}$$

This is consistent with the above definition of complex conjugation,

$$(\theta^\star\theta) = \frac{1}{2}(\theta_1\theta_1 + i\theta_2\theta_1 - i\theta_1\theta_2 + \theta_2\theta_2) = i\theta_2\theta_1 \tag{4.129}$$

and

$$(\theta^\star\theta) = \theta^\star\theta = -i\theta_1\theta_2 = i\theta_2\theta_1. \tag{4.130}$$

Integration can be similarly defined,

$$\int d\theta^\star d\theta f(\theta,\theta^\star) = \int d\theta^\star d\theta (A + B\theta + C\theta^\star + D\theta\theta^\star) = D. \tag{4.131}$$

Differentiation is again analogous.
This concludes our brief interlude.

Let us now calculate the **propagator**. Therefore we first consider

$$\begin{aligned}
\langle 0|\,\Psi_\xi(x)\,\bar\Psi_{\xi'}(y)\,|0\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2p^0}\left[\sum_s (u_s)_\xi\,(\bar u_s)_{\xi'}\right]\mathrm{e}^{-\mathrm{i}p(x-y)} \\
\langle 0|\,\Psi_\xi(x)\,\bar\Psi_{\xi'}(y)\,|0\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2p^0}(\not p + m)_{\xi\xi'}\,\mathrm{e}^{-\mathrm{i}p(x-y)} \\
&= (\mathrm{i}\not\partial_x + m)_{\xi\xi'}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2p^0}\,\mathrm{e}^{-\mathrm{i}p(x-y)}
\end{aligned} \tag{4.132}$$

and

$$\langle 0|\,\bar\Psi_{\xi'}(y)\,\Psi_\xi(x)\,|0\rangle = -\,(\mathrm{i}\not\partial_x + m)_{\xi\xi'}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2p^0}\,\mathrm{e}^{-\mathrm{i}p(y-x)}\,. \tag{4.133}$$

Note, that the two integrals correspond to the scalar propagator (see Eq. (3.86)) without the θ-function from time ordering. Also note the global minus sign in Eq. (4.133), which implies that the order of $\Psi(x)$, $\bar\Psi(y)$ is important (which was not the case for the scalar field)! With time ordering we find in analogy to Eq. (3.87) the

Feynman-propagator (for fermions)

$$\begin{aligned}
S_F(x-y) &= \langle 0|\,T\,\Psi_\xi(x)\,\bar\Psi_{\xi'}(y)\,|0\rangle \\
&= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{i}\,(\not p + m)_{\xi\xi'}}{p^2 - m^2 + \mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p(x-y)}\,,
\end{aligned} \tag{4.134}$$

with time ordering

$$\begin{aligned}
T\;\Psi(x)\bar\Psi(y) &= \theta(x^0 - y^0)\,\Psi(x)\,\bar\Psi(y) - \theta(y^0 - x^0)\,\bar\Psi(y)\,\Psi(x) \\
&= -T\;\bar\Psi(y)\,\Psi(x)
\end{aligned} \tag{4.135}$$

Again, note the relative minus sign, when comparing to Eq. (3.32).
We can now formulate the **Feynman rules for fermions**. We can directly take over the results for the scalar theory (section III- V), but we have to take care of the anti-symmetry of fermions. We have already introduced

$$T\,\Psi\,\bar\Psi = -T\,\bar\Psi\,\Psi\,.$$

Accordingly, if we define contractions as in the scalar theory, it follows

$$\begin{aligned}
\contraction{}{\Psi}{(x)\,}{\bar\Psi}\Psi(x)\,\bar\Psi(y) &= \langle 0|\,T\,\Psi(x)\,\bar\Psi(y)\,|0\rangle \\
&= S_F(x-y) \\
&= -\contraction{}{\bar\Psi}{(y)\,}{\Psi}\bar\Psi(y)\,\Psi(x)\,.
\end{aligned} \tag{4.136}$$

Then

$$\cdots \contraction{}{\Psi}{\,\Psi^n\,\bar{\Psi}^m\,}{\bar{\Psi}}\Psi\,\Psi^n\,\bar{\Psi}^m\,\bar{\Psi}\cdots = (-1)^{n+m}\cdots \contraction{}{\Psi}{}{\bar{\Psi}}\Psi\bar{\Psi}\,\Psi^n\,\bar{\Psi}^m\,. \tag{4.137}$$

Also it holds for normal ordering:

$$\begin{aligned} :a\,a^\dagger: &= -:a^\dagger\,a: = -a^\dagger\,a \\ \Rightarrow \quad :\Psi_1\cdots\Psi_n\Psi_{n+1}\cdots: &= -:\Psi_1\cdots\Psi_{n+1}\Psi_n\cdots: \\ \Rightarrow \quad :\Psi_1\cdots\Psi_n\bar{\Psi}_{n+1}\cdots: &= -:\Psi_1\cdots\bar{\Psi}_{n+1}\Psi_n\cdots: \end{aligned} \tag{4.138}$$

Similarly to Eq. (3.93), we obtain

Wick's theorem (for fermions)

$$T\ \Psi(x_1)\cdots\Psi(x_n)\,\bar{\Psi}(x_{n+1})\cdots\bar{\Psi}(x_{n+m}) = :\Psi(x_1)\cdots\Psi(x_n)\,\bar{\Psi}(x_{n+1})\cdots\bar{\Psi}(x_{n+m}) + \text{all contractions}\ :\ . \tag{4.139}$$

In the following, let us discuss the simplest **interacting theory** with fermions. Then we have a Lagrangian of the form

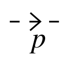
$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{I}}\,, \tag{4.140}$$

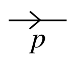
with the interaction Lagrangian of the *Yukawa theory*

$$\mathcal{L}_{\text{I}} = -h\,\bar{\Psi}\,\phi\,\Psi\,, \tag{4.141}$$

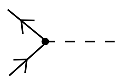
where h is the Yukawa coupling. We obtain

Propagators:

$\phi: \quad \contraction{}{\phi}{}{\phi}\phi\phi =$  $= \dfrac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon}$

$\Psi: \quad \contraction{}{\Psi}{}{\bar{\Psi}}\Psi\bar{\Psi} =$  $= \dfrac{\mathrm{i}(\not{p} + m_\Psi)}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}$

Vertex:

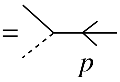
 $= -\mathrm{i}\,h$

(4.142)

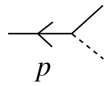
External leg contraction:

$$\begin{aligned} \contraction{}{\phi}{\,}{|}\phi\,|\mathbf{p}\rangle \quad &:= 1 =: \quad \langle\mathbf{p}|\,\phi \\ \Psi(x)\Big|_{\text{annihilation}}|\mathbf{p},s\rangle &= \int\frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{2p^0}{2q^0}}\sum_r\left[\mathrm{e}^{-\mathrm{i}qx}\,u_r(\mathbf{q})\,a_r(\mathbf{q})\,a_s^\dagger(\mathbf{p})\,|0\rangle\right] \\ &= \int\frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{2p^0}{2q^0}}\sum_r\left[\mathrm{e}^{-\mathrm{i}qx}\,u_r(\mathbf{q})\,\left\{a_r(\mathbf{q})\,,\,a_s^\dagger(\mathbf{p})\right\}|0\rangle\right] \\ &= \mathrm{e}^{-\mathrm{i}px}\,u_s(p)\,. \end{aligned} \tag{4.143}$$

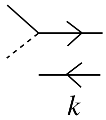
We drop the phase in Eq. (4.143) and find:

$$\Psi\,|\mathbf{p}, s\rangle := u_s(\mathbf{p}) =$$ 

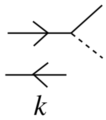
Analogously:

$$\langle \mathbf{p}, s|\,\bar\Psi := \bar u_s(\mathbf{p}) =$$ 

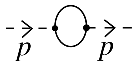
Anti-fermions:



$$= \bar\Psi\,|\mathbf{k}, s\rangle := \bar v^s(\mathbf{k})$$
$$\Vert$$
$$b_s^\dagger(\mathbf{k})\,|0\rangle$$

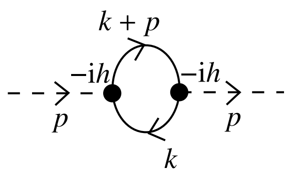


$$= \langle \mathbf{k}, s|\,\Psi := v^s(\mathbf{k})$$

Loops: e.g.  (vacuum polarisation)

$$\langle \mathbf{q}|\,(\phi\,\bar\Psi\,\Psi)\,(\phi\,\bar\Psi\,\Psi)\,|\mathbf{p}\rangle \sim -\langle \mathbf{q}|\,\phi\,(\Psi\,\bar\Psi\,\Psi\,\bar\Psi)\,\phi\,|\mathbf{p}\rangle\ . \tag{4.144}$$

Consequently, closed fermionic loops lead to minus signs! The calculations for loop integrals and Dirac traces goes as follows. Consider



.

Then:

$$-(-\mathrm{i}\,h)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\,\langle \mathbf{q}|\,\phi_x\,\Psi_{x_\xi}\,\bar\Psi_{y_\eta}\,\Psi_{y_\eta}\,\bar\Psi_{x_\xi}\,\phi_y\,|\mathbf{p}\rangle\ . \tag{4.145}$$

Evidently, (4.145) contains a dirac trace,

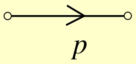
$$\Psi_{x_\xi}\,\bar\Psi_{y_\eta}\,\Psi_{y_\eta}\,\bar\Psi_{x_\xi} = S_{F_{\xi\eta}}(x-y)\,S_{F_{\eta\xi}}(y-x) = \mathrm{tr}_{\mathrm{Dirac}}\Big[S_F(x-y)\,S_F(y-x)\Big]. \tag{4.146}$$

It follows

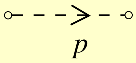
$$-(-\mathrm{i}\,h)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\,\langle \mathbf{q}|\,\phi_x\,\Psi_{x_\xi}\,\bar\Psi_{y_\eta}\,\Psi_{y_\eta}\,\bar\Psi_{x_\xi}\,\phi_y\,|\mathbf{p}\rangle = -h^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \mathrm{tr}\left(\frac{\not{k} + m_\Psi}{k^2 - m_\Psi^2 + i\,\epsilon}\,\frac{\not{k} + \not{p} + m_\Psi}{(k+p)^2 - m_\Psi^2 + i\,\epsilon}\right). \tag{4.147}$$

Let us summarise the

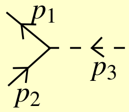
Feynman rules (in momentum space, for Yukawa theory)

i) 

$$= \frac{\mathrm{i}(\not p + m_\Psi)}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}$$



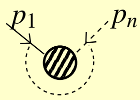
$$= \frac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon}$$

ii) 

$= -\mathrm{i}\,h$ and $p_2 = -(p_1 + p_3)$ (momentum conservation)

iii) $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$ for each loop

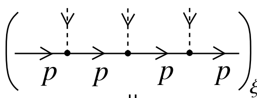
(−) for each fermion loop

iv) $(2\pi)^4\,\delta^4(\sum_i p_i)$ for 

.

(4.148)

When comparing to Eq. (3.117), we note, that there is *no symmetry factor* in Eq. (4.148), as $\mathcal{L}_I$ is built-up from 3 different fields. Also, now the *direction of the fermion line is important*. Along fermion lines Dirac indices are contracted, e.g.

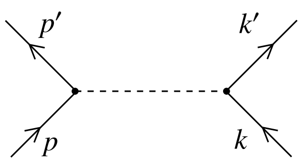


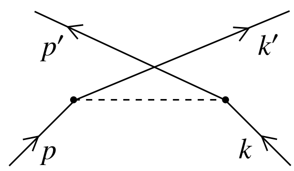
$$\left(\cdots\right)_{\xi\xi'} \simeq (-\mathrm{i}\,h)^3 \left(\frac{\mathrm{i}\,(\not p + m_\Psi)}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}\right)_{\xi\xi'} (-\mathrm{i}\,h) \left(\frac{\mathrm{i}\,(\not p + m_\Psi)}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}\right)_{\eta\xi'} .$$

$$\| $$

$$\bar\Psi\,\phi \left[\left(\Psi\,\bar\Psi\right)\,\phi\,\left(\Psi\,\bar\Psi\right)\right]_{\xi\xi'}\,\phi\,\Psi$$

Example 4-14: scattering process.



