

HEIDELBERG UNIVERSITY
INSTITUTE FOR THEORETICAL PHYSICS

Quantum Field Theory

LECTURE NOTES FROM 2022-2023

JAN M. PAWLOWSKI

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Part I.

Quantum field theory

1. Introduction

Quantum Field Theory (QFT) is the theoretical foundation of modern quantum theory. It underlies the description of the fundamental interactions of matter in terms of the Standard Model (SM) of particle physics, as well as for (Effective) field theories beyond the Standard Model (BSM). In terms of, e.g., asymptotically safe quantum gravity, lattice quantum gravity or loop quantum gravity it is also the theoretical foundations of some of the contenders for a unified quantum theory of matter and gravity, alongside with string theory. Non-relativistic quantum field theory also underlies the description of many-body phenomena in condensed matter and statistical physics, ranging from superconductivity and superfluidity to quantum phase transitions.

Quantum field theory can be understood as the many-body limit of quantum mechanical systems such as (an)harmonic oscillators. This is depicted in the lower horizontal map in [Figure 1.1](#). It can also be obtained from a quantisation of a classical field theory, depicted in the right vertical map in [Figure 1.1](#), historically called *second* quantisation. Evidently this simply is the quantisation introduced in quantum mechanics, as can be seen from [Figure 1.1](#).

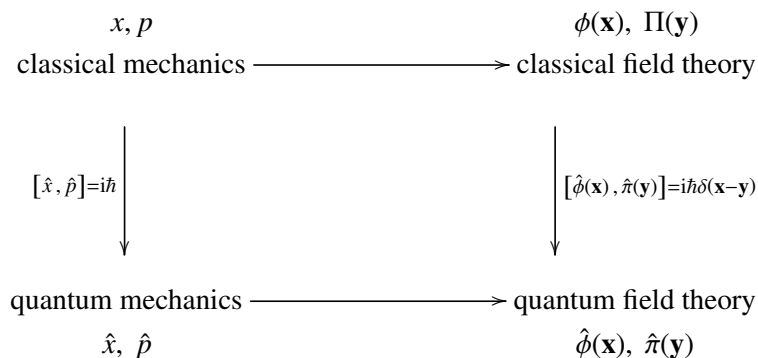


Figure 1.1.: Different paths from classical mechanics to quantum field theory.

As mentioned above, the applications of quantum field theory are manifold, and encompass all quantum systems from small to large scales. In particular modern theoretical particle physics is a great success story of quantum field theory. In the Standard Model the Higgs-boson corresponds to a scalar field (spin 0), leptons and quarks are described by fermion fields (spin 1/2), and vector (gauge) fields (spin 1) are used for photons, W^\pm and Z bosons, and gluons. More generically one distinguishes bosonic fields (spin n) and fermionic fields (spin $2n + 1/2$) with $n \in \mathbb{N}$. A prominent example for a higher spin field is the graviton (spin 2) and the higher spins come with the additional conceptual problems (perturbative non-renormalisability) that potentially renders the respective theories ill-defined for large momentum scales. While being highly interesting, the respective theories will not be considered in this lecture course.

Due to its pivotal importance for the descriptions of general quantum systems, quantum field theoretical methods has been continuous and rapid advances from its early beginnings in the 30ties and 40ties of the 20th century. From the very beginning one of the key methods used in QFT applications is *perturbation theory*, that is the expansion of the physics at hand about a non-interacting case in order (number) of interactions (scatterings). This is a Taylor expansion in the interaction strength of the theory, in QFT this series is an *asymptotic* one (to be explained later). This will be the main method used in the applications in the current lecture course, and its success and limits will be discussed in detail. As a side remark we mention that, despite its matureness,

even in perturbation theory there have been recently exciting new developments that go under the name of *resurgence*.

Despite its success the limits of perturbation theory are apparent: for strongly coupled or correlated systems an expansion in the number of scatterings may not be sufficient. A simple but relevant example is an observable $O(\lambda)$ that has the following dependence on the coupling,

$$O(\lambda) = \text{Pol}_0(\lambda) + \text{Pol}_1(\lambda) \exp\left\{-\frac{\text{cont.}}{\lambda^2}\right\}, \quad (1.1)$$

where $\text{Pol}_i(\lambda)$ with $i = 0, 1$ are polynomials or converging series in λ . Evidently, an expansion of (1.1) about $\lambda = 0$ will only catch part of $O(\lambda)$, namely $\text{Pol}_0(\lambda)$. Terms such as (1.1) arise typically from topological effects in QFT and quantum mechanics, and it is also here where the recent developments in resurgence have their merits.

Thus, the description of the physics of strongly correlated systems calls for non-perturbative methods. Prominent and important examples are the *lattice* approach to QFT, where the QFT is put on a space or space-time grid, and (functional) renormalisation group approaches, where the scale-dependence of the theory is resolved successively. Both approaches will be discussed briefly in *Part II* (second term) of the current lecture course, while more details are given for the latter, the renormalisation group, as it is also essential for perturbation theory. A full account of these approaches has to be subject of dedicated advanced QFT courses.

Let us now come back to the limits in Figure 1.1, which we want to elucidate with two examples, the *oscillating masses/string* and *electrodynamics*:

Example 1-1: Oscillating masses / string. We consider a chain of oscillating point masses with a fixed position in x on a string (harmonic forces between next neighbours). This situation is depicted in Figure 1.2.

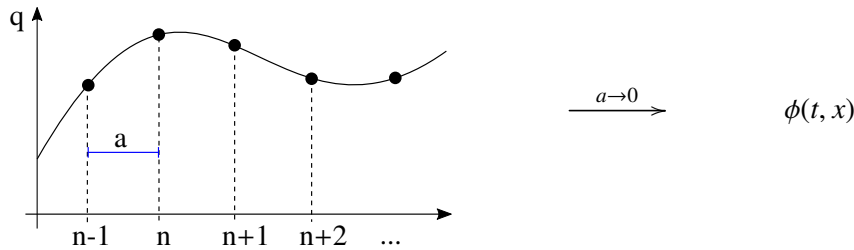


Figure 1.2.: Oscillating masses at a distance a on a string.

Now we take the limit of an infinitesimal *lattice* spacing a between the neighbouring point masses, thus storing more and more oscillating masses on a given interval with length L on the string. A mass point q_n on the string experiences a harmonic force $F(q_n, q_i)$ proportional to the distance d between q_n and its neighbouring points with $i = n - 1, n + 1$. For example, the force between q_n and q_{n-1} is given by

$$F(q_n, q_{n-1}) = -\frac{c^2}{a^2}(q_n - q_{n-1}), \quad (1.2)$$

with the spring constant c^2/a^2 , which pulls the mass point in the direction of q_{n-1} . Summing up the two forces $F(q_n, q_{n-1})$ and $F(q_{n+1}, q_n)$ leads us to the equation of motion,

$$\partial_t^2 q_n = -c^2 \frac{(q_n - q_{n-1}) + (q_n - q_{n+1})}{a^2}, \quad (1.3)$$

where we have dropped the mass m on the left hand side ($m = 1$). In the current one-dimensional example lattice spacing is seemingly a misnomer, but the example readily extends to d dimensions, where one typically

considers a rectangular lattice of point masses. In $d = 2$ this leads to a two-dimensional rectangular lattice, describing a membrane, in $d = 3$ we have a cubic lattice.

Coming back to our one-dimensional example, the *continuum* limit $a \rightarrow 0$ leads us to

$$\frac{q_n - q_{n-1}}{a} \rightarrow \partial_x \phi, \quad (1.4)$$

the differences turn into derivatives. We now also make this limit manifest in the position variable q_n . Instead of the single position variable we introduce the density of the mass points with

$$q_n \simeq \phi(na) \xrightarrow{a \rightarrow 0} \phi(x), \quad \text{with } x = na \in \mathbb{R}. \quad (1.5)$$

with a 'density' field $\phi(x)$. Collecting all these definitions and limits we are led to

$$\begin{aligned} \partial_t^2 q_n &= -c^2 \frac{(q_n - q_{n-1} + q_n - q_{n+1})}{a^2} \\ &\downarrow \\ \partial_t^2 \phi(t, x) &= c^2 \partial_x^2 \phi(t, x). \end{aligned} \quad (1.6)$$

Equation (1.6) entails that in the limit $a \rightarrow 0$ the difference interactions turn into derivative (kinetic) terms. As the basic object in quantum field theory we will make use of the action of the theory at hand. In the current example the action of this system of oscillating masses is given by

$$S[q] = \int dt \mathcal{L}(q(t), \dot{q}(t), t) = \int dt a \sum_i \frac{1}{2} \left[(\dot{q}_i)^2 - c^2 \frac{(q_{i+1} - q_i)^2}{a^2} \right], \quad (1.7)$$

with $\dot{q}_i = \partial_t q_i$. The Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_n} - \frac{\partial \mathcal{L}}{\partial q_n} = 0, \quad \text{with } \frac{\partial q_i}{\partial q_n} = \delta_{ni} = \frac{\partial \dot{q}_i}{\partial \dot{q}_n}, \quad (1.8)$$

for a mass point q_n lead to (1.3).

Performing the continuum limit $a \rightarrow 0$ in (1.7) the action turns into that of a classical scalar field theory

$$S[\phi] = \int dt \int dx \left((\partial_t \phi)^2 - c^2 (\partial_x \phi)^2 \right), \quad \text{with } \lim_{a \rightarrow 0} a \sum_n f(na) = \int dx f(x). \quad (1.9)$$

In general dimensions the action of a scalar field ϕ can be written as

$$S[\phi] = \int d^d x \left(\frac{1}{2} (\partial_t \phi)^2 - (\nabla \phi)^2 - V(\phi) \right), \quad (1.10)$$

where V denotes the potential, and we have used the common QFT notation, setting $c = 1$. We close this discussion with two remarks:

- (i) The problem can be simply described by a bunch of (coupled) harmonic oscillators.
- (ii) The action $S[\phi]$ has Poincaré invariance (to be discussed later).

The second example is used to briefly recapitulate some basic facts of the most important example of a classical field theory, electrodynamics. This is also used for establishing some notation. If you feel that you are not familiar with some of the parts in this example, please recapitulate these parts. *Quantum electrodynamics* is discussed in the first part (first term) of the lecture course, and serves as a simple example for the quantisation of gauge theories discussed in the second part of the lecture course. Some familiarity with it may come handy.

Example 1-2: Electrodynamics. Electrodynamics with only photons is a free field theory. It is formulated in terms of the gauge (vector) field A_μ , in quantum electrodynamics this describes the *photon*. Its classical action is given by

$$S[A_\mu] = \int d^4x \mathcal{L}(A_\mu(x), \partial_\mu A_\nu(x)), \quad (1.11)$$

where $x^0 = t$ and $(x^i) = \mathbf{x}$ for $i = 1, 2, 3$, and the Lagrangian \mathcal{L} is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (1.12)$$

with the electromagnetic fieldstrength $F_{\mu\nu}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \text{with} \quad F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} F_{\rho\sigma}, \quad (1.13)$$

where the flat Minkowski metric $\eta_{\mu\nu}$ is used for lowering and raising indices. The diagonal metric ($\eta^{\mu\nu}$) has $\det \eta = -1$ and $\eta^{00} = -\eta^{ii}$ (no sum over i). This leaves us with the options $\eta^{00} = \pm 1$, and in this lecture course we resort to that commonly used in QFT,

$$(\eta^{\mu\nu}) = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad \eta^{\nu\rho} \eta_{\mu\rho} = \eta_\mu{}^\nu = \delta_\mu{}^\nu, \quad (1.14)$$

where the latter relation in (1.14) is the orthogonality relation. Again we close with a few remarks:

- (i) The representation of the Minkowski metric with $\eta^{00} = -1$ is commonly used in general relativity and quantum gravity.
- (ii) For general curved space-times with metrics $g^{\mu\nu}$ the orthogonality relation in (1.14) still holds

$$g^{\nu\rho} g_{\mu\rho} = g_\mu{}^\nu = \delta_\mu{}^\nu, \quad (1.15)$$

In both examples the fields is quantised by inheriting the quantum mechanical quantisation from the underlying discrete systems, this can be depicted as

$$\begin{array}{ccc} q \rightarrow \phi, A_\mu & & q \rightarrow \hat{\phi}, \hat{A}_\mu \\ p \rightarrow \dot{\phi}, \dot{A}_\mu & \xrightarrow{\text{Quantisation}} & p \rightarrow \hat{\pi}_\phi, \hat{\pi}_{A_\mu}, \\ & & \parallel \\ & & \pi_\phi, \pi_{A_\mu} \end{array} \quad (1.16)$$

where the operators on the right hand side of 1 describe the annihilation and creation of particles:

- (i) The Hilbert space construction rests on the operator algebra spanned by e.g., $\hat{\phi}, \hat{\pi}_\phi$. These operators can be represented by a linear superposition of creation operator a^\dagger and annihilation operators a (similar to that defined in the harmonic oscillator). Then, all states in the Hilbert space are created by applying functions $f(a^\dagger)$ of the creation operators a^\dagger to the vacuum state $|\Omega\rangle$ (also noted by $|0\rangle$) which is annihilated by a with $a|\Omega\rangle = 0$. For example, the one particle state reads $|1\rangle \propto a^\dagger|\Omega\rangle$, and the annihilation operator reduces an n -particle state to an $n - 1$ particle state: $a|n\rangle \propto |n - 1\rangle$.
- (ii) Modern particle physics is described by *renormalisable* quantum field theories with spin 0, 1/2, 1 fields:
- scalar fields (spin 0): Higgs
 - fermion fields (spin 1/2): leptons; quarks
 - vector fields (spin 1): photon; W^\pm, Z ; gluons

where renormalisation will be discussed later. As discussed before, most quantum field theories used in statistical physics and condensed matter physics host (non-relativistic) spin 0, 1/2, 1 fields. In this lecture course we will discuss the quantisation of all the above fields, including their quantum phenomenology as well as the important aspect of quantum symmetries and renormalisation.

2. Free Scalar Field

In this chapter we discuss the quantum field theory of a free scalar field, which we have introduced in the first example with the continuum limit / many body limit of harmonic oscillators on a string, leading to the actions (1.9) and (1.10). The scalar field theory is the primary example for a quantum field theory, and also has many applications in physics, ranging from the Higgs field in the Standard Model to condensed matter and statistical physics systems, where it describes (composite) bosonic excitations such as Cooper pairs (Spin 0 & 1) or atomic dimers.

We discuss its classical field theory and the equation of motion in Section 2.1. Continuous Symmetries and the Noether theorem are reviewed in Section 2.2. The Noether theorem and its consequences is chiefly important in quantum field theory, and a deep understanding is pivotal for many applications in quantum field theory. The quantisation of the free scalar field is then discussed in Section 2.3. The canonical commutation relations for a free scalar field follow straightforwardly from the underlying quantum mechanical ones, see 2.3.1. The Hamiltonian operator of the theory is put forward in 2.3.2. The Hilbert space of the quantum field theory is, roughly speaking, the sum of the n -particle Hilbert spaces and is called the *Fock space*. Its construction is discussed in Section 2.3.3.

2.1. Classical Theory

For our first considerations we use a real scalar field $\phi(x)$. Here, scalar refers to the *Poincaré group* \mathcal{P} , and the scalar property entails that ϕ is invariant under *Poincaré transformations* $P \in \mathcal{P}$,

$$\phi(x) \rightarrow P(\phi(x)) = \phi(x), \quad \text{where} \quad P \in \mathcal{P}. \quad (2.1)$$

Poincaré transformations are (global) space-time transformations that leave the scalar product

$$(x - y)^2 = (x_\mu - y_\mu) \eta^{\mu\nu} (x_\nu - y_\nu), \quad (2.2)$$

of space-time differences invariant: To begin with, (2.2) is invariant under space-time reflections, the *discrete* Lorentz transformations, that are hence part of \mathcal{P} . Moreover, a translation of the coordinates with a constant a : $x, y \rightarrow x + a, y + a$ leaves the scalar product invariant as the shift drops out from the difference $x - y$. Finally, the *continuous* Lorentz transformations Λ , rotations and boosts, leave the Minkowski metric invariant,

$$(\Lambda^T \eta \Lambda) = \eta, \quad \text{in components:} \quad \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu}. \quad (2.3)$$

In summary, Poincaré transformations P are given by the pair,

$$P = (\Lambda, a) : \quad x^\mu \rightarrow P(x^\mu) = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad (2.4)$$

and the composition of Poincaré transformations is given by

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1). \quad (2.5)$$

Accordingly, the *Poincaré group* \mathcal{P} is a non-compact non-Abelian Lie group.

We will mostly consider Lorentz-invariant actions, and we exemplify this requirement within the standard scalar field theory, the ϕ^4 -theory. Its action is given by

$$S[\phi] = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)), \quad (2.6)$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \quad V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (2.7)$$

In (2.7), we have introduced a self-interaction term with the interaction strength λ . In view of the many-body limit, in which we have obtained the scalar field theory, the underlying quantum mechanical system is a simple anharmonic oscillator.

For the kinetic term, Lorentz invariance follows from

$$\partial_\mu \phi \partial^\mu \phi \rightarrow \partial_\nu \phi \Lambda_\mu^\nu \Lambda_\rho^\mu \partial^\rho \phi = \partial_\nu \phi \left(\Lambda^T \eta \Lambda \right)_\rho^\nu \partial^\rho \phi \stackrel{(2.3)}{=} \partial_\nu \phi \partial^\nu \phi, \quad (2.8)$$

while it trivially follows for the potential term due to the scalar property of the field ϕ ,

$$V(\phi) \rightarrow V(\phi). \quad (2.9)$$

In summary, the Lagrangian and hence also the action are Lorentz invariant,

$$\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \partial_\mu \phi), \quad \implies \quad S[\phi] \rightarrow S[\phi]. \quad (2.10)$$

We proceed by deriving the general solution of the *equation of motion* (EoM) in the free theory. This general solution turns out to be a simple superposition of plane wave solutions with general coefficients, the latter reflecting the fact, that the free theory is nothing but the continuum limit of a d -dimensional version of the string we have started with. The coefficients of the general solution characterise the density of these harmonic oscillators. This representation serves as the starting point for the canonical quantisation of the theory by using the quantum mechanical commutation relations for the single harmonic oscillators.

The equation of motion follows from the action (2.7) with $\lambda = 0$ from the extrema of the action δS , obtained from a general variation $\delta\phi$ of the field, $\phi \rightarrow \phi + \delta\phi$. We get

$$\begin{aligned} S[\phi + \delta\phi] - S[\phi] &= \int d^4x \left(\partial\phi \partial\delta\phi - m^2 \phi \delta\phi \right) + O(\delta\phi^2) \\ &\stackrel{O(\delta\phi)}{\longrightarrow} \int d^4x \left(\partial_\mu \phi (\partial_\nu \delta\phi) \eta^{\mu\nu} - m^2 \phi \delta\phi \right) \\ &= - \int d^4x \left(\eta^{\mu\nu} \partial_\mu \partial_\nu \phi + m^2 \phi \right) \delta\phi \quad (\text{using partial integration}) \\ &= - \int d^4x \delta\phi (\partial^2 + m^2) \phi, \end{aligned} \quad (2.11)$$

where we used the standard short hand notations $\partial^2 = \partial_\mu \partial^\mu$ and $(\partial\phi)^2 = \partial_\mu \phi \partial^\mu \phi$. We conclude that the scalar field satisfies the *Klein-Gordon* equation,

Klein-Gordon equation

$$(\partial^2 + m^2) \phi(x) = 0. \quad (2.12)$$

The Klein-Gordon equation in (2.12) is the equation of motion for a four-dimensional free scalar field. It describes the free propagation of the scalar field with the mass m^2 . A general solution ϕ of (2.12) is a linear superposition of all solutions of (2.12). Let us start with the solution for the 1+0 dimensional theory, and

subsequently generalise it to d dimensions. Note that 1+0 dimensional quantum field theory simply is quantum mechanics in disguise,

$$\phi(t, \mathbf{x}) \Big|_{1+0 \text{ dim}} = \phi(t) = q(t), \quad \mathcal{L} = \frac{1}{2} \dot{q}^2 - \frac{1}{2} m^2 q^2 - \frac{\lambda}{4} q^4. \quad (2.13)$$

The first two terms correspond to a harmonic oscillator and hence to a free theory, while the last anharmonic term is an anharmonic term is a self-interaction. The equation of motion is the Euler-Lagrange equation

$$\partial_t \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0. \quad (2.14)$$

Using the Lagrangian (2.13) in (2.14) we arrive at the EoM

$$\ddot{q} + m^2 q + \lambda q^3 = 0. \quad (2.15)$$

For $\lambda = 0$ this is the differential equation of a harmonic oscillator, which is solved by a plane wave

$$q(t) = A_0 e^{ikt} \quad \text{with} \quad k^2 - m^2 = 0, \quad (2.16)$$

and hence with $k = \pm m$. When extending to d dimensions (1 time + (d-1) spacial dimensions) to (2.12), the only change is $k^2 \rightarrow k_0^2 - \mathbf{k}^2$. Hence, instead of a discrete set of solutions the continuous set of solutions of the EoM are given by the *on-shell* condition

$$k_0 = \pm \sqrt{\mathbf{k}^2 + m^2}, \quad (2.17)$$

that are related by Lorentz transformations. The general solution ϕ of the Klein-Gordon equation (2.12) is given by a linear superposition of the single solutions (2.17). Hence, ϕ describes a density of coupled harmonic oscillators with

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(\alpha(\mathbf{k}) e^{-ikx} + \alpha^*(\mathbf{k}) e^{ikx} \right) \quad \text{with} \quad \omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}, \quad (2.18)$$

with the spatial momentum density $\alpha(k)$. The two terms in parenthesis are complex conjugates of each other and hence ϕ is real and satisfies (2.12), as the four-momentum $(\omega_{\mathbf{k}}, \mathbf{k})$ satisfies (2.17).

We close the derivation of the general solution of the Klein-Gordon equation with a remark on the properties of the momentum density $\alpha(k)$ under Lorentz transformations: To begin with, the normalisation of the spatial momentum integral with $1/\sqrt{\omega_{\mathbf{k}}}$ is a common but not unique choice, and the choices $1/\omega_{\mathbf{k}}$ and 1 are also found in the literature. As the spatial momentum measure $d^3 k$ and $\omega_{\mathbf{k}}$ are not Lorentz scalars, but the field $\phi(x)$ is, evidently the different choices affect the properties of the respective $\alpha(\mathbf{k})$ under Lorentz transformations as the Lorentz transformation Λ of $\alpha(\mathbf{k})$ has to compensate that of the rest, in short

$$\Lambda[\phi(x)] = \phi(x) \quad \rightarrow \quad \Lambda \left[\frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \alpha(\mathbf{k}) \right] = \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \alpha(\mathbf{k}), \quad (2.19)$$

as the phases $\exp(\pm ikx)$ are Lorentz scalars to begin with.

The rôle of $\omega_{\mathbf{k}}$ and the transformation property of $\alpha(\mathbf{k})$ is more easily seen after a reparametrising of the momentum integral in (2.18) as

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \left(\sqrt{2\omega_{\mathbf{k}}} \alpha(\mathbf{k}) e^{-ikx} + \sqrt{2\omega_{\mathbf{k}}} \alpha^*(\mathbf{k}) e^{ikx} \right) \quad (2.20)$$

Now we use that the spatial momentum integral measure, normalised with $1/2\omega_{\mathbf{k}}$, can be rewritten in a manifestly Lorentz-invariant way,

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} = \int \frac{d^4 k}{(2\pi)^4} (2\pi) \delta(k^2 - m^2) \theta(k^0). \quad (2.21)$$

More precisely, (2.21) is invariant under proper orthochronous Lorentz transformations: the four-momentum measure d^4k is invariant and so is the δ -function as its argument is a Lorentz scalar. Finally, the proper, orthochronous Lorentz transformations keep the sign of k_0 .

Equation (2.21) can be derived by using

$$\delta(g(x) - g(a)) = \frac{1}{|g'(a)|} \delta(x - a), \quad (2.22)$$

where $g(x)$ is any C^1 function. This entails that

$$\delta(k^2 - m^2) = \delta\left(\left(k^0\right)^2 - \mathbf{k}^2 - m^2\right) = \delta\left(k_0^2 - \omega_{\mathbf{k}}^2\right) = \frac{1}{|2\omega_{\mathbf{k}}|} \delta(k_0 - \omega_{\mathbf{k}}). \quad (2.23)$$

Inserting (2.23) on the right hand side of (2.21), we are led to the left hand side,

$$\int \frac{d^4k}{(2\pi)^4} (2\pi) \delta(k^2 - m^2) \theta(k^0) = \int \frac{d^4k}{(2\pi)^4} (2\pi) \frac{1}{|2\omega_{\mathbf{k}}|} \delta(k_0 - \omega_{\mathbf{k}}) \theta(k^0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}}. \quad (2.24)$$

Accordingly, $\sqrt{\omega_{\mathbf{k}}}\alpha(\mathbf{k})$ is a Lorentz scalar, or, put differently, $\alpha(\mathbf{k})\alpha^*(\mathbf{k})$ transforms inversely to the spatial momentum measure. This is important for the density interpretation of the α 's. For example, it follows that $\int \frac{d^3k}{(2\pi)^3} \alpha(\mathbf{k})\alpha^*(\mathbf{k})$ is a Lorentz scalar, and this quantity relates to the (electric) charge of a charged scalar field. In view of this application we close this Section with extending the real scalar field to a *complex scalar field* $\phi \in \mathbb{C}$. We know from classical electrodynamics, that charged fields or particles such as the electrons transform under the electromagnetic gauge group $U(1)$ by multiplication with a phase. In the present scalar case this requires a complex scalar field, and we define

$$\phi(x) = \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)], \quad (2.25)$$

where ϕ_1 and ϕ_2 are both real scalar fields and the phase transformation is given by $\phi \rightarrow e^{i\omega}\phi$ with a *global* phase ω : $\partial_\mu\omega = 0$. The free action is given by

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi), \quad (2.26)$$

with

$$\mathcal{L}(\phi, \partial_\mu\phi) = \partial_\mu\phi \partial^\mu\phi^* - m^2\phi\phi^* = \frac{1}{2} [(\partial\phi_1)^2 + (\partial\phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)]. \quad (2.27)$$

In short, (2.26) is the sum of the free actions of the fields ϕ_1 and ϕ_2 . The reader can readily convince themselves, that (2.26) is the only free action which is invariant under $U(1)$ rotations.

Then the general solution of (2.12) is given by

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} (\alpha(\mathbf{k}) e^{-ikx} + \beta^*(\mathbf{k}) e^{ikx}) \quad \text{with} \quad \omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}. \quad (2.28)$$

We close this Section with two remarks:

- (i) As mentioned above, the action (2.26) is invariant under multiplication of ϕ with a global phase $e^{i\omega}$. This global $U(1)$ symmetry implies a conserved charge, as will be discussed in the subsequent section. Moreover, if augmenting the free action with a self-interaction such as the ϕ^4 -term in (2.13), the $U(1)$ -symmetry dictates

$$V(\phi) = m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2, \quad (2.29)$$

the potential can only depend on the $U(1)$ -invariant $\phi^*\phi$.

- (ii) An action, that is invariant under a *local* $U(1)$ rotation, $\phi(x) \rightarrow e^{i\omega(x)}\phi(x)$ is *gauge invariant*. The action (2.26) is not gauge-invariant, this would require the introduction of a $U(1)$ gauge field, to be discussed later.

2.2. Noether Theorem

Symmetries play a pivotal rôle in quantum field theories. Their applications range from the direct deduction of physics simply from symmetry arguments, i.e. the exclusion of processes based on their lack of symmetry, to general construction principles (and hence restrictions) of quantum field theories. The latter is particularly relevant for the construction of Effective Field Theories (EFTs) in Beyond Standard Model (BSM) physics. Most of these powerful symmetry principles originate in Noether's theorem. Loosely speaking it states, that

Continuous symmetries of the action lead to a conserved current density and a conserved charge.

These symmetries can be space-time symmetries as translation invariance or rotation invariance already discussed in classical mechanics. The respective conserved charges in these cases were the (spatial) total momentum and (total) angular momentum. In the present case of Poincaré-invariant field theories these space time symmetries also includes boost invariance, and we will discuss the conserved currents and charges.

Moreover, we also have important internal symmetries such as the global $U(1)$ symmetry that we have briefly introduced for the complex scalar field at the end of the last Section. As indicated there, this symmetry relates to the conserved electric charge of this field, and we used this simple example for elucidating the Noether theorem for a continuous symmetry with one parameter.