$+$


$$\Rightarrow \mathrm{i}\,M = (-\mathrm{i}\,h)^2 \left[\bar u(\mathbf{p}')\,u(\mathbf{p})\,\frac{1}{(p-p')^2 - m_\phi^2}\,\bar u(\mathbf{k}')\,u(\mathbf{k}) - \bar u(\mathbf{p}')\,u(\mathbf{k})\,\frac{1}{(p-k)^2 - m_\phi^2}\,\bar u(\mathbf{k}')\,u(\mathbf{p})\right] . \quad (4.149)$$

Example 4-15: QED: couple electron Ψ_e to photon A_μ. The interaction Lagrangian is then contracted with a vector A_μ

$$\mathcal{L}_I = e\,\bar{\Psi}\,A_\mu\,\gamma^\mu\,\Psi\,. \tag{4.150}$$

Then

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{photon}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{I}}\,, \tag{4.151}$$

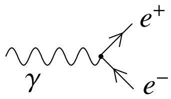
with

$$\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{I}} = \bar{\Psi}\,(\mathrm{i}\,\not{D} - m)\;\Psi\,, \tag{4.152}$$

where

$$D_\mu = \partial_\mu - \mathrm{i}\,e\,A_\mu \tag{4.153}$$

For the vertices it is



$= \mathrm{i}\,e$

To explicitly compute expressions as the above, we need the photon propagator. This will be subject to the subsequent chapter, in particular section II.

5. Gauge Fields

I. Gauge Symmetry

Consider the Dirac theory of e^+, e^-

$$\mathcal{L}_D = \bar{\Psi}(x)\,(\mathrm{i}\,\partial\!\!\!/ - m)\,\Psi(x)\,, \tag{5.1}$$

or complex scalar theory

$$\mathcal{L}_\phi = \partial_\mu\phi\,\partial_\mu\phi^* - m^2\,\phi\phi^* - V(\phi\phi^*)\;. \tag{5.2}$$

The Lagrangians in Eq. (5.1) and Eq. (5.2) are invariant under global U(1)-rotations, namely

$$\begin{aligned}\Psi &\;\rightarrow\; \mathrm{e}^{\mathrm{i}\alpha}\,\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi}\,\mathrm{e}^{-\mathrm{i}\alpha} \\ \phi &\;\rightarrow\; \mathrm{e}^{\mathrm{i}\alpha}\,\phi, \quad \phi^* \rightarrow \phi^*\,\mathrm{e}^{-\mathrm{i}\alpha}\,,\end{aligned} \tag{5.3}$$

which corresponds to a global rotation in field space. Let us require the invariance of the theory under **local rotations** (*gauge symmetry*), e.g.

$$\Psi(x) \rightarrow \mathrm{e}^{\mathrm{i}\alpha(x)}\,\Psi(x)\,. \tag{5.4}$$

We see, that $\mathcal{L}_D$ is not invariant, as

$$\begin{aligned}\mathcal{L}_D &\;\rightarrow\; \mathcal{L}_D - \bar{\Psi}(\partial\!\!\!/\,\alpha)\Psi \\ &\;=\; \mathcal{L}_D - \partial_\mu\alpha\, j^\mu\,,\end{aligned} \tag{5.5}$$

with

$$j^\mu = \bar{\Psi}\,\gamma^\mu\,\Psi\,. \tag{5.6}$$

Hence, if we add a term $A_\mu j^\mu$ to $\mathcal{L}_D$, and demand invariance, it follows

$$\begin{aligned}\mathcal{L}_D + A_\mu j^\mu &\;\rightarrow\; \mathcal{L}_D - \partial_\mu\alpha\, j^\mu + A'_\mu j^\mu \\ &\;\stackrel{!}{=}\; \mathcal{L}_D + A_\mu j^\mu \\ \Rightarrow A_\mu(x) &\;\rightarrow\; A_\mu(x) + \partial_\mu\alpha(x)\,.\end{aligned} \tag{5.7}$$

Also note, that $\mathcal{L}$ is a Lorentz scalar:

$$A_\mu \;\xrightarrow{\Lambda}\; \Lambda^\mu{}_\nu A_\nu\,, \tag{5.8}$$

as $A_\mu j^\mu$ transforms as a scalar. Next, we write the invariant action

$$\mathcal{L}_D = \bar{\Psi}\,(\mathrm{i}\,D\!\!\!\!/ - m)\,\Psi\,, \tag{5.9}$$

with the *covariant* derivative

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu\,. \tag{5.10}$$

A_μ is also called a connection (German: "Zusammenhang"). It induces covariant transformation properties for D_μ:

$$D_\mu \quad \to \quad \mathrm{e}^{\mathrm{i}\alpha(x)}\, D_\mu\, \mathrm{e}^{-\mathrm{i}\alpha(x)} = \partial_\mu - \mathrm{i}A_\mu - \mathrm{i}\partial_\mu\alpha = \partial_\mu - \mathrm{i}A'_\mu$$

$$\Rightarrow D_\mu\Psi \quad \to \quad \mathrm{e}^{\mathrm{i}\alpha(x)}\, D_\mu\, \mathrm{e}^{-\mathrm{i}\alpha(x)}\, \mathrm{e}^{\mathrm{i}\alpha(x)}\, \Psi = \mathrm{e}^{\mathrm{i}\alpha(x)}\, D_\mu\Psi \quad \text{(transforms homog. as the field } \Psi\text{)}\,, \tag{5.11}$$

as well as $D_\mu\phi \to \mathrm{e}^{\mathrm{i}\alpha(x)}\, D_\mu\phi$. Similarly, we get that

$$\mathcal{L}_\phi = D_\mu\phi \left(D_\mu\phi\right)^* - m^2\, \phi\phi^* - V\left(\phi\phi^*\right) \tag{5.12}$$

is invariant under

$$\begin{aligned} \phi(x) \quad &\to \quad \mathrm{e}^{\mathrm{i}\alpha(x)}\, \phi(x) \\ A_\mu \quad &\to \quad A_\mu + \partial_\mu\alpha\,. \end{aligned} \tag{5.13}$$

To examine the dynamics of the gauge field A_μ, we start by constructing gauge-invariant scalar quantities from A_μ. This is easily done from D_μ, which transforms covariantly:

$$\left[D_\mu\,,\, D_\nu\right] = -\mathrm{i}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -\mathrm{i}\, F_{\mu\nu}\,, \tag{5.14}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, or $F_{\mu\nu} = \mathrm{i}\left[D_\mu\,,\, D_\nu\right]$. As shown in Eq. (5.8), A_μ transforms as a vector. Thus

$$F_{\mu\nu} \quad \to \quad \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma} F_{\rho\sigma} \tag{5.15}$$

transforms as tensor. $F_{\mu\nu}$ can be interpreted as field strength, or curvature. $F_{\mu\nu}$ is gauge invariant:

$$\begin{aligned} F_{\mu\nu} \quad &\to \quad \mathrm{i}\left[\mathrm{e}^{-\mathrm{i}\alpha}\, D_\mu\, \mathrm{e}^{\mathrm{i}\alpha}\,,\, \mathrm{e}^{-\mathrm{i}\alpha}\, D_\nu\, \mathrm{e}^{\mathrm{i}\alpha}\right] \\ &= \quad \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\left[D_\mu\,,\, D_\nu\right]\mathrm{e}^{\mathrm{i}\alpha} = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha}\left(-\mathrm{i}\, F_{\mu\nu}\right)\mathrm{e}^{\mathrm{i}\alpha} \\ &= \quad F_{\mu\nu}\,. \end{aligned} \tag{5.16}$$

In summary this means, that $F_{\mu\nu}$ is gauge invariant, but is a Lorentz tensor. Thus, $F_{\mu\nu}F^{\mu\nu}$ is gauge invariant and a Lorentz scalar. Therefore, a gauge invariant Lagrangian can be written as

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4e^2}\, F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_D\,. \tag{5.17}$$

We re-parametrise $A_\mu \to eA_\mu$, with the electric charge e and obtain

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(\mathrm{i}\,\not{D} - m\right)\Psi\,, \tag{5.18}$$

with

$$D_\mu = \partial_\mu - \mathrm{i}\, eA_\mu\,. \tag{5.19}$$

Note, that this construction also goes through for

$$\begin{aligned} \Psi \quad &\to \quad U\,\Psi, \quad \text{for} \quad U \in \mathrm{SU(N)} \\ D_\mu \quad &\to \quad U\, D_\mu\, U^{-1} \\ F_{\mu\nu}F^{\mu\nu} \quad &\to \quad \sum_{a=1}^{N^2-1}\left(F_{\mu\nu}\right)^a\left(F^{\mu\nu}\right)^a\,, \end{aligned} \tag{5.20}$$

with

$$F_{\mu\nu} = \frac{\mathrm{i}}{e}\left[D_\mu\,,\, D_\nu\right] \sim \left[A_\mu\,,\, A_\nu\right]\,. \tag{5.21}$$

We remark, that we have quantum chromodynamics for $N = 3$ and weak interaction for $N = 2$.

II. Quantisation

To quantise gauge fields, we concentrate on the pure gauge field Lagrangian

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{5.22}$$

The equation of motion is

$$\partial_\mu \frac{\partial \mathcal{L}_\mu}{\partial_\mu A_\mu} = \partial_\mu F^{\mu\nu} = \left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right) A_\sigma = 0\,, \tag{5.23}$$

with the current $\partial_\mu F^{\mu\nu} = J^\nu$. Eq. (5.23) reflects a redundancy of the gauge field A_μ, because the EOM is invariant under

$$\begin{aligned} A_\mu &\to A_\mu + e\partial_\mu \alpha \\ \partial_\mu F^{\mu\nu} + e\left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right)\partial_\sigma \alpha &= \partial_\mu F^{\mu\nu} + 0\,, \end{aligned} \tag{5.24}$$

since

$$\begin{aligned} \partial_\mu \partial^\mu \eta^{\nu\sigma}\partial_\sigma \alpha - \partial^\nu \partial^\sigma \partial_\sigma \alpha &= \partial^2 \partial^\nu \alpha - \partial^2 \partial^\nu \alpha = 0 \\ \left(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma\right)\partial_\sigma &\,\hat{=}\, 0\,. \end{aligned} \tag{5.25}$$

In momentum space this writes

$$\left(p^2 \eta^{\nu\sigma} - p^\nu p^\sigma\right) p_\sigma \,\hat{=}\, 0\,, \tag{5.26}$$

where the term in the brackets is the transverse part, which we will discuss later in the section. Eq. (5.23) and Eq. (5.24) already entail, that A^μ cannot have canonical commutation relations! But what about the canonical momentum Π^μ:

$$\begin{aligned} \Pi^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -\frac{1}{4}\frac{\partial}{\partial(\partial_0 A_\mu)}\left(F_{\rho\sigma}F_{\gamma\delta}\eta^{\sigma\delta}\eta^{\rho\gamma}\right) \\ &= -\frac{1}{2} F_{\rho\sigma}\eta^{\sigma\delta}\eta^{\rho\gamma}\frac{\partial F_{\gamma\delta}}{\partial(\partial_0 A_\mu)} \\ &= F^{\mu 0}\,. \end{aligned} \tag{5.27}$$

In particular, it is

$$\Pi^0 \,\hat{=}\, 0\,, \tag{5.28}$$

which also reflects the redundancy. We remove the redundancy by **fixing the gauge**, e.g. with a Lorentz- or covariant gauge

$$\partial_\mu A^\mu = 0\,. \tag{5.29}$$

For these A^μ we can write

$$\begin{aligned} \mathcal{L}_A &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\epsilon}\left(\partial_\mu A^\mu\right)^2 \\ &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,, \end{aligned} \tag{5.30}$$

or

$$S[A] = \frac{1}{2}\int \mathrm{d}^4x\, A_\mu \left(\partial_\rho\partial^\rho\eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)A_\nu\,. \tag{5.31}$$

We split the gauge field in transverse and longitudinal parts

$$A_\mu = (A_\perp)_\mu + (A_L)_\mu\;, \tag{5.32}$$

with

$$\begin{aligned}\partial_\mu\,(A_\perp)_\mu &= 0\\ \left(\partial_\mu\partial^\mu\eta^{\nu\sigma} - \partial^\nu\partial^\sigma\right)(A_L)_\nu &= 0\,.\end{aligned} \tag{5.33}$$