Example 2-1: $U(1)$ -symmetry of action of a complex scalar field.

A simple example, that captures much of the general structure, is given by the field theory with a complex scalar with the action (2.26). The action is invariant the symmetry $\phi \rightarrow e^{i\omega}\phi$ as discussed at the end of the last chapter. An infinitesimal transformation with $\omega = \epsilon$ and $\epsilon \rightarrow 0$ is described by

$$e^{i\epsilon}\phi = \phi + \delta_\epsilon\phi + O(\epsilon^2), \quad \text{with} \quad \delta_\epsilon\phi = i\epsilon\phi, \quad \Delta\phi = \left. \frac{\partial\delta_\epsilon\phi}{\partial\epsilon} \right|_{\epsilon=0} = i\phi. \quad (2.30)$$

In (2.30), $\delta_\epsilon\phi$ is the infinitesimal transformation, and $\Delta\phi$ (or rather its ϕ -derivative) is the *generator* of the transformation. Evidently the action (2.26) is invariant under such the *global* transformation with $\partial_\mu\epsilon = 0$. Let us now consider a space-time dependent $\epsilon(x)$. Then, the action is shifted with

$$S[\phi(x)] \rightarrow S[\phi(x) + i\epsilon(x)\phi(x)] = S[\phi(x)] - i \int d^4x \partial_\mu\epsilon(x) [\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi] + O(\epsilon^2). \quad (2.31)$$

With a partial integration of the last term we arrive at

$$S[\phi(x) + i\epsilon(x)\phi(x)] \simeq S[\phi(x)] + \int d^4x \epsilon(x) \partial_\mu j^\mu, \quad \text{with} \quad j^\mu = i [\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi]. \quad (2.32)$$

This is an important result. It entails that for *global* symmetries the *local* variation of the action can be written as an integral of a total derivative $\partial_\mu j^\mu$, multiplied by $\epsilon(x)$. In turn, without the global symmetry, the non-invariant term cannot be written as a total derivative.

Clearly, the action is not invariant under the space-time dependent $U(1)$ -transformation. We also see, that for choosing constant ϵ , the ϵ -dependent terms reduce to a total derivative and vanishes upon integration, as it must. Now we use that $\delta_\epsilon\phi$ simply is a specific variation of the field in the direction of the symmetry. This entails already that the term linear in ϵ in (2.31) has to vanish on solutions of the equation of motion, as the latter are the stationary points of the action under a general variation. This leads us to

$$\partial_\mu j^\mu \Big|_{\text{EoM}} = 0, \quad (2.33)$$

and hence the theory exhibits a conserved current, the *Noether current*. Equation (2.33) follows with the definition (2.32) and the EoM (2.12). We leave the explicit proof of (2.33) to the reader.

The current j^μ can be easily derived from the Lagrangian as it is only generated from the terms dependent on $\partial_\mu\phi$. We have

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta\phi, \quad \text{or} \quad \partial_\mu j^\mu(x) = \left. \frac{\delta S[\phi]}{\delta\epsilon(x)} \right|_{\epsilon=0}, \quad (2.34)$$

up to a global minus sign between the two definitions in (2.34). The latter definition yields (2.32) including the global sign. It is the far more convenient one but requires some knowledge about functional derivatives. While not required at the present state of the lecture course, we suggest to the reader to get acquainted with functional derivatives as soon as possible, they facilitate quite some derivations and computations.

We also see that an additional term $\int d^4x \epsilon(x) \partial_\mu J^\mu$ in (2.31) would not have changed the existence of a conserved current. It simply would have led to a subtraction of J^μ on the right hand side of the first definition in (2.34), and would not have altered the second one. The latter fact again emphasises the naturality of using functional derivatives.

We emphasise again that (2.33) only holds true in the presence of a global symmetry. In turn, with a global symmetry (or for general variations) the right hand side is non-vanishing. An additional term occurs, namely $(\partial\mathcal{L}/\partial\partial_\mu\phi)\Delta\phi$.

The conserved current (2.33) leads to a *Noether charge* Q , which is conserved on the EoM. We define

$$Q = \int d^3x j^0(x) = i \int d^3x [\phi^* \partial_t \phi - (\partial_t \phi^*) \phi]. \quad (2.35)$$

Using $\partial_t j^0 = \nabla \mathbf{j}$ from (2.33) we are led to

$$\partial_t Q|_{\text{EoM}} = \int d^3x \partial_t j^0(x) \stackrel{\text{EoM}}{=} \int d^3x \nabla \mathbf{j}(x) = 0, \quad (2.36)$$

which can be also proven directly with the EoM. Up to a normalisation, the charge (2.35) is nothing but the electric charge of the complex scalar field.

We now proceed with the derivation of the general Noether theorem. Each step can be mapped back to our simple example discussed above. We consider an infinitesimal *global* symmetry transformation δ_ϵ with

$$\phi(x) \rightarrow \phi(x) + \delta_\epsilon \phi(x). \quad (2.37)$$

As has been discussed in our example, the global symmetry is described by a constant infinitesimal parameter ϵ with $\partial_\mu \epsilon = 0$. The transformation (2.37) is a symmetry of the action for

$$S[\phi(x)] \rightarrow S[\phi(x) + \delta_\epsilon \phi(x)] = S[\phi(x)]. \quad (2.38)$$

For the general case it is convenient to consider the transformation of the Lagrangian \mathcal{L} . Equation (2.38) holds if the Lagrangian transforms with

$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon \partial_\mu J^\mu(\phi), \quad (2.39)$$

as has been also discussed briefly in our introductory example below (2.34). Inserting the transformed Lagrangian on the right hand side of (2.39) into the action, the last term, $\partial_\mu J^\mu$, vanishes upon space-time integration as it is a divergence. Strictly speaking this necessitates the absence of surface terms, that is $(\|x\|^3 J^\mu[\phi])(\|x\| \rightarrow \infty) = 0$ in 1 + 3 dimensions, which we assume in the following. Then it follows

$$S[\phi] = \int d^4x \mathcal{L} \rightarrow \int d^4x \mathcal{L} + \epsilon \int d^4x \partial_\mu J^\mu(\phi) = S[\phi]. \quad (2.40)$$

In our simple example we have $J^\mu = 0$. Symmetries with $J^\mu \neq 0$ are e.g. space-time symmetries that lead to a conserved energy momentum tensor and are discussed later.

Let us now assume that the theory has a *global* symmetry leading to (2.39). We have already seen in our example, that this leads to a local current, that is conserved on the equations of motion. Accordingly, we consider the explicit *global* symmetry transformation of the Lagrangian \mathcal{L} ,

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\epsilon \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\epsilon \phi. \quad (2.41)$$

The last term on the right hand side of (2.41) depends on the derivative of the variation. We rewrite this term as a total derivative and a term, where the derivative hits the $\partial_\mu \phi$ -variation of the Lagrangian. This term combines with the first term on the right hand side of (2.41) to the equation of motion, indicated in red,

$$\begin{aligned} \mathcal{L} &\rightarrow \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_\epsilon \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\epsilon \phi \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta_\epsilon \phi \\ &= \mathcal{L} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\epsilon \phi \right) + (\text{EoM}) \delta_\epsilon \phi \\ &\stackrel{!}{=} \mathcal{L} + \epsilon \partial_\mu J^\mu, \end{aligned} \quad (2.42)$$

The steps in (2.42) are the same as for the derivation of the Euler-Lagrange equations for the field theory (with variations restricted to the global symmetry). It is the identification of the non-invariant terms with a divergence $\epsilon \partial_\mu J^\mu$ in the last line of (2.42), that only holds for global symmetries. In summary we are led to the relation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\epsilon \phi - \epsilon J^\mu \right) = -(\text{EoM}) \delta_\epsilon \phi. \quad (2.43)$$

that holds for general field configurations ϕ . For fields that satisfy the EoM, the right hand side vanishes and we are left with a conserved current, that is defined by the linear order in ϵ of the right hand side of (2.43). Using

$$\left. \frac{\partial \delta_\epsilon \phi}{\partial \epsilon} \right|_{\epsilon=0} = \Delta \phi, \quad (2.44)$$

we arrive at the general definition of the *Noether current* for a one-parameter global symmetry,

Conserved current (for a one-parameter global symmetry of a single scalar field)

$$j^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - J^\mu, \quad \text{with} \quad \partial_\mu j^\mu = 0. \quad (2.45)$$

The conservation law can also be expressed in terms of the

Noether charge (for a one-parameter global symmetry of a single scalar field)

$$Q(t) := \int d^3x j^0(t, \mathbf{x}) \quad \text{with} \quad \partial_t Q(t) = 0. \quad (2.46)$$

The Noether theorem extends readily to field theories with more than one field, indeed our example is such a case with (2.25). Then, the first term in j^μ needs to be replaced by a sum of the variations of the different fields.

Also, the generalisation to the case of symmetries with N parameters $r = 1, \dots, N$ is done by extending $\delta_\epsilon \rightarrow \delta_{\epsilon_r}$. This leads us to the generators of the symmetry, $\Delta_r \phi_i$ with

$$\Delta_r \phi_i = \left. \frac{\partial \delta_\epsilon \phi_i}{\partial \epsilon_r} \right|_{\epsilon=0}, \quad (2.47)$$

Repeating the derivation above in this general case leads us to the general definition of N Noether currents,

Conserved current (general case)

$$j_r^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta_r \phi_i - J_r^\mu, \quad \text{with} \quad \partial_\mu j_r^\mu = 0, \quad r = 1, \dots, N. \quad (2.48)$$

The N conserved currents give rise to N conserved Noether charges

Noether charge (general case)

$$Q_r(t) := \int d^3x j_r^0(t, \mathbf{x}), \quad \text{with} \quad \partial_t Q_r(t) = 0, \quad r = 1, \dots, N. \quad (2.49)$$

We close this derivation with a remark on general symmetries that involve both, transformation of the field as well as of its argument, the space-time variable x . This suggests to split $\delta_\epsilon \phi$ into an infinitesimal transformation of the field ϕ , $\delta_{\epsilon|_x} \phi$, and the symmetry variation of the space-time variable $\delta_{\epsilon} x_\mu$. We obtain

$$\delta_\epsilon \phi = \delta_{\epsilon|_x} \phi + \delta_\epsilon x_\mu \partial^\mu \phi, \quad \text{with} \quad \Delta_r^{(\phi)} \phi_i = \left. \frac{\partial \delta_{\epsilon|_x} \phi_i}{\partial \epsilon_r} \right|_{\epsilon=0}, \quad \Delta_r x = \left. \frac{\partial \delta_\epsilon x}{\partial \epsilon_r} \right|_{\epsilon=0}. \quad (2.50)$$

This split of the transformation translates into an according one for the Noether currents j^μ : they have a part that stems from the symmetry variation of the field, $\Delta_r^{(\phi)} \phi$, and the one which stems from the symmetry variation of the space-time variable x . Inserting the split (2.50) into the definition of the Noether current, (2.48), leads us to

General Noether current

$$j_r^\mu = \Delta_r^{(\phi)} \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \Delta_r x^\nu \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - J_r^\mu. \quad (2.51)$$

As a first relevant application of the Noether theorem we discuss the conservation of the *energy-momentum* or *stress-energy tensor*. The related conservation laws entail momentum and energy conservation. The underlying global symmetries are that of translation invariance of physics under a spatial as well as temporal shift of the laboratory system: the fundamental law of physics do not change with time or space. Hence we consider an infinitesimal global ($\partial_\mu \epsilon = 0$) space-time translation, $x \rightarrow x + \epsilon$, of the field

$$\phi(x) \rightarrow \phi(x + \epsilon) = \phi(x) + \epsilon^\mu \partial_\mu \phi(x) + \mathcal{O}(\epsilon^2), \quad \text{with} \quad \Delta_\mu \phi = \eta_\mu^\nu \partial_\nu \phi, \quad (2.52)$$

or $\Delta_\mu^{(\phi)} \phi = 0$ and $\Delta_\mu x^\nu = \eta_\mu^\nu$. Translations have four parameters, $r = \mu = 0, \dots, 3$. Note also that in (2.52) the *scalar* property of the field has been used, it is invariant under Poincaré transformations. This is different for fermion (spin 1/2) and vector fields (spin 1) to be considered later. We proceed by discussing the current J_r^μ . Applying the infinitesimal transformations (2.52) to the Lagrangian amounts to simply taking a space-time derivative of \mathcal{L} , as the only x -dependence of the Lagrangian resides in the fields and its derivatives. We get

$$\mathcal{L}(\phi(x), \partial_\mu \phi(x)) \rightarrow \mathcal{L} + \epsilon^\mu \partial_\mu \mathcal{L} = \mathcal{L} + \epsilon^\nu \partial_\mu \eta_\nu^\mu \mathcal{L}, \quad (2.53)$$

leading us to

$$J^\mu_\nu = \eta^\mu_\nu \mathcal{L}. \quad (2.54)$$

Inserting (2.54) in the definition of the Noether current (2.48) or (2.51) leads us to the energy-momentum tensor,

Energy-momentum tensor (or stress-energy tensor)

$$T^\mu_\nu := J^\mu_\nu = \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \eta^\mu_\nu \mathcal{L} \quad \text{with} \quad \partial_\mu T^{\mu\nu} = 0. \quad (2.55)$$

Consequently, we have four conserved Noether currents as well as the respective Noether charges,

$$P^\mu = \int d^3x T^{0\mu}, \quad (2.56)$$

the 4-momentum. The energy-density for time translations is given by the zeroth component of the 4-momentum, namely

$$\begin{aligned} P^0 &= \int d^3x T^{00} = \int d^3x \left((\partial^0 \phi) \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} - \mathcal{L} \right) = \int d^3x \left((\partial^0 \phi) \pi - \mathcal{L} \right) \\ &= \int d^3x \mathcal{H} = H, \end{aligned} \quad (2.57)$$

with the canonical momentum of the field,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}, \quad (2.58)$$

and the Hamiltonian density

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L}, \quad (2.59)$$

of the Hamiltonian H . The result (2.57) was to be expected, as P^0 is the Noether charge that originates from the invariance of the system under translations in time. The Hamiltonian of a theory generates the time evolution of the system. Note also that the Hamiltonian density and the Hamiltonian are positive definite functionals. For instance, the Hamiltonian density for the standard case of a scalar field theory with $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$ is

$$\mathcal{H} = \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi). \quad (2.60)$$

We also remark that the covariance of P^μ is not apparent. Note however, that $\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$ transforms as the 0-component of a contravariant vector. Hence, $\int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$ has the transformation properties of the measure $d^3x dx^0$, which is Lorentz-invariant. We leave the rest of the explicit proof of covariance of P^μ to the reader.

We proceed with the spatial components P^i of the Noether charges stemming from invariance of the system under space-time translations. We already discussed, that P^0 generates time translations, and the P^i are the generators of spatial translations of the fields within the *Poisson brackets*. To see this we use the explicit form of the P^i ,

$$P^i = \int d^3x T^{0i} = \int d^3x \pi \partial^i \phi = \left(\int d^3x \pi \nabla \phi \right)^i. \quad (2.61)$$

Inserting (2.61) into the Poisson brackets with the fields generates infinitesimal spatial translations,

Generator of spatial translations with Poisson brackets

$$\{\mathbf{P}(\mathbf{x}), \phi(\mathbf{x})\} = -\nabla\phi, \quad \text{with} \quad \{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) \quad \text{and} \quad \{\phi(\mathbf{x}), \phi(\mathbf{y})\} = 0, \quad (2.62)$$

with $(P^\mu) = (P^0, \mathbf{P})$. The property (2.62) is sustained in the quantisation, which promotes the Poisson brackets to commutators of the field (and momentum) operators. We close the discussion of the energy-momentum tensor and the respective Noether charges with three remarks:

- (i)* In general the canonical energy-momentum tensor is not symmetric, i.e. $T^{\mu\nu} \neq T^{\nu\mu}$. This originates in $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$. However, $T^{\mu\nu}$ can always be symmetrised by adding a divergence to the canonical EMT. Its symmetry is an important property for the coupling to gravity. An alternative -symmetric- definition results from the variation of the action with respect to the metric $g^{\mu\nu}$:

$$T_{sym}^{\mu\nu} = \frac{2}{\sqrt{-\det g}} \left. \frac{\delta S}{\delta g^{\mu\nu}} \right|_{g=\eta}, \quad (2.63)$$

with

$$\frac{\delta g^{\alpha\beta}(x)}{\delta g^{\mu\nu}(y)} = \frac{1}{2} (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu) \delta(x - y), \quad \text{and} \quad \frac{\delta \sqrt{-g(x)}}{\delta g^{\mu\nu}(y)} = -\frac{1}{2} \sqrt{-g(x)} g_{\mu\nu} \delta(x - y). \quad (2.64)$$

The derivation of the EMT from the variation of the metric inherits the symmetry of the latter.

- (ii)* We have already suggested the use of functional derivatives in the derivation of the Noether theorem instead of using variations. Functional derivatives are conveniently defined by

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x - y). \quad (2.65)$$

Note also that the scalar field is invariant under a combined transformation of field and space-time variable, i.e. $\phi'(x') = \phi(x)$, and so far we have only used $\phi(x) \rightarrow \phi(x')$. For the combined transformation we are led to

$$\begin{aligned} \Delta\phi &= 0 \\ \Delta_\rho x^\nu &= \eta_\rho^\nu \\ J_\rho^\mu &= 0 \quad (\mathcal{L}' = \mathcal{L}). \end{aligned} \quad (2.66)$$

- (iii) Finally, we briefly revisit the charge of a complex scalar field, used as an example in the beginning. The (free) Lagrangian of a complex scalar field, (2.25), is given by

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*, \quad (2.67)$$

and is invariant under global $U(1)$ rotations, see (2.30), leading to $J^\mu = 0$. We choose

$$\Delta\phi = -i\phi, \quad \Delta\phi^* = i\phi^*, \quad (2.68)$$

related to $\phi \rightarrow e^{-i\epsilon} \phi$, and we obtain the Noether current of the complex field,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^* = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} i\phi^* = i \left(\phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi \right), \quad (2.69)$$

in agreement with (2.32). The Noether current is the spatial integral of j^0 :

Noether charge of a complex scalar field

$$Q = \int d^3x j^0 = i \int d^3x \left(\phi^* \partial_t \phi - (\partial_t \phi^*) \phi \right). \quad (2.70)$$

We have already noted in our introductory example, that the (Noether) charge is conserved on the equation of motion. This is now checked explicitly,

$$\begin{aligned} \dot{Q} \Big|_{\text{EoM}} &= i \int d^3x \left(\dot{\phi}^* \dot{\phi} - \dot{\phi}^* \dot{\phi} + \phi^* \partial_t^2 \phi - (\partial_t^2 \phi^*) \phi \right) \\ (2.12) \rightarrow &= i \int d^3x \left(\phi^* (\nabla^2 - m^2) \phi - [(\nabla^2 - m^2) \phi^*] \phi \right) \\ &= i \int d^3x \left(\phi^* \nabla^2 \phi - \phi^* \nabla^2 \phi \right) = 0, \end{aligned} \quad (2.71)$$

where we have performed twice a partial integration for the last identity as well as assuming the absence of boundary terms. On the equation of motion we can also use (2.28) for rewriting the Noether charge (2.70) as a momentum space integral,

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(\alpha^*(\mathbf{p}) \alpha(\mathbf{p}) - \beta^*(\mathbf{p}) \beta(\mathbf{p}) \right), \quad (2.72)$$

and the global sign of (2.72) is changed by changing (2.68) to $\Delta\phi = i\phi$, $\Delta\phi^* = -i\phi^*$ as in (2.30).

In the next section we discuss the quantisation of the scalar theory. As already mentioned, in the quantisation procedure the coefficients α, α^* and β, β^* are elevated to annihilation and creation operators for particles and anti-particles, respectively. Then, their combination in (2.72) is simply the quantum field theoretical analogue of the number operator in quantum mechanics.

2.3. Quantisation

In the introduction, Chapter 1, we have argued, that quantum field theory can be understood as the field-theoretical limit of quantum mechanics. This entails, that the canonical quantisation relations of the position operator and the momentum operator of a single quantum mechanical system are simply carried over to the field theory, see Figure 2.1. This limit has been roughly described in the beginning of this lecture course, see Figure 1.1, and is put to work in the present section.

In Figure 2.1 we have set $\hbar = 1$ and $c = 1$ on the right hand side. This leads to convenient simplifications in all computations. The results are then presented in these natural units and the standard units such as SI can be easily restored. In thermal quantum field theory or quantum gravity these units are typically augmented with $k_B = 1$ (Boltzmann constant units) or $m_{\text{Planck}} = 1$ (Planck mass units).

2.3.1. Canonical commutation relations

The above general picture entails that the canonical quantisation in quantum field theory can be performed analogously to quantum mechanics. For emphasising this analogy, let us briefly recapitulate quantum mechanics as 1+0 dimensional quantum field theory. The generalisation to general dimensions is straightforward and is done subsequently.

Quantum Mechanics	\Rightarrow	Quantum Field Theory
$[\hat{q}, \hat{p}] = i\hbar$		$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$
$[\hat{q}, \hat{q}] = 0 = [\hat{p}, \hat{p}]$		$[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = 0 = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})]$

Figure 2.1.: Quantum field theory from the many body limit of quantum mechanics. The canonical quantisation relations of QM are transported to the canonical quantisation relations of QFT.

Example 2-2: Quantum mechanics as 1+0-dimensional quantum field theory.

Reducing the dimensions to $d = 1 + 0$, the spatial integration in the action is removed. This integration is an integration over the density of (quantum) mechanical systems. We will stick to our QFT convention $\hbar = 1$, $c = 1$, as this facilitates the generalisation to quantum field theory. In summary we are left with the action of a harmonic oscillator,

$$S[q] = \int dt \mathcal{L} = \int dt \left(\frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \right), \quad (2.73)$$

where, as before, we also set the mass to unity. The corresponding Hamiltonian reads

$$H = p\dot{q} - \mathcal{L} = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2, \quad \text{with} \quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}. \quad (2.74)$$

The quantisation entails $p, q \rightarrow \hat{p}, \hat{q}$, with the canonical commutation relation $[\hat{q}, \hat{p}] := i$ (with $\hbar = 1$). We introduce creation operators a^\dagger and annihilation operators a as

$$\hat{q} = \frac{1}{\sqrt{2\omega}} (a + a^\dagger), \quad \hat{p} = -i \sqrt{\frac{\omega}{2}} (a - a^\dagger). \quad (2.75)$$

The (canonical) commutation relations for the creation and annihilation operators follow from that of \hat{q} and \hat{p} ,

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0. \quad (2.76)$$

The Hamilton operator can then be written in terms of the creation and annihilation operators,

$$\hat{H} = \left(a^\dagger a + \frac{1}{2} \right) \omega, \quad (2.77)$$

where $(1/2)\omega$ corresponds to the vacuum energy. In the Heisenberg picture the operators evolve with time, whereas the states are stationary, i.e.

$$i \frac{\partial}{\partial t} \hat{O}(t) = [\hat{O}(t), \hat{H}], \quad \text{with} \quad \hat{O}(t) = e^{i\hat{H}t} \hat{O}(0) e^{-i\hat{H}t}. \quad (2.78)$$

Now we use (2.78) for showing that the commutation relation do not evolve in time: the position and momentum operators always admit canonical commutation relations,

$$[\hat{q}(t), \hat{p}(t)] = e^{i\hat{H}t} [\hat{q}, \hat{p}] e^{-i\hat{H}t} = i. \quad (2.79)$$

A final remark concerns a first step into the direction of a quantum field theory, namely the extension of the present harmonic operator to the superposition of many harmonic oscillators as discussed in the beginning of

the lecture course. Then, the total Hamiltonian is a sum of the single ones, in the most general case also with different frequencies ω_i , that is $\sum_i H(\hat{q}_i, \hat{p}_i; \omega_i) = \hat{H}(\hat{a}_i^\dagger, \hat{a}_i; \omega_i)$. The commutation relation of the operators stay the same, and operators stemming from the different subsystems commute. This leads us to

$$[\hat{q}_i, \hat{p}_j] := i \delta_{ij}, \quad [a_i, a_j^\dagger] = \delta_{ij}. \quad (2.80)$$

With this remark we close the brief recapitulation of some basic properties of the harmonic oscillator.

The astute reader may have noticed that all derivations and properties discussed in the quantum mechanical example above carry over to quantum field theory: the only difference of a field operator $\hat{\phi}$ in QFT and the position operator \hat{q} in QM is the integration over spatial momentum in the former. This is a linear operation and we expect that we simply have to change the Kronecker- δ 's in (2.80) into δ -functions. With this introductory remark we proceed to the 1+3 dimensional theory, which serves as the generic case. From now on we shall drop the hat marking the operators, it is understood implicitly. The Hamiltonian density for a *real scalar field* operator is

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = \frac{1}{2} \left[\pi(t, \mathbf{x})^2 + \phi(t, \mathbf{x}) (-\Delta + m^2) \phi(t, \mathbf{x}) \right], \quad (2.81)$$

with the Laplacian $\Delta = \nabla^2$ and the field momentum operator π . The Hamiltonian H is the spatial integral of the density \mathcal{H} . For example in 3 (spatial) dimensions we have $H = \int d^3x \mathcal{H}$, see (2.57). The respective Lagrangian is given by a Legendre transform and reads,

$$\mathcal{L} = \frac{1}{2} \left(\partial_0 \phi \partial^0 \phi - (\nabla \phi)^2 - m^2 \phi^2 \right), \quad \text{with} \quad \pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)}(t, \mathbf{x}) = \partial^0 \phi = \dot{\phi}. \quad (2.82)$$

the commutation relations of the field operator ϕ with the momentum operator π follows from that in the quantum mechanical case: for a system of harmonic oscillators we have, see (2.80),

$$[q_i, \pi_j] = i \delta_{ij}. \quad (2.83)$$

In the many-body limit discussed in the beginning the operators q_i and π_j turn into spatial densities of quantum mechanical operators. More precisely their products q^2, p^2, qp in the quantum mechanical Lagrangian or Hamiltonian turn into densities proportional to the inverse spatial volume element $1/a^{d-1}$. For example we have in one spatial dimension,

$$\frac{p_i^2(t)}{a} \xrightarrow{a \rightarrow 0} \pi(t, x)^2, \quad a \sum_{i=-\infty}^{\infty} \xrightarrow{a \rightarrow 0} \int dx, \quad (2.84)$$

where we have already assumed a system with infinite spatial extent. In $d-1$ spatial dimensions the index i turns into a vector $\mathbf{i} = (i_1, \dots, i_{d-1})^T$, and we have

$$\sum_{\mathbf{i}} p_{\mathbf{i}}(t) p_{\mathbf{i}}(t) = a^{d-1} \sum_{\mathbf{i}} \left(\frac{p_{\mathbf{i}}(t)}{a^{\frac{d-1}{2}}} \right) \left(\frac{p_{\mathbf{i}}(t)}{a^{\frac{d-1}{2}}} \right) \xrightarrow{a \rightarrow 0} \int d^{d-1}x \pi(t, \mathbf{x}) \pi(t, \mathbf{x}), \quad (2.85)$$

with $\pi(t, \mathbf{x})$ being defined analogously to the one dimensional example.

Consequently, the quantum mechanical commutation relations (2.83) have to be multiplied with a^{d-1} , where the exponent simply is the (inverse) dimension of space-time. Thus, in the many-body limit the product $1/a^{d-1} \delta_{ij}$ turns into the spatial δ -function, $\delta(\mathbf{x} - \mathbf{y})$, and we obtain canonical commutation relations for the field operators,

Canonical commutation relations

$$\begin{aligned} [\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] &= i \delta(\mathbf{x} - \mathbf{y}) \\ [\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] &= [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0. \end{aligned} \quad (2.86)$$

The field operators ϕ, π , that satisfy the canonical commutation relations (2.86) operators, define the free scalar

quantum field theory. We add a few remarks:

- (i) The field operator ϕ satisfies the EoM, as do its matrix elements $\langle A|\phi|B\rangle$ of time-independent states $|A\rangle, |B\rangle$.
- (ii) The free field theory describes a (coupled) set of harmonic oscillators due to the presence of $\phi\Delta\phi$ in the action. In Fourier space this term turns into $\phi(-p)\mathbf{p}^2\phi(p)$. Consequently, we can *diagonalise* \mathcal{L} and \mathcal{H} in momentum space. This is done analogously to the 1+0 dimensional theory.