It follows, that

$$\begin{aligned}\left(\partial_\mu\partial^\mu\eta^{\nu\sigma} - \partial^\nu\partial^\sigma\right)(A_\perp)_\nu &= \frac{1}{\xi}\partial^\sigma\partial_\nu A_L^\nu\\ &= 0\,,\end{aligned} \tag{5.34}$$

because the left-hand side is solely transverse, and the right-hand side solely longitudinal. The EOM is given by

$$\partial_\mu F^{\mu\nu} = -\frac{1}{\xi}\partial^\nu\left(\partial_\mu A^\mu\right) = 0\,. \tag{5.35}$$

Note, that $\left(\partial_\rho\partial^\rho\eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\nu\partial^\sigma\right)$ is invertible, and specifically simple for $\xi = 1$ (Feynman gauge $\partial_\mu\partial^\mu\eta^{\nu\sigma}$). With Eq. (5.29) we obtain the

EOM for the Lorentz gauge

$$\partial_\rho\partial^\rho A^\nu = 0\,, \tag{5.36}$$

which is similar to the Klein-Gordon equation (Eq. (2.11)). Eq. (5.36) suggests a quantised field

$$A_\mu(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\left(\mathrm{e}^{-\mathrm{i}kx}\,a_\mu(\mathbf{k}) + \mathrm{e}^{\mathrm{i}kx}\,a_\mu^\dagger(\mathbf{k})\right)\,, \tag{5.37}$$

with the commutation relations

$$\begin{aligned}\left[a_\mu(\mathbf{k})\,,\,a_\nu^\dagger(\mathbf{k}')\right] &= -\eta_{\mu\nu}\,(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\\ \left[a_\mu(\mathbf{k})\,,\,a_\nu(\mathbf{k}')\right] &= 0 = \left[a_\mu^\dagger(\mathbf{k})\,,\,a_\nu^\dagger(\mathbf{k}')\right]\,.\end{aligned} \tag{5.38}$$

Note, that the η in the first equation is necessary for Lorentz-symmetry. However, Eq. (5.37) and Eq. (5.38) are not compatible with Eq. (5.29), as

$$\begin{aligned}\partial_\mu A^\mu &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{-\mathrm{i}}{\sqrt{2k^0}}\left(\mathrm{e}^{-\mathrm{i}kx}\,k_\mu\,a^\mu(\mathbf{k}) + \mathrm{e}^{\mathrm{i}kx}\,k_\mu\left(a^\dagger\right)^\mu(\mathbf{k})\right)\\ &\overset{!}{=} 0\,.\end{aligned} \tag{5.39}$$

This entails, that $k_\mu a^\mu(\mathbf{k}) \stackrel{!}{=} 0$, because if Eq. (5.39) fails, the EOM is *not* satisfied

$$\partial_\mu F^{\mu\nu} = -\partial^\nu \partial_\mu A^\nu \,. \tag{5.40}$$

However,

$$\begin{aligned} k^\mu \left[a_\mu(\mathbf{k})\,,\, a^\dagger_\nu(\mathbf{k}')\right] &= -k^\nu\,(2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') \\ &\neq 0\,. \end{aligned} \tag{5.41}$$

Indeed one can show, that it is *not* possible to quantise the gauge field A_μ with canonical commutation relations *and* $\partial_\mu A^\mu = 0$, or other gauge conditions; If using A^μ in Eq. (5.37) and Eq. (5.38), the gauge $\partial_\mu A^\mu$ has to be implemented on the states! We will target this problem later in this section, but prior to this, we construct the **Fock space** $\mathcal{F}$ based on Eq. (5.37) and Eq. (5.38). We define the vacuum state $|0\rangle$ with

$$\langle 0|0\rangle = 1\,. \tag{5.42}$$

One-particle states are given by

$$\sqrt{2k^0}\, a^\dagger_\mu(\mathbf{k})\, |0\rangle \ , \tag{5.43}$$

with norm

$$\sqrt{2k^0 2\,(k')^0}\ \langle 0|\, a_\nu(\mathbf{k}')\, a^\dagger_\mu(\mathbf{k})\, |0\rangle = -\eta_{\mu\nu}\,(2\pi)^3\, 2k^0\, \delta(\mathbf{k}-\mathbf{k}')\,. \tag{5.44}$$

Thus, we have positive norm states for $\mu = \nu = i$, and negative norm states for $\mu = \nu = 0$. Consequently, $\mathcal{F}$ is not the physical Hilbert space $\mathcal{H}$, as it does not allow for probability interpretation. We remark, that $\eta_{\mu\nu} \to \eta^{\mu\nu}$ does not solve the problem of negative norm states (leave aside the wrong commutators $\left[A^i\,,\,\Pi^i\right]$). But separating the positive norm subspace of $\mathcal{F}$, will solve all problems of quantisation. This is the **Gupta-Bleuler quantisation**. We demand, that the EOM is satisfied on

physical states

$$\left\langle \text{physical states}'\right|\partial_\mu F^{\mu\nu}\left|\text{physical states}\right\rangle \ \stackrel{!}{=}\ 0\,, \tag{5.45}$$

that is, its matrix elements vanish. Eq. (5.45) is satisfied for

$$k^\mu a_\mu(\mathbf{k})\left|\text{physical states}\right\rangle = 0\,, \tag{5.46}$$

which is trivially satisfied on the vacuum. The above suggests to rewrite A_μ in Eq. (5.37) as

field operator (Lorentz gauge)

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \sum_{\lambda=0}^{3} \left(\alpha_\lambda(\mathbf{k})\, \epsilon^\lambda_\mu(k)\, e^{-ikx} + \alpha^\dagger_\lambda(\mathbf{k}) \left(\epsilon^\lambda_\mu\right)^* (k)\, e^{ikx} \right), \tag{5.47}$$

where the ϵ^λ_μ introduce unitary rotations[1] from a_μ to a_λ with

$$\begin{aligned} \epsilon^\lambda_\mu(k)\, \epsilon^{\lambda'\mu^*}(k) &= \eta^{\lambda\lambda'} \\ \epsilon^\lambda_\mu(k)\, \epsilon^*_{\lambda\nu}(k) &= \eta_{\mu\nu}\,. \end{aligned} \tag{5.48}$$

Hence, we write the "new" operators α as linear combination of the "old" operators a:

$$\alpha_\lambda(\mathbf{k}) = a_\mu(\mathbf{k}) \epsilon_\lambda^{\ \mu}(k)\,. \tag{5.49}$$

Now, we choose our coordinate system without loss of generality, such that $k\cdot\epsilon^0 = k^0 = k\cdot\epsilon^3$ and $k\cdot\epsilon^i = 0$, for $i = 1, 2$. The ϵ's are also called *polarisation vectors*. Eq. (5.46) now reads with $\alpha_\pm = \frac{1}{\sqrt{2}}(\alpha_0 \pm \alpha_3)$

$$\alpha_+ \left|\text{physical states}\right\rangle = 0\,, \tag{5.50}$$

with $\alpha_0 + \alpha_3 \simeq k^\mu a_\mu$. In the frame with $(k^\mu) = \left(k^0, 0, 0, k^0\right)$ we have

$$\left(\epsilon^\lambda\right)_\mu = \delta^\lambda_{\ \mu}\,. \tag{5.51}$$

The α's have the same commutation relations as the a's, as we have used unitary rotations (see also Eq. (5.48) and Eq. (5.49)). It follows, with $i = 1, 2$:

$$\begin{aligned} \left[\alpha_i(\mathbf{k})\,,\, \alpha_i^\dagger(\mathbf{k}')\right] &= (2\pi)^3\, \delta(\mathbf{k}-\mathbf{k}') \\ \left[\alpha_+(\mathbf{k})\,,\, \alpha_-^\dagger(\mathbf{k}')\right] &= -(2\pi)^3\, \delta(\mathbf{k}-\mathbf{k}') \\ \left[\alpha_\pm(\mathbf{k})\,,\, \alpha_\pm^{(\dagger)}(\mathbf{k}')\right] &= 0 = \left[\alpha_\pm(\mathbf{k})\,,\, \alpha_i^{(\dagger)}(\mathbf{k}')\right]. \end{aligned} \tag{5.52}$$

Let us now examine the **physical Hilbert space** $\mathcal{H}$. It is the physical subspace $\mathcal{F}_{\text{phys}} \subset \mathcal{F}$ with

$$|\Psi\rangle \in \mathcal{F}_{\text{phys}} \Rightarrow \alpha_+ |\Psi\rangle = 0\,. \tag{5.53}$$

It follows

$$|\Psi\rangle \in \mathcal{F}_{\text{phys}} \Rightarrow \alpha_i^\dagger |\Psi\rangle \in \mathcal{F}_{\text{phys}}, \quad \text{for} \quad i = 1, 2\,, \tag{5.54}$$

with

$$\alpha_+ \alpha_i^\dagger |\Psi\rangle = \alpha_i^\dagger \alpha_+ |\Psi\rangle = 0\,. \tag{5.55}$$

[1]unitary rotations keep the canonical commutation relations

Also

$$a_+^\dagger \left|\Psi\right\rangle \in \mathcal{F}_{\text{phys}}\,, \tag{5.56}$$

with

$$\alpha_+\,\alpha_+^\dagger \left|\Psi\right\rangle = \alpha_+^\dagger\,\alpha_+ \left|\Psi\right\rangle = 0. \tag{5.57}$$

This indicates, that everything, that commutes with α_+, is in the physical subspace. Therefore

$$\alpha_-^\dagger \left|\Psi\right\rangle \notin \mathcal{F}_{\text{phys}}\,, \tag{5.58}$$

since

$$\begin{aligned} \alpha_+\,\alpha_-^\dagger \left|\Psi\right\rangle &= \alpha_-^\dagger\,\alpha_+ \left|\Psi\right\rangle + \left[\alpha_+\,,\,\alpha_-^\dagger\right]\left|\Psi\right\rangle \\ &= \left[\alpha_+\,,\,\alpha_-^\dagger\right]\left|\Psi\right\rangle \sim \left|\Psi\right\rangle \neq 0\,. \end{aligned} \tag{5.59}$$

We conclude, that

$$\boxed{\mathcal{F}_{\text{phys}} = \text{span}\left[\left(a_+^\dagger\right)^{n_+}\,\left(a_1^\dagger\right)^{n_1}\,\left(a_2^\dagger\right)^{n_2}\left|0\right\rangle\right]\,.} \tag{5.60}$$

$\mathcal{F}_{\text{phys}}$ contains only states with semi-positive norm

$$\left\langle\Psi|\Psi\right\rangle \geq 0\,. \tag{5.61}$$

Indeed, it is

$$\begin{aligned} \|\,\alpha_+^\dagger \left|\Psi\right\rangle\|^2 &= \left\langle\Psi\right|\alpha_+\,\alpha_+^\dagger\left|\Psi\right\rangle \\ &= \left\langle\Psi\right|\alpha_+^\dagger\,\alpha_+\left|\Psi\right\rangle = 0\,, \end{aligned} \tag{5.62}$$

and

$$\|\left(\alpha_1^\dagger\right)^{n_1}\,\left(\alpha_2^\dagger\right)^{n_2}\left|0\right\rangle\| > 0\,, \tag{5.63}$$

with $\left[\alpha_i^\dagger\,,\,\alpha_i\right] = +(2\pi)^3\,2k^0\delta$. If we identify two states $\left|\Psi_1\right\rangle, \left|\Psi_2\right\rangle$ with $\|\left|\Psi_1\right\rangle - \left|\Psi_2\right\rangle\| = 0$, every matrix element of an operator $O\left(\alpha_i^{(\dagger)}, \alpha_+^{(\dagger)}\right)$ vanishes, and $\left\langle\Psi\right|O\left(\left|\Psi_1\right\rangle - \left|\Psi_2\right\rangle\right)$ vanishes. This means, that we define the physical Hilbert space as the space of equivalence classes

$$\mathcal{H} = \mathcal{F}_{\text{phys}}/\sim\,, \tag{5.64}$$

with $\left|\Psi_1\right\rangle \sim \left|\Psi_2\right\rangle$ for $\|\left|\Psi_1\right\rangle - \left|\Psi_2\right\rangle\| = 0$. For $\left|\Psi\right\rangle \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle\Psi|\Psi\right\rangle > 0, \quad &\text{for} \quad \left|\Psi\right\rangle \neq 0 \\ &\alpha_+\left|\Psi\right\rangle = 0\,, \end{aligned} \tag{5.65}$$

and hence the EOMs are satisfied, since

$$\left\langle\Psi'\right|\partial_\mu F^{\mu\nu}\left|\Psi\right\rangle = \left\langle\Psi'\right|\partial_\nu\partial_\mu A^\mu\left|\Psi\right\rangle = 0\,. \tag{5.66}$$