The field and momentum operators can be written as spatial momentum integrals by elevating the classical solutions (2.18) to operators,

Field operator

$$\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{-ipx} + a^\dagger(\mathbf{p}) e^{ipx} \right) \\ \pi(x) = \partial^0\phi(x) &= -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a(\mathbf{p}) e^{-ipx} - a^\dagger(\mathbf{p}) e^{ipx} \right),\end{aligned}\quad (2.87)$$

with the on-shell frequency (2.18),

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \quad (2.88)$$

In (2.87) the coefficients a, a^\dagger are operators that inherit their commutation relations from (2.86). For the derivation of the commutation relations we go to momentum space. The Fourier transform is defined as

$$\begin{aligned}\tilde{\phi}(p) &= \int d^4x e^{ipx} \phi(x) \\ \phi(x) &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{\phi}(p).\end{aligned}\quad (2.89)$$

With the spatial Fourier transform ($t = 0$) we get the representation of the field operator and its canonical momentum, $\tilde{\phi}$ and $\tilde{\pi}$ respectively, in momentum space,

Field operator in momentum space

$$\begin{aligned}\tilde{\phi}(\mathbf{p}) &= \int d^3x e^{-ipx} \phi(x) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) + a^\dagger(-\mathbf{p}) \right), \\ \tilde{\pi}(\mathbf{p}) &= \int d^3x e^{-ipx} \partial^0\phi(x) = -i \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a(\mathbf{p}) - a^\dagger(-\mathbf{p}) \right).\end{aligned}\quad (2.90)$$

For the derivation of the commutation relations of the operators a, a^\dagger we insert (2.90) into (2.86), and derive the commutation relations of $\tilde{\phi}(\mathbf{p}), \tilde{\pi}(\mathbf{p})$,

$$\begin{aligned}[\tilde{\phi}(\mathbf{p}), \tilde{\pi}(\mathbf{q})] &= \int d^3x d^3y e^{-i(\mathbf{p}\mathbf{x} + \mathbf{q}\mathbf{y})} [\phi(\mathbf{x}), \pi(\mathbf{y})] \\ &= \int d^3x d^3y e^{-i(\mathbf{p}\mathbf{x} + \mathbf{q}\mathbf{y})} i \delta(\mathbf{x} - \mathbf{y}) = i \int d^3x e^{i(\mathbf{p} + \mathbf{q})\mathbf{x}} \\ &= i (2\pi)^3 \delta(\mathbf{p} + \mathbf{q}),\end{aligned}\quad (2.91)$$

and

$$[\check{\phi}(\mathbf{p}), \check{\phi}(\mathbf{q})] = 0 = [\check{\pi}(\mathbf{p}), \check{\pi}(\mathbf{q})]. \quad (2.92)$$

Equations (2.91) and (2.92) entail that $\check{\pi}(\mathbf{q})$ is conjugate to $\check{\phi}(-\mathbf{q})$. Now we use that the creation and annihilation operators are related to sums of the field and momentum operators similarly to quantum mechanics,

Creation and annihilation operators

$$\begin{aligned} a(\mathbf{p}) &= \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \check{\phi}(\mathbf{p}) + i \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \check{\pi}(\mathbf{p}) \\ a^\dagger(-\mathbf{p}) &= \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \check{\phi}(\mathbf{p}) - i \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \check{\pi}(\mathbf{p}). \end{aligned} \quad (2.93)$$

From the commutation relations (2.91) and (2.92) of ϕ, π we deduce that for the creation and annihilation operators:

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = -\frac{i}{2} \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}} [\check{\phi}(\mathbf{p}), \check{\pi}(-\mathbf{q})] - \frac{i}{2} \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}} [\check{\phi}(-\mathbf{q}), \check{\pi}(\mathbf{p})] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \quad (2.94)$$

and

$$[a(\mathbf{p}), a(\mathbf{q})] = 0 = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})]. \quad (2.95)$$

This concludes our discussion of the canonical commutation relations in a free scalar field theory.

2.3.2. Hamiltonian of the free scalar field

In the 1+0-dimensional quantum mechanical example in the beginning of this section and in the classical scalar field theory we have seen, that the Hamiltonian density can be diagonalised. In terms of quantum mechanical annihilation and creation operators it could be rewritten in terms of the number operator. We now follow this derivation in the scalar quantum field theory. To that end we diagonalise the Hamiltonian density in momentum space. We start with the kinetic term,

$$-\int d^3x \phi(x) \Delta \phi(x) = \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{-ix(p+q)} \check{\phi}(\mathbf{p}) \mathbf{q}^2 \check{\phi}(\mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \check{\phi}(\mathbf{p}) \mathbf{p}^2 \check{\phi}(-\mathbf{p}), \quad (2.96)$$

where we have used that $\Delta e^{-iqx} = -\mathbf{q}^2 e^{-iqx}$. Analogously we get

$$m^2 \int d^3x \phi^2(x) = m^2 \int \frac{d^3p}{(2\pi)^3} \check{\phi}(\mathbf{p}) \check{\phi}(-\mathbf{p}), \quad \int d^3x \pi^2(x) = \int \frac{d^3p}{(2\pi)^3} \check{\pi}(\mathbf{p}) \check{\pi}(-\mathbf{p}). \quad (2.97)$$

All terms are diagonal in momentum space. Note that this complete diagonalisation is specific to free theories. Local interactions such as the ϕ^4 -term cannot be diagonalised in momentum space. In the free case we arrive at the diagonal Hamiltonian

Diagonal Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left[\tilde{\pi}(\mathbf{p}) \tilde{\pi}(-\mathbf{p}) + \omega_{\mathbf{p}}^2 \tilde{\phi}(\mathbf{p}) \tilde{\phi}(-\mathbf{p}) \right] \quad \text{with} \quad \omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2. \quad (2.98)$$

The physics interpretation of H is best done in terms of the annihilation and creation operators a , a^\dagger . Hence, we use (2.90) to rewrite

$$\begin{aligned} H &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left\{ a^\dagger(\mathbf{p}) a(\mathbf{p}) + \frac{1}{2} \left(a^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p}) + a(\mathbf{p}) a(-\mathbf{p}) \right) \right. \\ &\quad \left. + a^\dagger(\mathbf{p}) a(\mathbf{p}) - \frac{1}{2} \left(a^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p}) + a(\mathbf{p}) a(-\mathbf{p}) \right) + [a(\mathbf{p}), a^\dagger(\mathbf{p})] \right\} \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}) + \frac{1}{2} \mathcal{V} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}}, \end{aligned} \quad (2.99)$$

where we have used (2.95) for the last equality. We also used

$$[a(\mathbf{p}), a^\dagger(\mathbf{p})] = (2\pi)^3 \delta(\mathbf{0}) = \int d^3x e^{i\mathbf{p}\mathbf{x}} \Big|_{\mathbf{p}=0} = \mathcal{V}, \quad (2.100)$$

where V is the volume of the spatial \mathbb{R}^3 . The second term in the last line of (2.99) is sometimes called the vacuum term, and contains *two infinities*. As in physics energy differences are measured and not total energies this infinite constant can be conveniently dropped. However, it reflects one of the many divergences we will encounter on the way and we add a few remarks:

- (i) The volume of \mathbb{R}^3 is an *infrared* infinity related to considering arbitrarily large distances or wavelengths. It occurs in the infinite volume limit and can be dealt with by putting the theory in a finite volume, i.e. a box \mathcal{B} , a sphere S^3 or a torus T^3 (periodic box, no artificial curvature or boundary). All these choices are used in QFT:

The box or other compact spaces occurs in physics situations, where we discuss quantum systems in the laboratory or for the discussion of the Casimir effect, best known for the attractive force between two plates triggered by quantum effects buried in the 'vacuum' term $1/2 \mathcal{V} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}}$.

The sphere is mostly used for topological considerations, as then the fields can be understood as maps from S^3 or more generally S^{d-1} or S^d into the field space or the vacuum manifold, that is the subspace of solutions of the (quantum) equation of motion. In the case of N real scalar fields $\phi^T = (\phi_1, \dots, \phi_N)$, or N , we can define potentials that is invariant under general rotations of the fields into each other, $\phi \rightarrow O\phi$ with $O \in O(N)$. Such a potential is given by

$$V(\phi) = \mu\rho + \frac{\lambda_\phi}{2} \rho^2, \quad \rho = \frac{1}{2} (\phi_1^2 + \dots + \phi_N^2). \quad (2.101)$$

For $N = 1$ we are back to the real scalar field, and the vacuum manifold is simply a point, $\phi = 0$ ($\mu > 0$), or it is $\{\pm\phi_0\} \simeq Z_2$ ($\mu < 0$ and $\phi_0^2 = -2\mu/\lambda_\phi$). For the complex scalar field it is $\{\phi_0 \exp\{i\omega\}\} \simeq U(1) \simeq S^1$, and for N copies of the scalar field the vacuum manifold is isomorphic to $O(N)$. We leave the proof

of this statement to the reader. In all cases, the vacuum manifold carries the symmetry group \mathcal{G} of the action. Naturally, then the characterisation of maps $S^{d-1} \rightarrow \mathcal{G}$ is important for the theory at hand.

Finally, the torus T^d is used in lattice formulations of QFT, mostly out of numerical convenience. A further interesting property is, that the torus neither has a boundary such as the box nor does it introduce an artificial curvature to space-time such as the sphere.

All these choices lead to finite volume factor, and all of them, and others, have been considered in QFT.

- (ii) The second infinity is given by the vacuum energy density and occurs as the integral $\int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}}$ diverges. It is related to the limit of large momenta or small wave lengths and hence is an *ultraviolet* infinity. It can be dealt with by *regularising* the momentum integral, e.g. with the constraint $\mathbf{p}^2 \leq \Lambda^2$ with some finite ultraviolet cutoff Λ .
- (iii) Although we ave argued that the constant term can be dropped as only energy *differences* can be measured, it has to be considered in general: it plays a role at finite temperature or more generally for QFT with boundary conditions. The latter has an interesting and measurable application with the Casimir effect in QED already mentioned above: it introduces an attractive force between conducting plates, more generally the strength of the Casimir force and even the sign depends on the considered geometry. Last but not least the constant term is important for QFT in curved space-time and/or coupled to gravity, specifically but not exclusively for the *cosmological constant problem*.

For now we continue with our derivation and simply drop the vacuum term. The remaining Hamiltonian is diagonal, and is given by a spatial momentum integral over harmonic oscillators with frequencies $\omega_{\mathbf{p}}$.

Hamiltonian

$$H = \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}), \quad \text{with} \quad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \quad (2.102)$$

As anticipated in the beginning, (2.102) is nothing but an integral over an operator density of harmonic oscillators. As in the quantum mechanical case the operator is the number operator: a , a^\dagger are annihilation and creation operators respectively, and the combination $a^\dagger(\mathbf{p}) a(\mathbf{p})$ is the momentum density of the number operator. Accordingly, H simply counts the number of particles with a given momentum \mathbf{p} and integrates over their energies $\omega_{\mathbf{p}}$.

2.3.3. Fock space of scalar quantum field theory

This leads us directly to the question of the Hilbert space of the quantum field theory which we have to construct in terms of its operators. In quantum mechanics the Hilbert space is given by the span of all states that can be created from the vacuum by applying sums of powers of creation operators. The same construction works here, and the Hilbert space of quantum field theory is the *Fock space*, which is basically a sum of a set of Hilbert spaces of the n -particle states. The Fock space is systematically constructed from the vacuum state and the operator algebra given by a, a^\dagger .

(i) Vacuum & generic states: The vacuum is the state with the lowest energy, and we define

Vacuum state

$$H|0\rangle = 0 \quad \text{with} \quad a(\mathbf{p})|0\rangle = 0 \quad \text{and} \quad \langle 0|0\rangle = 1. \quad (2.103)$$

Note that it is the definition (2.103) that leads to the interpretation of a and a^\dagger as annihilation and creation operators respectively. With (2.103) we now create all states in the Hilbert space by applying a , a^\dagger on the vacuum state $|0\rangle$. Indeed it is sufficient to only consider a^\dagger , as any a can be commuted through to the right where it finally hits the vacuum state. We also remark that a general state is given by applying a sum of products of creation operators to the vacuum,

$$|\mathbf{f}\rangle = \sum_{N=0}^{\infty} \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{p_1}}} \cdots \frac{d^3 p_N}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{p_N}}} f(\mathbf{p}_1, \dots, \mathbf{p}_N) a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N) |0\rangle, \quad (2.104)$$

where $\mathbf{f} = (f_0, f_1, \dots)$ is the infinite-dimensional vector of the coefficient functions $f_n(\mathbf{p}_1, \dots, \mathbf{p}_N)$.

(ii) General one-particle state: In order to construct such a general state and for discussing its properties, we first consider the one-particle state with momentum \mathbf{p} , which is proportional to $a^\dagger(\mathbf{p})|0\rangle$. The normalised state is given by

One-particle state

$$|\mathbf{p}\rangle = \sqrt{2\omega_{\mathbf{p}}} a^\dagger(\mathbf{p}) |0\rangle, \quad \text{with} \quad H|\mathbf{p}\rangle = \omega_{\mathbf{p}} |\mathbf{p}\rangle. \quad (2.105)$$

The combination $\sqrt{2\omega_{\mathbf{p}}} a^\dagger(\mathbf{p})$ is Lorentz-invariant, as discussed below (2.20). Moreover, the state in (2.105) is normalised and is an eigenstate of the Hamiltonian with the energy $\omega_{\mathbf{p}}$. Both these properties are proven below: Let us first consider the latter property. Applying the Hamiltonian (2.102) to the state (2.105) leads us to

$$\begin{aligned} H|\mathbf{p}\rangle &= \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} a^\dagger(\mathbf{p}') a(\mathbf{p}') \sqrt{2\omega_{\mathbf{p}}} a^\dagger(\mathbf{p}) |0\rangle \\ &= \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} a^\dagger(\mathbf{p}') \sqrt{2\omega_{\mathbf{p}}} \left([a(\mathbf{p}'), a^\dagger(\mathbf{p})] + a^\dagger(\mathbf{p}) a(\mathbf{p}') \right) |0\rangle \\ &= \omega_{\mathbf{p}} a^\dagger(\mathbf{p}) \sqrt{2\omega_{\mathbf{p}}} |0\rangle = \omega_{\mathbf{p}} |\mathbf{p}\rangle. \end{aligned} \quad (2.106)$$

In (2.106) we have used the canonical commutation relations (2.95) as well as the vacuum property $a|0\rangle = 0$, see (2.103).

We now prove the first property: the states (2.105) are orthonormal with respect to the Lorentz invariant measure $\int d^3 p / (2\pi)^3 1 / (2\omega_p)$. First of all the scalar product of two states with momenta \mathbf{p} and \mathbf{q} is proportional to the spatial momentum δ -function. Moreover, it is scalar and hence the factor has to involve $\omega_{\mathbf{p}}$. This leads us to the normalisation of the momentum states with $(2\pi)^3 \omega_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{q})$. We note in passing that the scalar product of the momentum state is Lorentz-invariant. Indeed, integrated over the Lorentz-invariant measure $\int d^3 p / (2\pi)^3 1 / (2\omega_p)$ the total normalisation is unity, trivially being Lorentz-invariant. With (2.105) we arrive at

$$\langle \mathbf{p} | \mathbf{q} \rangle = 2 \sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}} \langle 0 | a(\mathbf{p}) a^\dagger(\mathbf{q}) | 0 \rangle = 2 \sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}} \langle 0 | [a(\mathbf{p}), a^\dagger(\mathbf{q})] | 0 \rangle = 2\omega_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}). \quad (2.107)$$

and

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \langle \mathbf{p} | \mathbf{q} \rangle = 1. \quad (2.108)$$

The state defined in (2.105) is not only an Eigenstate of the Hamiltonian, but also an Eigenstate of the momentum operator, which is discussed later, see in particular (2.137).

With (2.105), the general one-particle state is given by a weighted momentum integral of $|\mathbf{p}\rangle$,

General one-particle state

$$|f\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p}) a^\dagger(\mathbf{p}) |0\rangle, \quad (2.109)$$

where $f(\mathbf{p})$ denotes the distribution of momenta present in the state. The norm of the general one-particle state is given by

$$\langle f | f \rangle = \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}} f^*(\mathbf{p}) f(\mathbf{q}) \langle 0 | a(\mathbf{p}) a^\dagger(\mathbf{q}) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} f^*(\mathbf{p}) f(\mathbf{p}). \quad (2.110)$$

If the state is normalised to unity, $f^* f$ (or rather $f^* f / (2\omega_{\mathbf{p}})$) is nothing but the propability distribution of a given one-particle state in momentum space. While the momentum eigenstate (2.105) is also an eigenstate of the Hamiltonian, the general one-particle is not due to the momentum integral involved. We find

$$H |f\rangle = \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} a^\dagger(\mathbf{p}') a(\mathbf{p}') \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p}) a^\dagger(\mathbf{p}) |0\rangle = \int \frac{d^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} a^\dagger(\mathbf{p}) f(\mathbf{p}) |0\rangle, \quad (2.111)$$

the energy is distributed according to the momentum distribution of the state. If sandwiched with this state this becomes even more obvious,

$$\langle f | H | f \rangle = \int \frac{d^3 p}{(2\pi)^3} \left[\frac{1}{2\omega_{\mathbf{p}}} f^*(\mathbf{p}) f(\mathbf{p}) \right] \omega_{\mathbf{p}}, \quad (2.112)$$

for normalised states $|f\rangle$ this is nothing but the energy distribution following from the probability $f^*(\mathbf{p}) f(\mathbf{p})$.

(iii) N -particle state: The analysis done above for the one-particle state extends straightforwardly to the n -particle state. First we note that particle momentum states are eigenstates of the Hamiltonian and energy-momentum is additive. Assume now that we have a state $|\beta\rangle$ with $H |\beta\rangle = E_\beta |\beta\rangle$. Then $a^\dagger(\mathbf{p}) |\beta\rangle$ is a state with one additional particle with momentum \mathbf{p} and the energy $E_\beta + \omega_{\mathbf{p}}$,

$$H \left(a^\dagger(\mathbf{p}) |\beta\rangle \right) = a^\dagger(\mathbf{p}) H |\beta\rangle + [H, a^\dagger(\mathbf{p})] |\beta\rangle = (E_\beta + \omega_{\mathbf{p}}) \left(a^\dagger(\mathbf{p}) |\beta\rangle \right). \quad (2.113)$$

Accordingly, we simply have to consider N creation operators acting on the vacuum. This leads us to N -particle momentum states:

N -particle state

$$|\mathbf{p}_1 \cdots \mathbf{p}_N\rangle = \prod_{i=1}^N \sqrt{2\omega_{\mathbf{p}_i}} a^\dagger(\mathbf{p}_i) |0\rangle, \quad \text{with} \quad H |\mathbf{p}_1 \cdots \mathbf{p}_N\rangle = \left(\sum_{i=1}^N \omega_{\mathbf{p}_i} \right) |\mathbf{p}_1 \cdots \mathbf{p}_N\rangle, \quad (2.114)$$

where the latter property follows by starting with the vacuum, recursively multiplying the creating operators and using (2.113). Note also, that the N -particle states have Bose symmetry, i.e.

$$|\mathbf{p}_1 \cdots \mathbf{p}_i \mathbf{p}_{i+1} \cdots \mathbf{p}_N\rangle = |\mathbf{p}_1 \cdots \mathbf{p}_{i+1} \mathbf{p}_i \cdots \mathbf{p}_N\rangle, \quad (2.115)$$

as $[a^\dagger(\mathbf{p}_i), a^\dagger(\mathbf{p}_{i+1})] = 0$. While the energy relation in (2.114) follows concisely from (2.113), we re-derive it with the explicit form of the state and the Hamiltonian operator for further exemplifying the use of the commutation relation. This leads us to

$$H |\mathbf{p}_1 \cdots \mathbf{p}_N\rangle = \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} \prod_{i=1}^N \sqrt{2\omega_{\mathbf{p}_i}} \left(a^\dagger(\mathbf{p}') a(\mathbf{p}') a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N) |0\rangle \right). \quad (2.116)$$

In the next step we shift all creation operators to the left and all annihilation operators to the right, which is called *normal ordering*. Then we use (2.103) and the canonical commutation relations for the creation and annihilation operators, (2.95), to simplify the equation,

$$\begin{aligned} a(\mathbf{p}') \left(a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N) \right) |0\rangle &= \left([a(\mathbf{p}'), a^\dagger(\mathbf{p}_1)] a^\dagger(\mathbf{p}_2) \cdots a^\dagger(\mathbf{p}_N) + \dots \right. \\ &\quad \dots + a^\dagger(\mathbf{p}_1) [a(\mathbf{p}'), a^\dagger(\mathbf{p}_2)] a^\dagger(\mathbf{p}_3) \cdots a^\dagger(\mathbf{p}_N) + \dots + \dots \\ &\quad \left. \dots + a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_{N-1}) [a(\mathbf{p}'), a^\dagger(\mathbf{p}_N)] \right) |0\rangle. \end{aligned} \quad (2.117)$$

With (2.117) it follows that $|\mathbf{p}_1 \cdots \mathbf{p}_N\rangle$ is an energy eigenstate with the energy $\sum \omega_{\mathbf{p}_i}$, see (2.114). The ordering of operators of the final expression, namely all annihilation operators to the right, is called *normal ordering*. We have already used it implicitly in the definition of the Hamiltonian (2.102), and it will be discussed in more detail later.

Analogously to the general one-particle state, the general N -particle state is then given by a momentum integral of a distribution $f(\mathbf{p}_1, \dots, \mathbf{p}_N)$ of momentum eigenstates similarly to (2.109):

General N -particle state

$$|f_N\rangle = \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_1}}} \cdots \frac{d^3 p_N}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_N}}} f(\mathbf{p}_1, \dots, \mathbf{p}_N) a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N) |0\rangle. \quad (2.118)$$

This leads us directly to the definition of generic states in the Fock space of the scalar quantum field theory as a sum of general N -particle states for all N as defined in (2.104). We close the discussion of the N -particle state with a remark on the generation of an $(N-1)$ -particle state from an N -particle state. This is naturally achieved by applying an annihilation operator to the N -particle state, which is illustrated with the example of $N=1$ below,

Example 2-3: Annihilation operator applied to general one particle state.

$$\begin{aligned} a(\mathbf{p}) |f\rangle &= a(\mathbf{p}) \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}'}}} f(\mathbf{p}') a^\dagger(\mathbf{p}') |0\rangle \\ &= \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}'}}} f(\mathbf{p}') [a(\mathbf{p}), a^\dagger(\mathbf{p}')] |0\rangle \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} f(\mathbf{p}) |0\rangle \end{aligned} \quad (2.119)$$

(iv) Field operator ϕ and coherent states: Let us now discuss the interpretation of the field operator $\phi(\mathbf{x})$ in (2.87). We first remark that applying ϕ to the vacuum state yields a one particle state,

$$\phi(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{i\mathbf{p}\mathbf{x}} |\mathbf{p}\rangle. \quad (2.120)$$

Equation (2.120) is the Fourier transformation of a momentum one-particle state, hence a position state. Testing this state with a momentum one-particle state results in

$$\langle \mathbf{p} | \phi(\mathbf{x}) | 0 \rangle = e^{i\mathbf{p}\mathbf{x}}, \quad (2.121)$$

is a plane wave with the momentum \mathbf{p} and reminiscent of non-relativistic QM, as $\langle \mathbf{p} | x \rangle = e^{i\mathbf{p}\mathbf{x}}$. Applying ϕ to the dual (bra-vector) vacuum vector $\langle 0 |$ leads to a similar conclusion for the annihilation operator part of ϕ . We conclude that the field operator $\phi(\mathbf{x})$ creates and annihilates a particle at the spatial position \mathbf{x} . Moreover, states with defined particle number have a vanishing expectation value of ϕ . In particular this applies to the vacuum,

$$\langle 0 | \phi(\mathbf{x}) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \langle 0 | a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} | 0 \rangle = 0, \quad (2.122)$$

with $\langle 0 | a | 0 \rangle = 0 = \langle 0 | a^\dagger | 0 \rangle$. Similarly it follows, that

$$\begin{aligned} \langle \mathbf{p} | \phi(\mathbf{x}) | \mathbf{p} \rangle &= 0 \\ &\vdots \\ \langle \mathbf{p}_1 \cdots \mathbf{p}_N | \phi(\mathbf{x}) | \mathbf{p}_1 \cdots \mathbf{p}_N \rangle &= 0, \end{aligned} \quad (2.123)$$

by using

$$|\mathbf{p}_1 \cdots \mathbf{p}_N\rangle \simeq a^\dagger(\mathbf{p}_1) \cdots a^\dagger(\mathbf{p}_N) |0\rangle. \quad (2.124)$$

Let us now concentrate on the annihilation part of the field. A potential eigenvector for this part is one for the annihilation operator,

$$a(\mathbf{p}) |\alpha\rangle = \alpha(\mathbf{p}) |\alpha\rangle, \quad \text{with} \quad \langle \alpha | \alpha \rangle = 1. \quad (2.125)$$

The state is unchanged by the annihilation (detection) of a particle with momentum \mathbf{p} and the eigenvalue is the amplitude $\alpha(\mathbf{p})$. Equation (2.125) defines a *coherent state*, heuristically one with minimal uncertainty, hence a 'classical' state. If the field operator is sandwiched with $|\alpha\rangle$, this interpretation is very suggestive. We find

$$\langle \alpha | \phi(\mathbf{x}) | \alpha \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{i\mathbf{p}\mathbf{x}} \alpha(\mathbf{p}) + e^{-i\mathbf{p}\mathbf{x}} \alpha^*(\mathbf{p}) \right) = \phi_{\text{cl}}(\mathbf{x}). \quad (2.126)$$

We emphasise, that $\alpha(\mathbf{p})$, $\alpha^*(\mathbf{p})$ are no operators, and (2.126) is equivalent to (2.18), i.e. the classical real scalar field $\phi_{\text{cl}}(\mathbf{x})$. It is left to explicit construct $|\alpha\rangle$. Its defining property (2.125) implies that it must be a sum of N -particle states $|\alpha_N\rangle$ that are mapped into $|\alpha_{N-1}\rangle$ if hit by an annihilation operator a . This leads us to

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \sum_{N=0}^{\infty} |\alpha_N\rangle, \quad \text{with} \quad a(\mathbf{p}) |\alpha_N\rangle = \alpha(\mathbf{p}) |\alpha_{N-1}\rangle, \quad (2.127)$$

with $|\alpha_{-1}\rangle = 0$ and a normalisation $\mathcal{N}(\alpha)$. Such an N -particle state is given by

$$|\alpha_N\rangle = \frac{1}{N!} \prod_{i=1}^N \left(\int \frac{d^3p_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_i}}} \alpha(\mathbf{p}_i) \right) |\mathbf{p}_1 \cdots \mathbf{p}_N\rangle, \quad (2.128)$$

and the property in (2.127) is shown with successively using

$$a(\mathbf{p}) |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle = \sum_{i=1}^n (2\pi)^3 \sqrt{2\omega_{\mathbf{p}_i}} |\mathbf{p}_1 \cdots \mathbf{p}_{i-1} \mathbf{p}_{i+1} \cdots \mathbf{p}_n\rangle \delta(\mathbf{p} - \mathbf{p}_i), \quad (2.129)$$

more details can be found in [Appendix A](#). It is left to compute the normalisation $\mathcal{N}(\alpha)$ necessary for $\langle \alpha | \alpha \rangle = 1$ in (2.127). This computation is deferred to [Appendix A](#). This concludes our construction of the coherent state. The sum in (2.127) with the N -particle states $|\alpha_N\rangle$ defined in (2.128) is the exponential series and we write conveniently

Coherent state

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \exp\left(\int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \alpha(\mathbf{p}) a^\dagger(\mathbf{p})\right) |0\rangle, \quad \text{with} \quad \mathcal{N}(\alpha) = \exp\left(\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} |\alpha(\mathbf{p})|^2\right). \quad (2.130)$$

Coherent states are not orthogonal which can be deduced from their scalar product,

$$\begin{aligned} \langle \beta | \alpha \rangle &= \frac{1}{\mathcal{N}(\alpha) \mathcal{N}(\beta)} \exp\left(\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \beta^* \alpha(\mathbf{p})\right) \\ &= \frac{1}{\mathcal{N}(\alpha) \mathcal{N}(\beta)} \exp\left\{-\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left[|\alpha(\mathbf{p})|^2 + |\beta(\mathbf{p})|^2 - 2\beta^* \alpha(\mathbf{p})\right]\right\}, \end{aligned} \quad (2.131)$$

where we used the Baker–Campbell–Hausdorff formula

$$\exp(A) \exp(B) = \exp(B) \exp(A) \exp([A, B]) \quad \text{for} \quad [A, [A, B]] = 0 = [B, [B, A]]. \quad (2.132)$$

Indeed the set of coherent states is overcomplete. Note that in quantum mechanics (1+0 dimensional theory) we can easily show that

$$\frac{1}{\pi} \int d^2 \alpha |\alpha\rangle \langle \alpha| = \mathbb{1}, \quad (2.133)$$

where $\mathbb{1}$ is the unity operator in the Fock space. This completes our discussion of coherent states and the interpretation of the field operator.