We can now introduce the **Feynman rules** for gauge fields. For the propagator we find, for $x^0 > y^0$

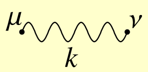
$$
\begin{aligned}
\langle 0|\, A_\mu(x)\, A_\nu(y)\, |0\rangle &= \langle 0| \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\,(k')^0}}\, e^{-ikx+ik'y} \left[a_\mu(\mathbf{k})\,,\, a_\nu^\dagger(\mathbf{k}')\right] |0\rangle \\
&= \langle 0| \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\,(k')^0}}\, e^{-ikx+ik'y}\, (2\pi)^3\, \delta(\mathbf{k}-\mathbf{k}')\, |0\rangle \\
&= -\eta_{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0}\, e^{-ik(x-y)}\,, \qquad (5.67)
\end{aligned}
$$

where similar to Eq. (4.133) the last integral corresponds to the scalar propagator $\mathcal{D}_F(x-y)$ (Eq. (3.86)). With analogous arguments we find the

Feynman-propagator (for gauge fields)

$$\langle 0|\, T\, A_\mu(x)\, A_\nu(y)\, |0\rangle = -\eta_{\mu\nu}\, \mathcal{D}_F(x-y)\,, \qquad (5.68)$$

or



$$= -\frac{i\,\eta_{\mu\nu}}{k^2 + i\epsilon}\,.$$

Initial and final states are given by

$$|\mathbf{k}, \epsilon\rangle = \sqrt{2k^0}\, \alpha^\dagger(\mathbf{k})\, |0\rangle\ . \qquad (5.69)$$

Note, that $\alpha^\dagger = \epsilon_\mu^* \left(a^\dagger\right)^\mu$ (Eq. (5.49)). Hence, we have

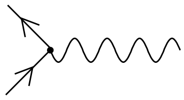
$$
\begin{aligned}
A_\mu\Big|_{\text{annihil.}} |\mathbf{k}, \epsilon\rangle &= \int \frac{d^3k'}{(2\pi)^3} \sqrt{\frac{2k^0}{2\,(k')^0}}\, e^{ik'x}\, a_\mu(\mathbf{k}')\, \alpha^\dagger(\mathbf{k}')\, |0\rangle \\
\text{(drop phase)} \quad \rightarrow \quad &\simeq \epsilon_\mu^*(k)\,. \qquad (5.70)
\end{aligned}
$$

That is

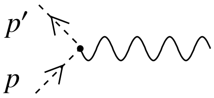
$$
\begin{aligned}
A\, |\mathbf{k}, \epsilon\rangle &:= \epsilon^* \\
\langle \mathbf{k}, \epsilon|\, A &= \epsilon\,. \qquad (5.71)
\end{aligned}
$$

At the vertices we have:

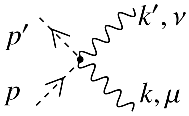
a) $\mathcal{L}_I = e\,\bar{\Psi}\,\not{A}\,\Psi$:


$= -\mathrm{i}\,e\,\gamma^{\mu}$

b) $\mathcal{L}_I = D_\mu\phi\;(D^\mu\phi)^* - \partial_\mu\phi\,\partial^\mu\phi^*$:



$= -\mathrm{i}\,e\,\left(p_\mu + p'_\mu\right)$,



$= 2\mathrm{i}\,e^2\,\eta^{\mu\nu}$.

(5.72)

Let us next discuss **gauge independence and Feynman rules**. We can add a longitudinal part to the field A_μ, without changing the physics:

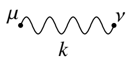
$$A_\mu \quad\rightarrow\quad A_\mu + \alpha\partial_\mu \frac{1}{\partial^\rho\partial^\rho}\partial^\nu A_\nu\,, \tag{5.73}$$

or in the Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2\ . \tag{5.74}$$

Then the propagator is

$\langle 0|\,T\ A_\mu A_\nu\,|0\rangle\,(k)$

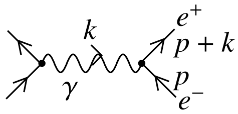


$$\begin{aligned}
&= \;\; = -\mathrm{i}\left(\frac{\eta_{\mu\nu}}{k^2+\mathrm{i}\epsilon} - (1-\xi)\frac{k_\mu\,k_\nu}{(k^2+\mathrm{i}\epsilon)^2}\right)\\
&= -\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu\,k_\nu}{k^2+\mathrm{i}\epsilon}\right).
\end{aligned} \tag{5.75}$$

The equation of motion in Fourier space with the Lagrangian in Eq. (5.74) is

$$\left(k^2\eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)A_\nu(k) = 0\,. \tag{5.76}$$

Note, that for $\xi = 1$ we have the Feynman gauge, $\xi = 0$ the Landau gauge and for $\xi = \infty$ the unitary gauge. When considering the scattering amplitudes, ξ drops out. E.g.:



$\simeq \langle 0|\,T\ A_\mu A_\nu\,|0\rangle\,(k)\,\bar{v}(p+k)\,\gamma^\mu\,u(p)\,.$

This only holds on-shell, i.e. internal Feynman diagrams are ξ dependent. We have used, that:

$$\begin{aligned}
\xi\,k_\mu\,k_\nu\,\bar{v}(p+k)\,\gamma^\mu\,u(p) \quad &= \quad \xi\,k_\nu\,\bar{v}(p+k)\,\not{k}\,u(p)\\
\text{Eq. (4.87)} \quad \rightarrow \quad &= \quad \xi\,k_\nu\,\bar{v}(p+k)\,(\not{k}+\not{p}-\not{p})\,u(p)\\
\text{Eq. (4.87)} \quad \rightarrow \quad &= \quad \xi\,k_\nu\,\bar{v}(p+k)\,(\not{k}+\not{p}-m)\,u(p)\,.
\end{aligned} \tag{5.77}$$

We now look at **gauge invariant observables**, for example E and B-fields. We have

$$\begin{aligned} E^i &= -F^{0i} = -\left(\partial^0 A^i - \partial^i A^0\right) \\ B^i &= \epsilon^{ijk} F_{jk} \, . \end{aligned} \tag{5.78}$$

Using Eq. (5.47) they read

$$\begin{aligned} \mathbf{E} &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2k^0} \, \mathrm{i}k^0 \Big[\left(\mathbf{a} - \frac{\mathbf{k}}{k^0} \, a_0\right) \mathrm{e}^{-\mathrm{i}kx} - \left(\mathbf{a}^\dagger - \frac{\mathbf{k}}{k^0} \, a_0^\dagger\right) \mathrm{e}^{\mathrm{i}kx} \Big] \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2k^0} \, \mathrm{i}k^0 \Big[(\epsilon_1 \, \alpha_1 + \epsilon_2 \, \alpha_2) \, \mathrm{e}^{-\mathrm{i}kx} - \left(\epsilon_1 \, \alpha_1^\dagger + \epsilon_2 \, \alpha_2^\dagger\right) \mathrm{e}^{\mathrm{i}kx} \Big] \\ &\qquad - \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2k^0} \, \mathrm{i}k^0 \Big[\frac{\mathbf{k}}{k^0} \, \alpha_+ \, \mathrm{e}^{-\mathrm{i}kx} - \frac{\mathbf{k}}{k^0} \, \alpha_+^\dagger \, \mathrm{e}^{\mathrm{i}kx} \Big] , \end{aligned} \tag{5.79}$$

with the physical polarisations $\epsilon_{1,2}$.
Analogously, we find

$$B^i(x) = \epsilon^{ijl} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \, \mathrm{i}k^j \Big[\left(\epsilon_1^l \, \alpha_1 + \epsilon_2^l \, \alpha_2\right) \, \mathrm{e}^{-\mathrm{i}kx} - \left(\epsilon_1^l \, \alpha_1^\dagger + \epsilon_2^l \, \alpha_2^\dagger\right) \, \mathrm{e}^{\mathrm{i}kx} \Big] . \tag{5.80}$$

(We note that l is summed over despite the fact that it appears three times in the expression.)
It follows that only $\alpha_{1,2}$ and α_+ appear in $\mathbf{E}$ and $\mathbf{B}$. Sandwiched between physical states $|\Psi\rangle \in \mathcal{H}$, α_+ drops out. The Hamiltonian reads, with $\Pi^i = E^i$

$$\begin{aligned} H &= \int \mathrm{d}^3x \Big[\mathbf{\Pi} \, (\partial_0 \mathbf{A} - \nabla A_0) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Big] \\ &= \int \mathrm{d}^3x \Big[\frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2\right) + \nabla \, (\mathbf{E} \, A_0) \Big] \\ \nabla \mathbf{E} = 0 \quad \to \quad &= \frac{1}{2} \int \mathrm{d}^3x \, \left(\mathbf{E}^2 + \mathbf{B}^2\right) , \end{aligned} \tag{5.81}$$

where we have used

$$\begin{aligned} \nabla \mathbf{E} &= \left(-\partial^0 \partial^i A^i + \left(\partial^i\right)^2 A^0\right) \\ \partial_\mu A^\mu = 0 \quad \to \quad &= \Big[-\left(\partial^0\right)^2 + \left(\partial^i\right)^2 \Big] A^0 = 0 \, . \end{aligned} \tag{5.82}$$

We insert the E-B-field operators Eq. (5.79) and Eq. (5.80) and arrive at

$$\begin{aligned} P^0 &= H \simeq \frac{1}{2} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k_0}{2k^0} \, k^0 \sum_{i=1}^{2} \left(\alpha_i \, \alpha_i^\dagger + \alpha_i^\dagger \, \alpha_i\right) \\ &\simeq \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k_0}{2k^0} \, k^0 \sum_{i=1}^{2} \alpha_i^\dagger(\mathbf{k}) \, \alpha_i(\mathbf{k}) \, , \end{aligned} \tag{5.83}$$

where we have dropped the α_+-terms in the first line, and the vacuum terms in the second line. Similarly, we get for **P**

$$\begin{aligned}\mathbf{P} &= \int \mathrm{d}^3x\,\mathbf{E}\times\mathbf{B} \\ &\simeq \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathbf{k}\sum_{i=1}^{2}\alpha_i^\dagger(\mathbf{k})\,\alpha_i(\mathbf{k})\,,\end{aligned} \tag{5.84}$$

where we have dropped the vacuum terms.

6. QED

In this chapter we discuss Quantum Electro Dynamics (QED) as an application. We use the following notation:

Dirac fields	electrons, positrons:	e^-, e^+	Ψ_e
Leptons	myons:	μ^-, μ^+	Ψ_μ
	tau:	τ^-, τ^+	Ψ_τ
Gauge field	photons:	γ	A_μ

Note, that the photon is the gauge boson of the U(1)-symmetry, with the Noether charge being the electric charge (see chapter 5).