(v) Conserved energy-momentum tensor: Let us now discuss the fate of the classical conservation laws in the quantisation procedure. To begin with, as nothing in the derivation of the Noether theorem made use of the nature of the field (function or operator), the conservation laws should also hold in the quantised theory. Indeed we have already discussed the Hamiltonian as the generator of time translations. This is the 0-component of the conserved current $P^\mu = \int d^3 x T^{0\mu}$, see (2.56). Corresponding to (2.56) we now calculate the spatial momentum operator as

$$\begin{aligned} P^i &= \int d^3 x T^{0i} = \int d^3 x \pi \partial^i \phi \\ &= \int d^3 x \left\{ (-i) \int \frac{d^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} - a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right) \right. \\ &\quad \left. \times \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \left(a(\mathbf{q}) (-iq^i) e^{i\mathbf{q}\mathbf{x}} + a^\dagger(\mathbf{q}) (iq^i) e^{-i\mathbf{q}\mathbf{x}} \right) \right\} \\ &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} p^i \left\{ -a(\mathbf{p}) a(-\mathbf{p}) + a(\mathbf{p}) a^\dagger(\mathbf{p}) + a^\dagger(\mathbf{p}) a(\mathbf{p}) - a^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p}) \right\}. \end{aligned} \quad (2.134)$$

where we have used the Fourier representation of the field and its canonical momentum, (2.87), and have pulled out the common factor ip^i . Note also that $\partial^i(\mathbf{p}\mathbf{x}) = -\partial^i(x_j p^j) = -p^i$. The first and last term in the momentum integral proportional to $p^i a(\mathbf{p}) a(-\mathbf{p})$ and $p^i a^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p})$ vanish: $a(\mathbf{p}) a(-\mathbf{p})$ is even under $\mathbf{p} \rightarrow -\mathbf{p}$ as the operators commute. This follows similarly for $a^\dagger(\mathbf{p}) a^\dagger(-\mathbf{p})$. Hence, the total integrands are odd under $\mathbf{p} \rightarrow -\mathbf{p}$ and the integrals vanish, i.e. $\int d^3 p p^i a(\mathbf{p}) a(-\mathbf{p}) = 0$. We *normal order* the operators (all annihilation operator to the right) and arrive at

$$\mathbf{P} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(2 a^\dagger(\mathbf{p}) a(\mathbf{p}) + [a^\dagger(\mathbf{p}), a(\mathbf{p})] \right) = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a^\dagger(\mathbf{p}) a(\mathbf{p}) - \frac{1}{2} (2\pi)^3 \delta(\mathbf{0}) \right). \quad (2.135)$$

Analogously to the Hamiltonian in (2.100), the second term in the last line of (2.135) is ill-defined. In contradistinction to the Hamiltonian, it contains an infinity, $(2\pi)^3 \delta(\mathbf{0}) = \mathcal{V}$ and a zero (after regularisation of the momentum integral with an ultraviolet cutoff, $\mathbf{p}^2 \leq \Lambda^2$). Then the second term reads

$$\int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (2\pi)^3 \delta(\mathbf{0}) \rightarrow \mathcal{V} \int_{\mathbf{p}^2 \leq \Lambda^2} \frac{d^3 p}{(2\pi)^3} \mathbf{p} = 0, \quad (2.136)$$

and can be safely dropped. This already indicates that divergences can be avoided if the theory is first defined in a regularised way. Here, regularisation is a well-defined mathematical procedure known from distribution theory (functional analysis) and is used for tempered distributions. With (2.136) we are led to

4-momentum operator

$$\begin{aligned} \mathbf{P}^0 &= H \\ \text{(spatial momentum)} \quad \mathbf{P} &= \int d^3 x \pi \nabla \phi \\ &\simeq \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a^\dagger(\mathbf{p}) a(\mathbf{p}) \quad \text{with} \quad \mathbf{p} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle. \end{aligned} \quad (2.137)$$

(vi) Lorentz symmetry: We close the construction of the Fock space with discussing *Lorentz symmetry*. Let $U(\Lambda)$ denote the unitary Fock space representation of a Lorentz transformation Λ . Then

$$\begin{aligned} U(\Lambda) |0\rangle &= |0\rangle \\ U(\Lambda) |\mathbf{p}\rangle &= |\Lambda \mathbf{p}\rangle. \end{aligned} \quad (2.138)$$

Note, that

$$\langle \mathbf{q} | \mathbf{p} \rangle = 2\omega_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \quad (2.139)$$

is Lorentz-invariant, as

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} = \int \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \theta(p^0) \quad (2.140)$$

is invariant under proper orthochronous Lorentz transformations ($\det \Lambda = 1, \Lambda^0_0 > 0$), and

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} 2\omega_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) = 1. \quad (2.141)$$

With this, we have completed the Fock space construction.

A final remark concerns the causality of the theory. For this purpose we recall that $\phi(x)$ generates a superposition of one-particle states from the vacuum (2.121). This entails, that the propagation of a particle (or state) in the theory and hence its causality is encoded in the propagator

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle. \quad (2.142)$$

As the underlying differential equation is of second order, (2.142) is only one of two independent solutions. Typically, we will deal with *retarded*, *advanced*, *time ordered* or *causal* propagator, all of them convenient choices in specific situations. Note that all the different propagators can be constructed from two of that mentioned above, for example from the retarded and advanced propagators. This is further discussed in **Chapter 3, Section 3.1**. In summary we have constructed and discussed the Hilbert space for the free real scalar field, the Fock space, in the paragraphs (i) - (vi).

The quantisation as well as the construction of the Fock space readily carries over to complex scalar field and $O(N)$ -theories, $\phi = (\phi_1, \dots, \phi_N)$. We refrain from repeating the identical steps of the construction and simply quote some of the important result. The action is given by

$$S[\phi] = \int d^4x \left(\partial_\mu \phi \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger \right), \quad \text{with} \quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad (2.143)$$

where we have written the action of the complex scalar in terms of real scalars ϕ_i with $i = 1, 2$ with (2.87) with respective creation and annihilation operators a_1, a_1^\dagger and a_2, a_2^\dagger . If rewriting $S[\phi]$ in terms of the ϕ_1, ϕ_2 we arrive at

$$S[\phi] = \sum_{i=1}^2 \frac{1}{2} \int d^4x \left(\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i \phi_i \right). \quad (2.144)$$

In the representation (2.144) it is apparent, that the action encounters no operator ordering problems. Hence, it is suggestive to use this representation of the complex scalar field in order to avoid operator ordering problems within the quantisation of composite operator expression such as the action, Hamiltonian and Noether currents. Evidently, the complex field and its canonical momentum can be written in terms of two pairs of creation and annihilation operators. Instead of a_1, a_2 and their adjoints one usually combines them in complex pairs that reflect the representation of ϕ by ϕ_1 and ϕ_2 ,

$$\begin{aligned} \phi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + b^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right), \\ \pi(\mathbf{x}) = \partial^0 \phi^\dagger(x) &= -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(b(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} - a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right), \end{aligned} \quad (2.145)$$

with

$$a = \frac{1}{\sqrt{2}}(a_1 + ia_2), \quad b = \frac{1}{\sqrt{2}}(a_1 - ia_2), \quad \leftrightarrow \quad a_1 = \frac{1}{\sqrt{2}}(a + b), \quad a_2 = \frac{i}{\sqrt{2}}(b - a), \quad (2.146)$$

It can be checked readily, that (2.146) and the commutation relations (2.94) for a_i, a_i^\dagger , lead to canonical commutation relations for the complex field,

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta(\mathbf{x} - \mathbf{y}), \\ [a(\mathbf{p}), a^\dagger(\mathbf{q})] &= (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \\ [b(\mathbf{p}), b^\dagger(\mathbf{q})] &= (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (2.147)$$

and all other commutators vanish.

We identify a, a^\dagger with the annihilation and creation operators for particles, while b, b^\dagger are the annihilation and creation operators for anti-particles. This can be also deduced from the respective Noether charges for energy conservation and $U(1) \simeq O(2)$ invariance, H and the (electric charge) Q respectively:

The Hamiltonian operator is given by

$$H = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[\tilde{\pi}(\mathbf{p}) \tilde{\pi}^\dagger(\mathbf{p}) + \omega_{\mathbf{p}}^2 \tilde{\phi}(\mathbf{p}) \tilde{\phi}^\dagger(\mathbf{p}) \right], \quad (2.148)$$

and reads in the diagonal momentum representation

$$H \simeq \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left[a^\dagger(\mathbf{p}) a(\mathbf{p}) + b^\dagger(\mathbf{p}) b(\mathbf{p}) \right], \quad (2.149)$$

where the integrand corresponds to the sum of the energy of particles and antiparticles, and we have dropped the vacuum term.

Finally, the Noether charge operator Q is derived from the classical Noether charge (2.70) by inserting field operators and their derivatives. In contradistinction to the Hamiltonian we encounter operator ordering problems, which are typically circumvented by using the normal ordered form of the Noether charge. However, for illustration we keep the full expressions. As suggested above we use the representation (2.144) for the derivation of the Noether current and charge. The infinitesimal $O(2)$ rotation is given by

$$\phi_1 \rightarrow \phi_1 - \epsilon \phi_2, \quad \phi_2 \rightarrow \phi_2 + \epsilon \phi_1, \quad (2.150)$$

and the Noether charge reads

$$Q = \int d^3 x \left(\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2 \right), \quad (2.151)$$

without any operator ordering problems. In (2.151) we have used the fact that ϕ_i commutes with $\partial_t \phi_j$ for $i \neq j$, which allows us to bring the derivative terms to right. We remark, that (2.151) is manifestly symmetric under $1 \leftrightarrow 2$. Hence, if expressing (2.151) in terms of ϕ, ϕ^* or $a, a^\dagger, b, b^\dagger$, this expression is necessarily symmetric in $a \leftrightarrow b$ and $\phi \leftrightarrow \phi^\dagger$. Note these transformations exchanges the rôle of creation and annihilation operators, which should be a symmetry of the theory: the definition of a, a^\dagger as annihilation and creation operator came via our choice of the vacuum $a|\Omega\rangle = 0$. We could as well have chosen a^\dagger as annihilation operator.

Inserting (2.87) into (2.151) and using (2.146) leads us to

$$Q = \int \frac{d^3 p}{(2\pi)^3} \left[a^\dagger(\mathbf{p}) a(\mathbf{p}) - b^\dagger(\mathbf{p}) b(\mathbf{p}) \right], \quad (2.152)$$

which has the symmetries discussed above. Note also that no vacuum term is present in contradistinction to the Hamiltonian. This property originates in the symmetry between particles and anti-particles, being correlated to the symmetry between $a, a^\dagger, b, b^\dagger$: a vacuum term corresponds to an either positive or negative vacuum 'charge' and hence has to be absent in a symmetric formulation.

The computation of the Noether charge (2.152) within the ϕ_1, ϕ_2 representation is straightforward but a bit tedious. It is facilitated by translating (2.151) into a form with $\phi, \pi, \phi^\dagger, \pi^\dagger$ and using (2.145). We arrive at

$$Q = \frac{i}{2} \int d^3 x \left(\{ \phi^\dagger, \pi^\dagger \} - \{ \phi, \pi \} \right) = i \int d^3 x \left(\phi^\dagger \pi^\dagger - \phi \pi \right), \quad (2.153)$$

where $\{A, B\} = AB + BA$ is the anti-commutator of operators, being pivotal in the quantisation of fermionic fields discussed later. The second relation is obtained by bringing all momentum operators π, π^\dagger to the right. This generates two vacuum charges with opposite sign which cancel each other which again reflects the symmetry between annihilation and creation operators or particles and anti-particles. Moreover, (2.153) is hermitian

for the same reason: to begin with, Q^\dagger has the same form as (2.153), but with π, π^\dagger to the left. However, bringing both to the right produces two vacuum terms that annihilate each other.

With (2.153) we are led to (2.152),

$$Q = i \int d^3x \left(\phi^\dagger \pi^\dagger - \phi \pi \right) \\ = i \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}} \quad (2.154)$$

$$\times \left[\left(a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} + b(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} \right) \left(-i\omega_{\mathbf{q}} a(\mathbf{q}) e^{i\mathbf{q}\mathbf{x}} + i\omega_{\mathbf{q}} b^\dagger(\mathbf{q}) e^{-i\mathbf{q}\mathbf{x}} \right) \right. \\ \left. - \left(a(\mathbf{q}) e^{i\mathbf{q}\mathbf{x}} + b^\dagger(\mathbf{q}) e^{-i\mathbf{q}\mathbf{x}} \right) \left(i\omega_{\mathbf{p}} a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} - i\omega_{\mathbf{p}} b(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} \right) \right] \\ = \int \frac{d^3p}{(2\pi)^3} \left[a^\dagger(\mathbf{p}) a(\mathbf{p}) - b^\dagger(\mathbf{p}) b(\mathbf{p}) \right], \quad (2.155)$$

The 'classical' charge in a coherent state is given by

$$\langle \alpha | Q | \alpha \rangle = \int \frac{d^3p}{(2\pi)^3} \left[\alpha^* \alpha(\mathbf{p}) - \beta^* \beta(\mathbf{p}) \right], \quad (2.156)$$

which agrees with the classical Noether charge derived in (2.70).