I. Action and Feynman rules

The action is a sum of the Dirac actions of e, μ, τ and the gauge field action of the photon (see Eq. (5.18))

$$S_{\text{QED}}[A, \Psi_e, \Psi_\mu, \Psi_\tau] = S_D[A, \Psi_e] + S_D[A, \Psi_\mu] + S_D[A, \Psi_\tau] + S_A[A] + S_{gf}[A]\,, \tag{6.1}$$

with the Dirac actions

$$S_D[A, \Psi_{e,\mu,\tau}] = \int d^4x\, \bar{\Psi}_{e,\mu,\tau} \left(\mathrm{i}\, \not{D} - m_{e,\mu,\tau}\right) \Psi_{e,\mu,\tau}, \quad \text{with} \quad D_\mu = \partial_\mu - \mathrm{i}eA_\mu\,, \tag{6.2}$$

and the gauge field action

$$S_A[A] = -\frac{1}{4} \int d^4x\, F_{\mu\nu}F^{\mu\nu}, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{6.3}$$

The gauge fixing term $S_{gf}[A]$ in the covariant gauge is

$$S_{gf}[A] = -\frac{1}{2\xi} \int d^4x \left(\partial_\mu A^\mu\right)^2\,, \tag{6.4}$$

with the gauge fixing parameter ξ. The gauge transformations are

$$\begin{aligned} \Psi(x) \quad &\rightarrow \quad e^{\mathrm{i}\alpha(x)}\, \Psi(x) = \Psi^\alpha(x) \\ A_\mu(x) \quad &\rightarrow \quad A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) = A^\alpha_\mu(x)\,, \end{aligned} \tag{6.5}$$

with

$$S_{\text{QED}}[A^\alpha, \Psi^\alpha] = S_{\text{QED}}[A, \Psi] + \frac{1}{\xi}\frac{1}{e} \int d^4x\, \partial_\mu A^\mu \partial_\rho \partial^\rho \alpha\,, \tag{6.6}$$

where

$$\Psi = \begin{pmatrix} \Psi_e \\ \Psi_\mu \\ \Psi_\tau \end{pmatrix}. \tag{6.7}$$

Next, we consider the Feynman rules. It is

$$S_{\mathrm{QED}} = S_{\mathrm{free}} + S_{\mathrm{I}}\,, \tag{6.8}$$

with

$$S_{\mathrm{free}} = S_A[A] + S_{gf}[A] + \int \mathrm{d}^4x\,\bar{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi\,, \tag{6.9}$$

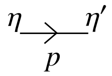
and

$$S_{\mathrm{I}} = e\int \mathrm{d}^4x\,\bar{\Psi}A\!\!\!/\Psi\,. \tag{6.10}$$

We remark, that any other coupling of leptons and photon introduces dimension-full couplings to the theory, e.g. spin-coupling

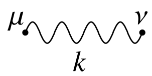
$$\frac{e}{\Lambda}\bar{\Psi}\,\sigma^{\mu\nu}\,\Psi\,F_{\mu\nu}\,\Psi\,, \tag{6.11}$$

where Λ carries momentum dimension one. Such a term makes the theory non-renormalisable. We obtain the propagators for
Leptons (Eq. (4.142)):



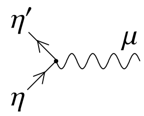
$$= \mathrm{i}\left(\frac{p\!\!\!/ + m_\Psi}{p^2 - m_\Psi^2 + \mathrm{i}\epsilon}\right)_{\eta\eta'}$$

and photon (Eq. (5.75)):



$$= -\frac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu\,k_\nu}{k^2 + \mathrm{i}\epsilon}\right).$$

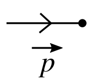
At the vertices it holds (see Eq. (5.72)



$$= -\mathrm{i}\,e\,(\gamma_\mu)_{\eta\eta'}\,. \tag{6.12}$$

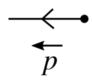
Note, that here the sign is irrelevant, as $A_\mu \to -A_\mu$. Eq. (6.12) has been deduced simply by analogy to the derivation of the scalar self-interaction. Further, we have

incoming lepton:



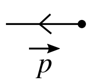
$$= u(p)$$

outgoing lepton:



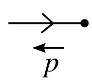
$$= \bar{u}(p)$$

incoming anti-lepton:



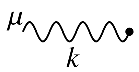
$$= \bar{v}(p)$$

outgoing anti-lepton:



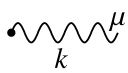
$$= v(p)$$

incoming photon:



$$= \epsilon_\mu(k)$$

outgoing photon:



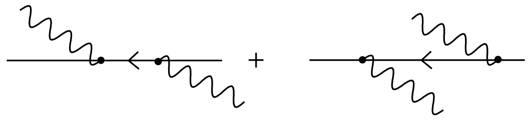
$$= \epsilon^*_\mu(k)\,. \tag{6.13}$$

See also Eq. (4.143) ff. and recall the minus sign for fermion loops.

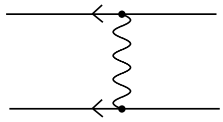
II. Elementary Processes

This section deals with elementary processes. Consider

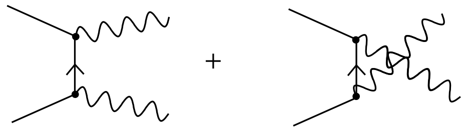
i) Compton scattering: $e^- \gamma \to e^- \gamma$



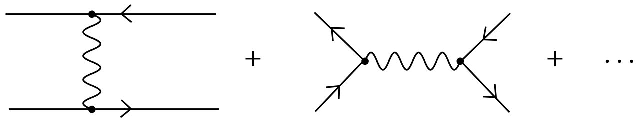
ii) Elastic $e^- e^-$-scattering:



iii) Pain-annihilation/creation: $e^+ e^- \to \gamma \gamma$



iv) Bhaba-scattering: $e^+ e^- \to e^+ e^-$



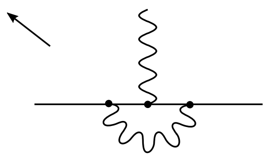
tree level processes

v) light-by-light scattering: (non-linear electrodynamics)

effective four photon vertex

vi) Landé factor (gyromagnetic ratio):

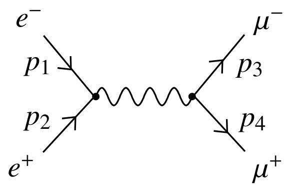
$$i\not{D} - m_e \to i\not{D} - m_e + \frac{\Delta g}{2} \frac{e}{4m_e} \sigma_{\mu\nu} F^{\mu\nu}, \qquad \Delta g = \frac{\alpha}{\pi}, \; \alpha = \frac{e^2}{4\pi}$$



loop effects

Let us compute an example of a tree level process in detail.

Example 6-16: electron-positron annihilation into muon-antimuon pair ($e^+ e^- \to \mu^+ \mu^-$). We choose this example, because there exists only a single Feynman diagram for this process, namely:



As we look at the highly relativistic case of 2-2 scattering we can use Eq. (3.154):

$$\mathrm{d}\sigma = \frac{1}{2s}\left|M_{fi}\right|^2 \int \mathrm{d}\Pi_2\,, \tag{6.14}$$

with

$$\int \mathrm{d}\Pi_2 = \frac{1}{2}\frac{s/4}{(2\pi)^2\,4p_3^0\,p_4^0}\,\mathrm{d}\Omega\,, \quad s = (p_1+p_2)^2\,. \tag{6.15}$$

We find

$$|M|^2 = \frac{1}{2}\sum_r \frac{1}{2}\sum_{r'}\sum_{s,s'} |M(r,r',s,s')|^2\,, \tag{6.16}$$

where we computed the average by summing over r, r' and summed over all possible splits $s, s,'$. The scattering amplitude is read off by the Feynman rules:

$$\mathrm{i}M = \bar{u}_{\mu_s}(p_3)\left(\mathrm{i}\,e\,\gamma_\rho\right)v_{\mu_{s'}}(p_4)\left[\frac{\eta^{\rho\sigma}}{s}\right]\bar{v}_{e_{r'}}(p_2)\,(\mathrm{i}\,e\,\gamma_\sigma)\,u_{e_r}(p_1)\,, \tag{6.17}$$

where the term in square brackets corresponds to the (on-shell) propagator. Therefore, the gauge fixing parameter ξ drops out (see Eq. (5.77)). The terms in front of and behind the propagator correspond to right- and left-hand side in the diagram, respectively. It follows that

$$|M|^2 = \frac{e^4}{4s^2}(T_\mu)_{\alpha\beta}\,(T_e)^{\alpha\beta}\,, \tag{6.18}$$

with

$$\begin{aligned}
(T_\mu)_{\alpha\beta} &= \sum_{s,s'}\bar{u}_{\mu_s}(p_3)\,(\mathrm{i}\,e\,\gamma_\alpha)\,v_{\mu_{s'}}(p_4)\cdot\left[\bar{u}_{\mu_s}(p_3)\left(\mathrm{i}\,e\,\gamma_\beta\right)v_{\mu_{s'}}(p_4)\right]^*\\
(T_e)^{\alpha\beta} &= \sum_{r,r'}\bar{v}_{e_{r'}}(p_2)\,(\mathrm{i}\,e\,\gamma^\alpha)\,u_{e_r}(p_1)\cdot\left[\bar{v}_{e_{r'}}(p_2)\left(\mathrm{i}\,e\,\gamma^\beta\right)u_{e_r}(p_1)\right]^*\,.
\end{aligned} \tag{6.19}$$

We use Eq. (4.100), i.e.

$$\begin{aligned}
\sum_s u_{\mu_s}(p_3)\,\bar{u}_{\mu_s}(p_3) &= (\not{p}_3 + m_\mu)\\
\sum_s v_{\mu_{s'}}(p_4)\,\bar{v}_{\mu_{s'}}(p_4) &= (\not{p}_4 - m_\mu)\,,
\end{aligned} \tag{6.20}$$

to compute

$$\sum_{s.s'}\bar{u}_{\mu_s}(p_3)\,\gamma_\alpha\left[v_{\mu_{s'}}(p_4)\,\bar{v}_{\mu_{s'}}(p_4)\right]\gamma_\beta^*\,u_{\mu_s}(p_3) = \mathrm{tr}\left(\not{p}_3 + m_\mu\right)\gamma_\alpha\left(\not{p}_4 - m_\mu\right)\gamma_\beta\,, \tag{6.21}$$

with

$$\begin{aligned}
\left[\bar{u}_s(p)\gamma_\alpha v_{s'}(q)\right]^* &= v_{s'}^\dagger(q)\,\gamma^0\gamma^0\,\gamma_\alpha^\dagger\,\gamma^0\gamma^0\,\bar{u}_s^\dagger(p)\,, \quad \gamma^0\gamma^0 = \mathbb{1}\\
&= \bar{v}_{s'}(q)\,\gamma_\alpha\,u_s(p)\,.
\end{aligned} \tag{6.22}$$

As we consider the highly relativistic limit, we drop m_μ, m_e. Then

$$\begin{aligned} (T_\mu)_{\alpha\beta} &= \operatorname{tr}\left(\not p_3 + m_\mu\right)\gamma_\alpha\left(\not p_4 - m_\mu\right)\gamma_\beta \\ \operatorname{tr}\gamma^{2n+1} = 0 \quad \to \quad &= \operatorname{tr}\not p_3\gamma_\alpha\not p_4\gamma_\beta + \operatorname{tr}\gamma_\alpha\gamma_\beta\, m_\mu^2 \,. \end{aligned} \tag{6.23}$$

We use

$$\begin{aligned} \operatorname{tr}\gamma^\rho\gamma^\sigma &= \frac{1}{2}\operatorname{tr}\{\gamma^\rho\,,\,\gamma^\sigma\} = \frac{1}{2}\operatorname{tr}(2\eta^{\rho\sigma}) = 2\eta^{\rho\sigma} \\ \operatorname{tr}\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta &= 2\eta^{\rho\sigma}\operatorname{tr}\gamma^\alpha\gamma^\beta - \operatorname{tr}\gamma^\sigma\gamma^\rho\gamma^\alpha\gamma^\beta = 8\eta^{\rho\sigma}\eta^{\alpha\beta} - \operatorname{tr}\gamma^\sigma\gamma^\rho\gamma^\alpha\gamma^\beta \\ &= \cdots = 4\left(\eta^{\rho\sigma}\eta^{\alpha\beta} - \eta^{\rho\alpha}\eta^{\beta\sigma} + \eta^{\rho\beta}\eta^{\alpha\sigma}\right)\,, \end{aligned} \tag{6.24}$$

and obtain

$$\begin{aligned} (T_\mu)_{\alpha\beta} &= 4\left(p_{3_\alpha}p_{4_\beta} + p_{3_\beta}p_{4_\alpha} - \eta_{\alpha\beta}p_3p_4\right) - 4\eta_{\alpha\beta}m_\mu^2 \\ s \gg m_\mu^2 \quad \to \quad &\simeq 4\left(p_{3_\alpha}p_{4_\beta} + p_{3_\beta}p_{4_\alpha} - \eta_{\alpha\beta}(p_3p_4)\right)\,. \end{aligned} \tag{6.25}$$

Similarly, we get

$$(T_e)^{\alpha\beta} \simeq 4\left(p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha - \eta^{\alpha\beta}(p_1p_2)\right)\,, \tag{6.26}$$

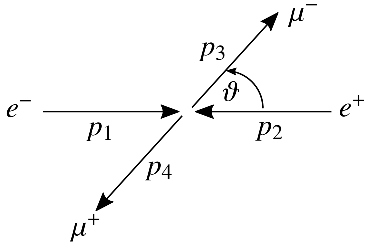
and arrive at

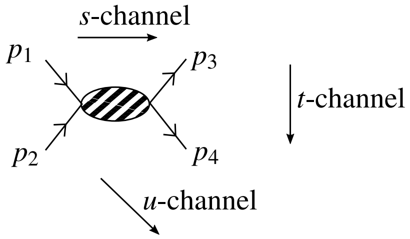
$$|M|^2 = \frac{e^4}{4s^2}\cdot 2\cdot 16\Big[(p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_3)(p_2\cdot p_4)\Big] = \frac{8e^4}{s^2}\cdot\Big[(p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_3)(p_2\cdot p_4)\Big]\,. \tag{6.27}$$

In summary, and after inserting Eq. (6.27) in Eq. (6.14), we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\alpha^2}{p_3^0p_4^0 s}\Big[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4)\Big]\,, \quad \alpha = \frac{e^2}{4\pi}\,. \tag{6.28}$$

Note, that this expression depends on the scattering angle ϑ. Furthermore, $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ are called *Mandelstam variables*.





The scattering angle is given by

$$\cos\vartheta = \frac{\mathbf{p}_1\mathbf{p}_3}{|\mathbf{p}_1||\mathbf{p}_3|}\,. \tag{6.29}$$

In the highly relativistic limit, it holds

$$\begin{aligned} p_1\cdot p_3 &= p_1^0p_3^0 - \mathbf{p}_1\mathbf{p}_3 \simeq \frac{1}{4}s - \frac{1}{4}s\,\cos\vartheta = \frac{1}{4}s(1-\cos\vartheta) = p_2\cdot p_4 \\ p_1\cdot p_4 &= \frac{1}{4}s(1+\cos\vartheta) = p_2\cdot p_3 \\ \Rightarrow (p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) &= \frac{1}{16}s^2\left(2+2\cos^2\vartheta\right) = \frac{1}{8}s^2\left(1+\cos^2\vartheta\right)\,. \end{aligned} \tag{6.30}$$

The final result for $|M|^2$ is

$$|M|^2 = e^4 \left(1 + cos^2\vartheta\right) = 16\pi^2\alpha^2 \left(1 + cos^2\vartheta\right) , \quad \alpha = \frac{e^2}{2\pi} . \tag{6.31}$$

If we compare Eq. (6.31) with that for scalar 2-2 scattering (Eq. (3.37)), $|M|^2 = \lambda^2$, we see, that for fermions the scattering angle is important, whereas for scalar fields it is not. Inserting Eq. (6.31) in Eq. (6.14) and using $4p_3^0 p_4^0 \simeq s$ yields the cross section

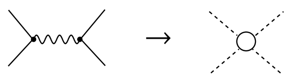
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1 + \cos\vartheta) . \tag{6.32}$$

Again, compare this to the cross section of scalar 2-2 scattering (Eq. (3.154)), i.e. $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2}\frac{\lambda^2}{4\pi}$.

We remark, that the intermediate virtual photon was chosen in the Feynman gauge, i.e. $\xi = 1$. However, we have shown in Eq. (5.77), that any choice of ξ leads to the same result, in particular $\xi = 0$. Further, in the high energy limit also $(p_1 - p_2)_\mu \bar{v}(p_2)\,\gamma^\mu\, u(p_1) \overset{\sim m_e}{\approx} 0$. Only the physical polarisations ϵ_1 and ϵ_2 play a role, ϵ_3 drops out (see Eq. (5.48)). This argument also applies to $\bar{u}(p_3)\,\gamma^\nu\, v(p_4)$. In summary we have

$$\begin{aligned} \bar{u}(p_3)\,\gamma^\nu\, v(p_4)\,(p_{3/4})_\nu &\approx 0 \\ \bar{v}(p_2)\,\gamma^\mu\, u(p_1)\,(p_{1/2})_\mu &\approx 0 . \end{aligned} \tag{6.33}$$

So if $p_{3,4}$ are orthogonal to the beam axis, defined by $p_{1/2}$, the related polarisation ϵ_1 or ϵ_2 also 'drops out of the game'. In this case, $\alpha = \pi/2$, only one polarisation contributes to the scattering, for $\alpha = 0$, both. Lastly, note, that in the highly relativistic case and for $\alpha = \pi/2$:

"  " .

7. Renormalisation

As we have seen in the previous chapters, loop diagrams are divergent. In this chapter we discuss renormalisation, a mathematical approach, to cancel singularities from the integrals.

I. ϕ^4-theory

In ϕ^4-theory the action is given by (see Eq. (3.18))

$$S[\phi] = -\frac{1}{2}\int d^4x\, \phi_0 \left(\partial^2 + m_0^2\right)\phi_0 - \frac{\lambda_0}{4!}\int d^4x\, \phi_0^4\,, \tag{7.1}$$

with *bare* fields ϕ_0 and parameters/couplings m_0^2 and λ_0. We write

$$\begin{aligned} \phi_0 &= Z_\phi^{1/2}\,\phi \\ m_0^2 &= Z_m\, m^2 \\ \lambda_0 &= Z_\lambda\, \lambda\,, \end{aligned} \tag{7.2}$$

with *renormalised* or physical fields ϕ, parameters m^2, λ and *multiplicative* renormalisations Z_ϕ, Z_m, Z_λ. The Z's are expanded in powers of λ:

$$Z = 1 + \delta Z\,, \quad \delta Z = \delta Z_1 \lambda + \delta Z_2 \lambda^2 + \dots\,, \tag{7.3}$$

where the first part corresponds to classical theory and δZ to quantum corrections. Recalling the LSZ-formalism we use Eq. (3.169) with fields ϕ_0

$$\langle T\ \phi_0\, \phi_0\rangle(p)\big|_{\text{pole}} = \frac{iZ}{p^2 - m^2_{\text{phys}}} + \text{finite} = Z_\phi \langle T\ \phi\,\phi\rangle\big|_{\text{pole}}\,. \tag{7.4}$$

We demand $Z_\phi = Z$, which implies

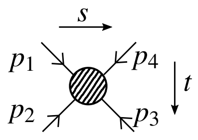
$$\langle T\ \phi\,\phi\rangle\big|_{\text{pole}} = \frac{i}{p^2 - m^2} + \text{finite}\,. \tag{7.5}$$

Here, we have implicitly fixed Z_ϕ such that $m^2 = m^2_{\text{phys}}$, i.e. $p^2 = m^2$. Eq. (7.4) and Eq. (7.5) can be cast into

renormalisation conditions (1-2)

$$\begin{aligned} \Big[\langle T\ \phi\phi\rangle(p)\Big]^{-1}_{p^2=m^2} &\overset{!}{=} 0 \\ \partial_{p^2}\Big[\langle T\ \phi\phi\rangle(p)\Big]^{-1}_{p^2=m^2} &\overset{!}{=} 1\,. \end{aligned} \tag{7.6}$$

This fixes the constants Z_ϕ and Z_m. More generally, we fix $\langle T\ \phi\phi\rangle$ at some scale $p^2 = \mu^2$, where μ is called renormalisation scale. The coupling renormalisation Z_λ is fixed, by fixing the amputated four-point function:



$$\Big|_{s=t=u=m^2} \stackrel{!}{=} -\mathrm{i}\,\lambda\,.$$

If we write this in terms of the Green function (using Eq. (7.5)), we obtain the third

renormalisation condition (3)

$$\prod_i \Big[\langle T\,\phi\,\phi\rangle\,(p_i) \Big]^{-1} \cdot \langle T\,\phi(p_1)\cdots\phi(p_4)\rangle \Big|_{s=t=u=m^2} = -\mathrm{i}\,\lambda\,, \tag{7.7}$$

where $\lambda = \lambda_{\text{phys}}\big|_{\text{symmetric point}}$. The renormalisation conditions Eq. (7.6) and Eq. (7.7) fix the map between the bare quantities ϕ_0, m_0, λ_0 to the renormalised (finite) quantities ϕ, m, λ. The finiteness of correlation functions of the renormalised fields ϕ follows from the finiteness of Eq. (7.6) and Eq. (7.7). Hence, the Z's have to cancel the loop divergences. Thus, the Z's carry the singularities. Note, that in (perturbatively) renormalisable theories, it is sufficient to introduce the Z's (and similar quantities) for getting a manifestly finite theory. The freedom of (re)-normalising fields and couplings also encodes, that Green functions are *not* by themselves physical observables. For example, we could have renormalised the theory at some other momentum scale $p^2 = \mu^2$ with the renormalisation conditions, with

$$\begin{aligned} \lambda &= \lambda_{\text{phys}}\big|_{p^2=\mu^2} \\ m^2 &= m^2_{\text{phys}}\big|_{p^2=\mu^2}\,. \end{aligned} \tag{7.8}$$

Physics is invariant under changing μ, which is expressed in the

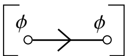
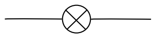
renormalisation group equation

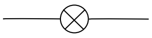
$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}(\text{phys. observables}) = 0\,. \tag{7.9}$$

Note, that the renormalisation conditions encode the **reparametrisation invariance of the theory** and the **insensitivity of physics** to the specific renormalisation scheme. μ is called *renormalisation group (RG) scale*. We remark, that the generator of the RG is $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$, and the RG is a one-parameter, Abelian semi group (see QFT II).

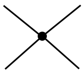
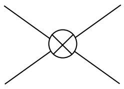
Let us now formulate the **Feynman rules** in terms of renormalised quantities (where we have dropped the $i\epsilon$):

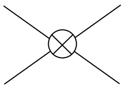
Propagator:

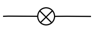
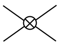
$\Big[$  $\Big]^{-1} = Z_\phi \dfrac{p^2 - Z_m m^2}{\mathrm{i}} = \left[\dfrac{\mathrm{i}}{p^2 - m^2}\right]^{-1} -$  ,

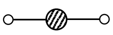
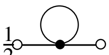
where  $= (-\mathrm{i})\Big[\underbrace{(1 - Z_\phi)}_{<1} p^2 - (1 - Z_\phi Z_m) m^2\Big].$

Vertex:

 $= -\mathrm{i}\, Z_\lambda Z_\phi^2 \lambda = -\mathrm{i}\,\lambda +$  ,

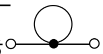
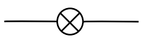
where  $= \mathrm{i}\,\lambda\,(1 - Z_\phi^2 Z_\lambda)\,.$

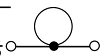
Note, that  ,  are called counter terms. Z_ϕ, Z_m, Z_λ cancel singularities, that are proportional to p^2, m^2 and λ, respectively. Next, we examine **renormalisation at one loops**. First, we consider the mass correction (see Eq. (3.193)).

 $=$ $+\frac{1}{2}$  $+ \quad \mathcal{O}(\lambda^2)$

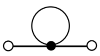
$$= \frac{\mathrm{i}}{p^2 - m^2} + \frac{\mathrm{i}}{p^2 - m^2}\Big[-\mathrm{i}\,\Pi(p)\Big]\frac{\mathrm{i}}{p^2 - m^2} + \ldots,$$

with self-energy:

$-\mathrm{i}\,\Pi(p) = \Big[\frac{1}{2}$  $+$  $\Big]$

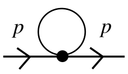
$= \underbrace{\Big[\frac{1}{2}$  $+\ \mathrm{i}\,(1 - Z_\phi)\,p^2 + \mathrm{i}\,(1 - Z_\phi Z_m)\,m^2\Big]}_{\text{finite}}$.

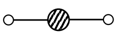
For the loop diagram, we have:

 $= -\mathrm{i}\,\lambda \displaystyle\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2 - m^2}\,.$

Consequently, the self-energy is

$$-\mathrm{i}\,\Pi(p) = -\frac{\mathrm{i}\,\lambda}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2 - m^2} - \mathrm{i}\,(1 - Z_\phi)\,p^2 + \mathrm{i}\,(1 - Z_\phi Z_m)\,m^2\,. \tag{7.10}$$

Note, that  has no dependence on the external momentum p. Therefore, $Z_\phi\big|_{\text{1-loop}} = 1$. Further, it is

$\Big[$  $\Big]^{-1}\Big|_{p^2=m^2} = 0\,.$

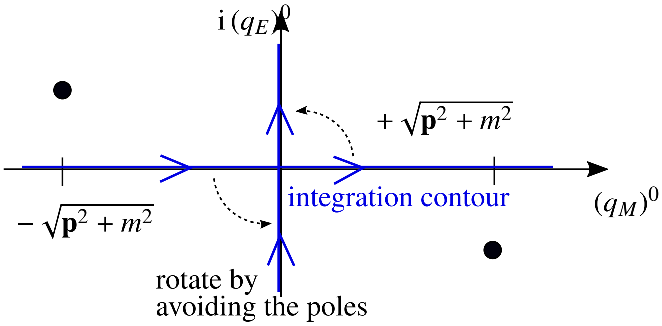


Figure 7.1.: Recall the iϵ of time-ordering: $\frac{1}{q^2-m^2+\mathrm{i}\,\epsilon}$. The rotated Euclidean contour runs $\mathrm{i}(q_E)^0$ from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$, or $(q_E)^0$ from $-\infty$ to $+\infty$. As this rotation does not swipe the poles, the integration stays the same.

The renormalisation condition in Eq. (7.6) implies

$$\Rightarrow \Pi(p)\big|_{\text{1-loop}} \hat{=} 0\,. \tag{7.11}$$

We conclude, that

$$\boxed{1 - Z_m = \frac{1}{2}\frac{\lambda}{m^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2}\,.} \tag{7.12}$$

It remains to compute

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\epsilon}\,, \tag{7.13}$$

where we have to encounter two problems, namely that the integrand diverges on-shell ($q^2 = m^2$), and that the integral diverges for $q^2 \to \infty$. The diverging integrand can be resolved with **Wick rotation**. For this, we rescale momentum in Minkowski space to Euclidean space as shown in figure 7.1. Then

$$\begin{aligned}
& & (q_M)^0 &= \mathrm{i}\,(q_E)^0 \\
(\eta_E)^{\mu\nu} = -\mathbb{1} \quad &\rightarrow & \Rightarrow \quad & (q_M)_\mu (q_M)^\mu = -(q_E)_\mu (q_E)^\mu = -(q_E)_\mu (q_E)_\mu \\
& & \Rightarrow \quad & \int_{\mathbb{R}^4} \mathrm{d}^4 q_M = \mathrm{i}\int_{\mathbb{R}^4} \mathrm{d}^4 q_E \\
& & \Rightarrow \quad & \int \frac{\mathrm{d}^4 q_M}{(2\pi)^4}\frac{\mathrm{i}}{q_M^2 - m^2} = \int \frac{\mathrm{d}^4 q_E}{(2\pi)^4}\frac{\mathrm{i}}{q_E^2+m^2}\,.
\end{aligned} \tag{7.14}$$

The divergent integral for $q^2 \to \infty$ is dealt with by **regularisation**. Consider for example a momentum cut-off Λ:

$$\int_{\mathbb{R}^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \rightarrow \int_{q^2 \le \Lambda^2} \frac{\mathrm{d}^4 q}{(2\pi)^4}\,. \tag{7.15}$$

Then

$$\int\limits_{q^2\leq\Lambda^2} \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2+m^2} = \int\limits_0^\Lambda \frac{\mathrm{d}q}{(2\pi)^4}\, q^3 \int \mathrm{d}\Omega\, \frac{1}{q^2+m^2} = \frac{1}{8\pi^2}\int\limits_0^\Lambda \mathrm{d}q\, \frac{q^3}{q^2+m^2}$$

$$= \frac{1}{16\pi^2}\left[\Lambda^2 + m^2 \ln\frac{m^2}{\Lambda^2+m^2}\right]. \tag{7.16}$$

Another example is dimensional regularisation. We rewrite the four-dimensional integral as

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2+m^2} = \left[\left(\bar\mu^2\right)^{\frac{4-d}{2}}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\right]\frac{1}{q^2+m^2}$$

$$= \frac{\Omega_d}{(2\pi)^d}\left(\bar\mu^2\right)^{\frac{4-d}{2}}\int\limits_0^\infty \mathrm{d}q\, q^{d-1}\frac{1}{q^2+m^2}\,, \tag{7.17}$$

with the angular volume Ω_d. Note, that the term in square brackets in the first line has dimension 4 due to the scaling factor in front of the d-dimensional integral. For $d<2$ the integral in the last line is finite, and we can compute Eq. (7.17), and then analytically extend the result. We use

$$\int\frac{\mathrm{d}\Omega_d}{(2\pi)^d}: \quad \sqrt{\pi}^d = \left(\int \mathrm{d}x\, e^{-x^2}\right)^d = \int \mathrm{d}^d x\, e^{-\mathbf{x}^2}\,, \quad \mathbf{x} = \begin{pmatrix} x_1\\ \vdots \\ x_d\end{pmatrix}$$

$$= \int \mathrm{d}\Omega_d \int\limits_0^\infty \mathrm{d}x\, x^{d-1}\, e^{-x^2} = \frac{1}{2}\int \mathrm{d}\Omega_d\, \Gamma\left(\frac{d}{2}\right), \tag{7.18}$$

where we have used an integral representation of the Γ-function: $\Gamma[t] = \int_0^\infty \mathrm{d}s\, s^{t-1}\exp\{-s\}$ with $s=x^2$ and $\mathrm{d}s = dx^2 = 2\mathrm{d}x\, x$. This leads us to

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\,, \tag{7.19}$$

and hence

$$\int\limits_0^\infty \mathrm{d}q\, q^{d-1}\frac{1}{(q^2+m^2)^n} = \frac{1}{2}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{m^2}\right)^{n-\frac{d}{2}} \tag{7.20}$$

$$\Rightarrow \int\limits_0^\infty \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q^2+m^2)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{m^2}\right)^{n-\frac{d}{2}}\,. \tag{7.21}$$

With this, we obtain

$$\left(\bar\mu^2\right)^{\frac{4-d}{2}}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{q^2+m^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right) m^2\left(\frac{\bar\mu^2}{m^2}\right)^2 - \frac{d}{2}\,. \tag{7.22}$$

Then, for $d = 4 - 2\epsilon$ with $\epsilon \to 0$, this is

$$\left(\bar{\mu}^2\right)^{\epsilon} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} = \frac{m^2}{16\pi^2}\left[\frac{1}{\epsilon} + \gamma - 1 + \ln 4\pi - \ln \frac{m^2}{\bar{\mu}^2}\right], \tag{7.23}$$

with

$$\begin{aligned}\Gamma(-1+\epsilon) &= \frac{1}{-1+\epsilon}\Gamma(\epsilon) \quad \leftarrow \quad x\Gamma(x) = \Gamma(x+1) \\ &= -\frac{1}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon),\end{aligned} \tag{7.24}$$

and the Euler-Mascheroni constant $\gamma = 0.577\ldots$. This allows us to determine $Z_m\big|_{\text{1-loop}}$. With cut-off regularisation (Eq. (7.16)) we get

$$Z_m = 1 - \frac{1}{2}\frac{1}{16\pi^2}\lambda\left(\frac{\Lambda^2}{m^2} + \ln \frac{1}{1+\frac{\Lambda^2}{m^2}}\right). \tag{7.25}$$

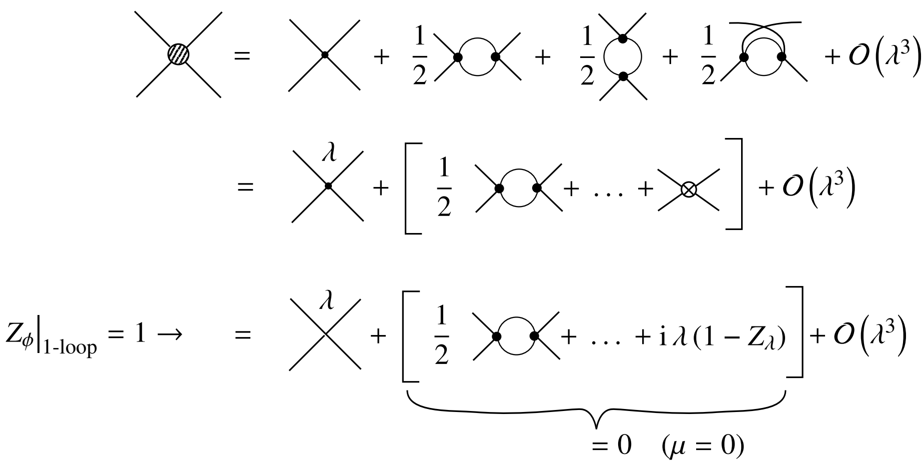
And with dimensional regularisation (Eq. (7.24)) we obtain

$$Z_m = 1 - \frac{1}{2}\frac{1}{16\pi^2}\lambda\left(-\frac{1}{\epsilon} + \gamma - 1 + \ln 4\pi - \ln \frac{m^2}{\bar{\mu}^2}\right). \tag{7.26}$$

Note, that in Eq. (7.25) and Eq. (7.26) the term $\frac{1}{16\pi^2}\lambda$ is the expansion coefficient in ϕ^4-theory. The equivalence between these equations is best seen with: $\ln \frac{1}{1+\frac{\Lambda^2}{m^2}} = \ln \frac{m^2}{\Lambda^2} + \ln \frac{1}{1+\frac{m^2}{\Lambda}}$.
Finally, in both cases at one loop we have

$$= \frac{\mathrm{i}}{p^2 - m^2} + \mathcal{O}(\lambda^2). \tag{7.27}$$

Next, we have to consider the coupling correction



The renormalisation condition for $t = s = u = 0$, i.e. $\mu^2 = 0$ becomes

$$1 - Z_\lambda = \frac{3\lambda}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{(q^2 - m^2)^2} \,. \tag{7.28}$$

Using Wick rotation (Eq. (7.21)) with dimensional regularisation ($n = 2, 2 - \frac{d}{2} = \epsilon$), we compute

$$\begin{aligned} -\frac{3\lambda}{2}\mu^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^2} &= -\frac{3\lambda}{2} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\epsilon)}{\Gamma(2)} \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \\ &= -\frac{3\lambda}{2} \frac{1}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2}\right) , \end{aligned} \tag{7.29}$$

where in the last line, we used the expansion: $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon)$. In summary, we state, that (at renormalisation group scale $\mu^2 = 0$)

$$Z_\lambda = 1 + \frac{3}{2} \frac{\lambda}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2}\right) . \tag{7.30}$$

With $Z_\phi = 1$ our theory is consistent at one loop. Also, it is renormalisable at one loop. We remark, that the renormalised correlation functions $\langle \phi(p_1)\, \phi(p_2) \rangle_{\text{1-loop}}$, $\langle \phi(p_1) \cdots \phi(p_4) \rangle_{\text{1-loop}}$ are *finite*, but depend on the renormalisation scale μ. Higher correlation functions at one loop are finite from the onset, e.g. the six-point function, as at $p_i = 0$:

$$\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle_{\text{1-loop}} \sim \lambda^3 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2)^3} \,, \tag{7.31}$$

is finite. Note, that a singularity in $\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle_{\text{1-loop}}$ would be disastrous, because there is no counter term for it. Hence, perturbative renormalisability (in ϕ^4-theory) implies, that all correlation functions *to all orders* in perturbation theory are finite, by adjusting Z_ϕ, Z_m, Z_λ. Also note, that 'Physics' does *not* depend on the renormalisation scheme, which yields the

renormalisation group invariance

$$\mu \frac{d}{d\mu} \text{observable} = 0 \,. \tag{7.32}$$

Moreover, it does *not* depend on the cut-off scale

$$\Lambda \frac{d}{d\Lambda} \text{observable} = 0 \,. \tag{7.33}$$

Evidently, the bare quantities know nothing of the renormalisation point. Hence

$$\mu \frac{d}{d\mu} \phi_0 = \mu \frac{d}{d\mu} m_0 = \mu \frac{d}{d\mu} \lambda_0 = 0 \,. \tag{7.34}$$

It follows, that

$$\begin{aligned} \mu \frac{d\phi}{d\mu} \frac{1}{\phi} &= -\frac{1}{2} \mu \frac{dZ_\phi}{d\mu} \frac{1}{Z_\phi} = -\gamma_\phi \\ \mu \frac{d\lambda}{d\mu} \frac{1}{\lambda} &= -\mu \frac{dZ_\lambda}{d\mu} \frac{1}{Z_\lambda} = \beta_\lambda \\ \mu \frac{dm^2}{d\mu} \frac{1}{m^2} &= -\mu \frac{dZ_m}{d\mu} \frac{1}{Z_m} = \gamma_m \,, \end{aligned} \tag{7.35}$$

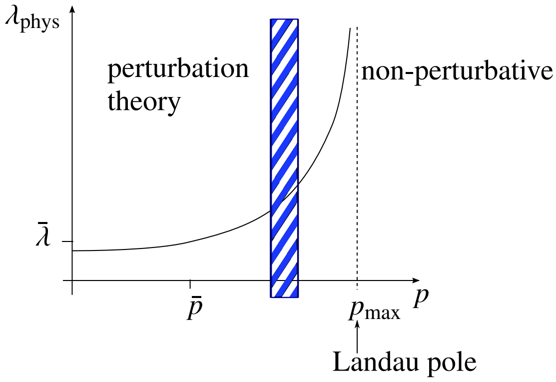


Figure 7.2.: Sketch of the running coupling.

which are also referred to as *beta-functions*. In turn, the renormalised quantities are insensitive to the cut-off. Hence

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}\phi = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}m = \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda}\lambda = 0\,. \tag{7.36}$$

Therefore, Λ and μ scaling are (asymptotically) directly related. Lastly, we discuss **renormalised and running coupling**. The renormalised coupling is *not* the physical coupling, as it runs with μ (see Eq. (7.35)):

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \lambda = \beta\,. \tag{7.37}$$

In our one loop case, we have

$$\boxed{\beta(\mu) = -\mu \frac{\mathrm{d} \ln Z_\lambda}{\mathrm{d}\mu} = \frac{3}{16\pi^2}\lambda\,.} \tag{7.38}$$

Note, that at $\mu = 0$ this is also equal to $-m\frac{\mathrm{d}\ln Z_\lambda}{\mathrm{d}m}$. Our renormalisation condition, however, fixed $\lambda = \lambda_{\text{phys}}$ at the momentum scale μ. Hence,

$$p\frac{\mathrm{d}}{\mathrm{d}p}\lambda_{\text{phys}}(p) \simeq \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\lambda\Big|_{\mu=p}\,. \tag{7.39}$$

We can integrate this equation at one loop and get

$$\lambda_{\text{phys}}(p) = \frac{\bar{\lambda}}{1 + \frac{3\bar{\lambda}}{16\pi^2}\ln\frac{\bar{p}}{p}}\,. \tag{7.40}$$

This is linked to the triviality of ϕ^4-theory: $\lambda_{\text{phys}}(p) < \infty$ for all p. It follows, that

$$\lambda_{\text{phys}} \hat{=} 0\,. \tag{7.41}$$

Note, that this has to be proven non-perturbatively (see figure 7.2). This is subject to QFT II, though.

II. QED

We have shown, how to renormalise scalar fields in the previous section. In this section, we will do the same for QED. For this purpose consider the action of QED (Eq. (6.1)), only with an electron:

$$S_{\text{QED}}[A,\psi] = \int \mathrm{d}^4x\, \bar{\psi}_0\,(\mathrm{i}\not{D} - m_0)\,\psi_0 - \frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}(A_0)F^{\mu\nu}(A_0) - \frac{1}{\xi_0}\int \mathrm{d}^4x\,\left(\partial_\mu A_0^\mu\right)^2\,, \tag{7.42}$$

with $D_\mu = \partial_\mu - \mathrm{i}e_0 A_{0\mu}$ and $\psi_0 = \psi_{0e}$. The action is gauge invariant under

$$\begin{aligned} A_{0\mu} &\to A_{0\mu} + \frac{1}{e_0}\partial_\mu \alpha \\ \psi_0 &\to e^{-\mathrm{i}\alpha}\,\psi_0\,, \end{aligned} \tag{7.43}$$

of the bare fields $A_{0\mu}$ and ψ_0 (see Eq. (6.5)). We introduce renormalised fields and parameters:

$$\begin{aligned} A_{0\mu} &= Z_A^{\frac{1}{2}}\,A_\mu \\ \psi_0 &= Z_\psi^{\frac{1}{2}}\,\psi \\ e_0 &= Z_e\,e \\ m_0 &= Z_m\,m \qquad [\xi_0 = Z_\xi\,\xi]\,. \end{aligned} \tag{7.44}$$

It can be shown, that gauge symmetry enforces the relation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(Z_A^{\frac{1}{2}}\,Z_e\right) = 0\,, \tag{7.45}$$

that is, $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}(e\,A_\mu) = 0$. This and similar relations for correlation functions are called *Ward-Takahashi identities* or *Slavnov-Taylor identities* and will be subject of QFT II and QCD, respectively. Here, we proceed with a heuristic argument for Eq. (7.45). Firstly, *physical gauge invariance* should apply to renormalised quantities, so the covariant derivative should read

$$D_\mu = \partial_\mu - \mathrm{i}\,e\,A_\mu\,. \tag{7.46}$$

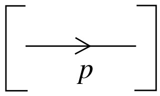
This implies Eq. (7.45). Secondly, we have gauge-fixed the *bar, classical* action (Eq. (7.42)). The previous argument only holds if this simple additive structure holds also on quantum level. To that end we evaluate

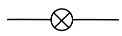
$$\langle S_{\mathrm{QED}}[A^\alpha,\psi^\alpha] - S_{\mathrm{QED}}[A,\psi]\rangle\big|_{O(\alpha)} = -\frac{1}{\xi}\int \mathrm{d}^4x\,\left\langle \partial_\rho A^\rho\right\rangle\,\partial_\rho\partial^\rho\alpha = 0\,, \tag{7.47}$$

due to the gauge fixing $\partial_\rho A^\rho = 0$. We conclude, that for general linear gauge fixings Eq. (7.45) holds, and

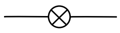
$$Z_\xi = Z_A\,. \tag{7.48}$$

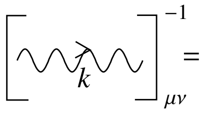
In turn, for non-linear gauge fixings and for non-Abelian gauge theories (strong and weak forces) Eq. (7.45) fails.
We will now present the Feynman rules in terms of renormalised quantities (see Eq. (6.12) ff.). For the propagators we find:



$$^{-1} = \frac{1}{\mathrm{i}}\, Z_\psi(\not{p} - Z_m m) \;\; = \left[\mathrm{i}\,\frac{\not{p} + m^2}{p^2 - m^2}\right]^{-1} -$$  ,

where:

 $= -\mathrm{i}\,(1 - Z_\phi)\,\not{p} + \mathrm{i}\,(1 - Z_\phi Z_m)\, m\,.$



$$^{-1}_{\mu\nu} = \mathrm{i}\, Z_A \left[k^2\, \eta_{\mu\nu} - k_\mu k_\nu \left(1 - \frac{1}{Z_A \xi}\right)\right]$$

$$= \left[-\frac{\mathrm{i}}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right)\right]^{-1} -$$  ,

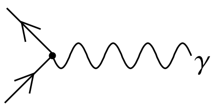
where:

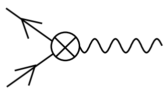
 $= -\mathrm{i}\,(1 - Z_A)(k^2\, \eta_{\mu\nu} - k_\mu k_\nu)\,.$

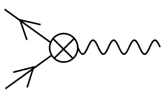
only transversal modes
get renormalised

(7.49)

For the vertices we get:



$$= \mathrm{i}\, Z_\phi\, Z_A^{\frac{1}{2}}\, Z_e\, e = \mathrm{i}\, e\, \gamma_\mu \quad +$$  ,

where:  $= -\mathrm{i}\, e\, \gamma_\mu \left(1 - Z_\psi\, Z_A^{\frac{1}{2}}, Z_e\right).$

(7.50)

A. Complementary Calculations

I. Coherent states

Here we provide some more details for the computations relevant for the discussion of the coherent state $|\alpha\rangle$ in Section III.3 . We start with the ansatz given in Section III

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{\infty} \left(\int \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p_i}}}} \alpha(\mathbf{p_i}) \right) |\mathbf{p_1} \cdots \mathbf{p_n}\rangle \ , \tag{A.1}$$

with

$$a(\mathbf{p}) \, |\mathbf{p_1} \cdots \mathbf{p_n}\rangle = \sum_{i=1}^{n} (2\pi)^3 \ \sqrt{2\omega_{\mathbf{p_i}}} \ |\mathbf{p_1} \cdots \mathbf{p_{i-1}} \, \mathbf{p_{i+1}} \cdots \mathbf{p_n}\rangle \ \delta(\mathbf{p} - \mathbf{p_i}) \, . \tag{A.2}$$

From Eq. (A.2) and with normal ordering (Eq. (2.109)) it follows

$$\begin{aligned} & \frac{1}{n!} \, a(\mathbf{p}) \left(\int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \, \alpha(\mathbf{p}') \, a^\dagger(\mathbf{p}') \right)^n |0\rangle \\ = \ & \frac{1}{n!} \, n \, \alpha(\mathbf{p}) \left(\int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \, \alpha(\mathbf{p}') \, a^\dagger(\mathbf{p}') \right)^{n-1} |0\rangle \\ = \ & \alpha(\mathbf{p}) \, \frac{1}{(n-1)!} \left(\int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \, \alpha(\mathbf{p}') \, a^\dagger(\mathbf{p}') \right)^{n-1} |0\rangle \end{aligned} \tag{A.3}$$

and similarly

$$\begin{aligned} & \langle 0| \, \frac{1}{n!} \left(\int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \, \alpha^*(\mathbf{p}') \, a(\mathbf{p}') \right)^n a^\dagger(\mathbf{p}) \\ = \ & \langle 0| \, \frac{1}{(n-1)!} \left(\int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \, \alpha^*(\mathbf{p}') \, a(\mathbf{p}') \right)^{n-1} \alpha^*(\mathbf{p}) \, . \end{aligned} \tag{A.4}$$

Now we can calculate

$$\begin{aligned} \langle \alpha | \alpha \rangle \ = \ & \frac{1}{\mathcal{N}^2(\alpha)} \sum_{n=0}^{\infty} \left(\frac{1}{n!} \right)^2 \int \prod_{i=1}^{\infty} \left(\frac{\mathrm{d}^3 p_i}{(2\pi)^3} \frac{\mathrm{d}^3 p_i'}{(2\pi)^3} \, \alpha^*(\mathbf{p_i}) \, \alpha(\mathbf{p_i'}) \right) \cdot \ldots \\ & \ldots \cdot \langle 0 | a(\mathbf{p_1}) \cdots a(\mathbf{p_n}) \, a^\dagger(\mathbf{p_n'}) \cdots a^\dagger(\mathbf{p_1'}) | 0 \rangle \\ & . \end{aligned} \tag{A.5}$$

With

$$
\begin{aligned}
& \langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_n})\, a^\dagger(\mathbf{p'_n})\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
= \;& \langle 0|a(\mathbf{p_1})\cdots\Big(\Big[a(\mathbf{p_n})\,,\, a^\dagger(\mathbf{p'_n})\Big] + a^\dagger(\mathbf{p'_n})\, a(\mathbf{p_n})\Big)\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
= \;& (2\pi)^3\,\delta(\mathbf{p_n}-\mathbf{p'_n})\;\langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_{n-1}})\cdots a^\dagger(\mathbf{p'_1})|0\rangle + ... \\
& ... + \langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_n})\, a(\mathbf{p_n})\, a^\dagger(\mathbf{p'_{n-1}})\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
\vdots \;& \quad \text{(continue normal ordering)} \\
= \;& (2\pi)^3\sum_{i=1}^{n}\langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_n})\cdots\widehat{a^\dagger(\mathbf{p'_i})}\cdots a^\dagger(\mathbf{p'_1})|0\rangle\;\cdot\delta(\mathbf{p_n}-\mathbf{p'_i})\,,
\end{aligned} \tag{A.6}
$$

where

$$
\widehat{a^\dagger(\mathbf{p'_i})} = 1\,. \tag{A.7}
$$

Finally we get

$$
\begin{aligned}
\langle\alpha|\alpha\rangle \;=\;& \frac{1}{\mathcal{N}^2(\alpha)}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\int\frac{\mathrm{d}^3 p_n}{(2\pi)^3}\alpha^*(\mathbf{p_n})\,\alpha(\mathbf{p_n})\cdot ... \\
& ...\cdot\int\prod_{i=1}^{n-1}\left(\frac{\mathrm{d}^3 p_i}{(2\pi)^3}\frac{\mathrm{d}^3 p'_i}{(2\pi)^3}\,\alpha^*(\mathbf{p_i})\,\alpha(\mathbf{p'_i})\right)\langle 0|a(\mathbf{p_1})\cdots a(\mathbf{p_{n-1}})\, a^\dagger(\mathbf{p'_{n-1}})\cdots a^\dagger(\mathbf{p'_1})|0\rangle \\
\vdots \;& \quad \text{(continue normal ordering)} \\
=\;& \frac{1}{\mathcal{N}^2(\alpha)}\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\alpha^*(\mathbf{p})\,\alpha(\mathbf{p})\Big)^n \\
=\;& \frac{1}{\mathcal{N}^2(\alpha)}\exp\left(\int\frac{\mathrm{d}^3 p}{(2\pi)^3}|\alpha(\mathbf{p})|^2\right) \\
\Rightarrow \mathcal{N}(\alpha) \;=\;& \exp\left(\frac{1}{2}\int\frac{\mathrm{d}^3 p}{(2\pi)^3}|\alpha(\mathbf{p})|^2\right)\,.
\end{aligned} \tag{A.8}
$$

Bibliography