3. Perturbation Theory

Perturbation theory is a standard computational method in quantum field theory. It considers **interactions as a perturbation of the free theory**. For example, a scattering event of such as $2 \rightarrow 2$ scattering in the ϕ^4 -theory, $\langle \mathbf{p}_1 \mathbf{p}_2 | \mathbf{p}_3 \mathbf{p}_4 \rangle$ in momentum space can be expanded in powers of single scatterings: the zeroth order describes just the propagation of two fields (either with $\mathbf{p}_1 = \mathbf{p}_3$ and $\mathbf{p}_2 = \mathbf{p}_4$ or $3 \leftrightarrow 4$), the next order describes a single scattering of the fields involved (in general with momentum transfer, so only momentum conservation holds, $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$) and is linear in the coupling λ . Two single scatterings are proportional to λ^2 and so on. In perturbation theory we assume the coupling to be sufficiently small, $\lambda \ll 1$, and expand the observables, e.g. scattering amplitudes such as described above, in orders of λ and strive for convergence.

We also remark, that strictly speaking perturbation theory is an amplitude expansion in the field amplitude as the coupling always comes with powers of the field. This differentiation is important for strong field physics, e.g. electrodynamics in strong fields. There, the coupling is small, the fine structure constant α is of the order 10^{-2} , but perturbation theory at least has to be resummed.

3.1. Interaction Picture

Perturbation theory in QFT is typically done in the *interaction picture*, which is a mixture of the *Heisenberg* and *Schrödinger picture* known from quantum mechanics (1+0 dimensional QFT). In short, in the interaction picture the operators evolve with the Hamiltonian of the free QFT, while the states are evolved with the interaction Hamiltonian. For the construction we first briefly recapitulate the Heisenberg and Schrödinger pictures.

In the previous chapter, the Fock space construction was performed in the Heisenberg picture. As mentioned above, in the Heisenberg picture the operators evolve in time, whereas the states are stationary,

$$\begin{aligned} i\partial_t |f\rangle &= 0, \\ i\partial_t O(t) &= [O(t), H], \end{aligned} \quad (3.1)$$

and the latter equation holds true for a time-independent Hamiltonian. The time evolution equation for the operator $O(t)$ in (3.1) has the simple solution

$$O(t) = e^{iHt} O e^{-iHt}, \quad (3.2)$$

with the *unitary* time evolution operator

$$U(t, t') = \exp(iH(t - t')), \quad (3.3)$$

that describes the time evolution over a time distance $t = t - t'$.

An important example is the field operator $\phi(x)$. In the discussion of the free field we only discussed equal time commutation relations and operators at $t = 0$ such as $\phi(\mathbf{x})$. Due to the simple time evolution of operators in (3.1), the commutation relations do not evolve in time, see (2.79) in Section 2.3.1. The proof there was based on the time translation invariance of the free theory or rather the free Hamiltonian.

The time evolution of the field operator $\phi(\mathbf{x})$ defined in (2.87) is determined by that of the creation and annihilation operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$. Their time evolution is given by (3.2) and we are led to

$$e^{iHt} a(\mathbf{p}) e^{-iHt} = a(\mathbf{p}) e^{-i\omega_{\mathbf{p}}t}, \quad \text{and} \quad e^{iHt} a^\dagger(\mathbf{p}) e^{-iHt} = a^\dagger(\mathbf{p}) e^{i\omega_{\mathbf{p}}t}, \quad (3.4)$$

the time evolution of the a, a^\dagger is imply given by phases with the on-shell frequency. For the derivation of (3.4) we use the series expansion of the exponential,

$$e^{iHt} = \sum \frac{1}{n!} t^n H^n, \quad (3.5)$$

which defines the exponential of an operator in the first place. First we consider the time evolution of the annihilation operator. For its computation we have to pull one of the exponentials to the other side. This task boils down to computing the commutator $[H^n, a(\mathbf{p})]$, which can be done by successively using the commutator $[H, a(\mathbf{p})]$,

$$[H, a(\mathbf{p})] = \int \frac{d^3q}{(2\pi)^3} \omega_{\mathbf{q}} \left(a^\dagger(\mathbf{q}) [a(\mathbf{q}), a(\mathbf{p})] + [a^\dagger(\mathbf{q}), a(\mathbf{p})] a(\mathbf{q}) \right) = -\omega_{\mathbf{p}} a(\mathbf{p}). \quad (3.6)$$

where we have used the canonical commutation relations (2.95). This can be conveniently rewritten as

$$H a(\mathbf{p}) = a(\mathbf{p}) (H - \omega_{\mathbf{p}}) \quad \longrightarrow \quad H^n a(\mathbf{p}) = a(\mathbf{p}) (H - \omega_{\mathbf{p}})^n \quad (3.7)$$

Then, the time evolution of the annihilation operator in (3.4) follows straightforwardly from (3.5) and (3.7) with

$$e^{iHt} a(\mathbf{p}) e^{-iHt} = a(\mathbf{p}) e^{i(H-\omega_{\mathbf{p}})t} e^{-iHt} = a(\mathbf{p}) e^{-i\omega_{\mathbf{p}}t}. \quad (3.8)$$

The time evolution of the creation operator in (3.4) follows similarly. This allows us to derive the time-dependent field operator $\phi(x)$, using (3.2) and (3.4),

$$\begin{aligned} \phi(x) &= e^{iHt} \phi(\mathbf{x}) e^{-iHt} \\ &= e^{iHt} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}} + a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \right) e^{-iHt} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{-i(\omega_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} + a^\dagger(\mathbf{p}) e^{i(\omega_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{-ipx} + a^\dagger(\mathbf{p}) e^{ipx} \right), \end{aligned} \quad (3.9)$$

with $p_0 = \omega_{\mathbf{p}}$. Equation (3.9) is nothing but the quantised version of the time-dependent classical solution of the equation of motion, (2.18), and the field operator in (3.9) collapses to the time-dependent classical free field if sandwiched with a coherent state (2.130), $\langle \alpha | \phi(x) | \alpha \rangle = \phi_{\text{cl}}(x)$ with $\phi_{\text{cl}}(x)$ being defined in (2.18). This closes our brief recapitulation of the Heisenberg picture.

In the *Schrödinger picture* the states evolve in time and the operators are stationary,

$$\begin{aligned} i\partial_t |f(t)\rangle &= H |f\rangle, \\ i\partial_t O &= 0, \end{aligned} \quad (3.10)$$

where the first equation holds true for a time-independent Hamiltonian. As for the time evolution of the operators in the Heisenberg picture, the time evolution of the states in the Schrödinger picture have a simple solution in terms of the unitary time evolution operator $\exp(-iHt)$,

$$|f(t)\rangle = e^{-iHt} |f\rangle. \quad (3.11)$$

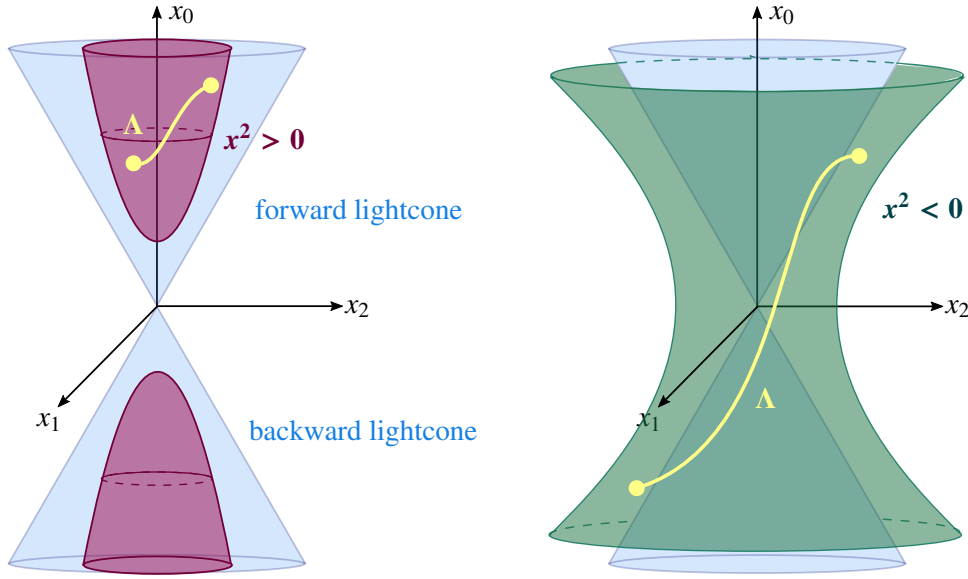


Figure 3.1.: Lorentz transformations connect all points on a $x^2 = \text{const.}$ surface in the Minkowski diagram. Thus, for space-like separations we find a Lorentz transformation Λ with $\Lambda(x - y) = -(x - y)$, sketched in the figure on the right hand side. In turn, for time-like separations this is not possible, see the figure on the left hand side.

Hence, the time evolution operator $U(t, t')$ defined in (3.3) either acts on the operators (Heisenberg) *or* the states (Schrödinger); the expectation values are the same.

This leads us to an important property, the *causality* of a *local* Poincaré-invariant QFT: Causality is a necessary requirement of any physics description, and has to be present in a QFT on the microscopic level. In a *local* QFT we have point-like interactions, a necessary requirement for the causality of the QFT. This property has to be carried by the field operator ϕ . In particular, ϕ should not connect space-like regions. This leads to the requirement

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0. \quad (3.12)$$

Equation (3.12) entails that two measurements with a space-like distance have no impact on each other, and hence the field operators should commute at space-like distances. We proceed with proving (3.12), using the explicit representation of the field operator $\phi(x)$ in terms of creation and annihilation operators, (3.9),

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}} \left([a(\mathbf{p}), a^\dagger(\mathbf{q})] e^{-i(p x - q y)} + [a^\dagger(\mathbf{p}), a(\mathbf{q})] e^{i(p x - q y)} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{-i p(x-y)} - \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{i p(x-y)} \\ &= \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) e^{-i p(x-y)} - \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) e^{i p(x-y)}, \end{aligned} \quad (3.13)$$

with $p_0 = \sqrt{\vec{k}^2 + m^2} \geq 0$, and using (2.21) in the last step. Now we utilise the manifest Lorentz-invariant form of the momentum measures that in (3.13): For space-like separations $(x - y)^2 < 0$, there is a *orthochronous* Lorentz transformation with

$$\Lambda(x - y) = -(x - y), \quad (3.14)$$

see [Figure 3.1](#). Hence, for $(x - y)^2 < 0$ we have

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) e^{ip(x-y)} &= \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) e^{ip \Lambda(x-y)} \\ &= \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) e^{-ip(x-y)}. \end{aligned} \quad (3.15)$$

With (3.15), the two terms in (3.13) cancel, and we are led to the causality relation (3.12). Evidently, the derivation carries over directly to a complex scalar field, or more generally, to N copies of the real scalar field, $\phi^T = (\phi_1, \dots, \phi_N)$. We find

$$[\phi_i(x), \phi_j(y)] = 0, \quad \text{for } (x - y)^2 < 0 \quad \text{and} \quad \forall i, j \in \{1, \dots, N\}. \quad (3.16)$$

In summary, local (point-like) interactions (trivially preserving causality), and the causal propagation of particles or fields as is inherited from the free field operators $\phi(x)$ lead to the necessary causality of the QFT on the microscopic level.

We now proceed to the *interaction picture*. As already mentioned above, in the interaction picture we expand the theory in powers of interaction about the free theory, the latter being formulated in the Heisenberg picture. In turn, the states evolve with the interaction Hamiltonian. For this construction we decompose the Lagrangian density in a free and an interaction part

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_{\text{int}}(\phi) = \frac{1}{2} \phi(x) (-\partial^2 - m^2) \phi(x) + \mathcal{L}_{\text{int}}(\phi), \quad (3.17)$$

where the interaction Lagrangian is given by the local point-like interaction $V(\phi)$ with,

$$\mathcal{L}_{\text{int}}(\phi) = -V(\phi), \quad (3.18)$$

where $V(\phi)$ is a polynomial in $\phi(x)$ at the same space-time point x . This decomposition carries over to the Hamiltonian density

$$\mathcal{H}(\pi, \phi) = \mathcal{H}_0(\pi, \phi) + \mathcal{H}_{\text{int}}(\phi) = \frac{1}{2} \pi(x)^2 + \frac{1}{2} \phi(x) (-\Delta + m^2) \phi(x) + \mathcal{H}_{\text{int}}(\phi), \quad (3.19)$$

where the canonical momentum solely occurs in the free part and

$$\mathcal{H}_{\text{int}}(\phi) = V(\phi). \quad (3.20)$$

For the time being we restrict ourselves to the ϕ^4 -interaction,

Interaction part of the Hamiltonian density

$$\mathcal{H}_{\text{int}}(\phi) = V(\phi) = \frac{\lambda}{4!} \phi(x)^4. \quad (3.21)$$

Note, that the normalisation factor varies in literature, common choices are of $1/(4!)$ as in (3.21), or $1/8$, commonly used in $O(N)$ -theories. Let us also add a few remarks:

- (i) Higher order terms are excluded by lack of renormalisability (predictivity) in four space-time dimensions.
- (ii) ϕ^3 terms are not bounded from below, which leads to an unstable theory. Moreover, they lack the discrete symmetry under $\phi \rightarrow -\phi$. Still, these models are used within perturbation theory, mostly due to the fact, that they allow for simple computations and the quantum corrections in a ϕ^3 -theory show some properties of more physical theories. For example, they incorporate 1-to-2 scattering, also present in QED, QCD and the Standard Model.

(iii) As the simplest interacting and potentially stable QFT, the ϕ^4 theory is the 'workhorse' and 'guinea pig' of QFT.

In the interaction picture, the operators evolve in time with the free Hamiltonian

$$\begin{aligned} i\partial_t O &= [O, H_0] \\ \Rightarrow O(t) &= e^{iH_0 t} O e^{-iH_0 t}, \end{aligned} \quad (3.22)$$

with

$$H_0 = \int d^3x \mathcal{H}_0. \quad (3.23)$$

Accordingly, the evolution of the operators is well and analytically under control, as we have solved the free theory in [Chapter 2](#), and all operators can be written in terms of the creation and annihilation operators of the free theory.

The states evolve with the interaction Hamiltonian

$$i\partial_t |f\rangle = H_{\text{int}} |f\rangle. \quad (3.24)$$

[Equation \(3.21\)](#) looks deceptively simple, and for a time-independent interaction Hamiltonian the time evolution operator would be given by [\(3.3\)](#). However, the interaction Hamiltonian does not commute with the free one, and hence it follows with [\(3.22\)](#), that H_{int} is time-dependent,

$$i\partial_t H_{\text{int}} = [H_0, H_{\text{int}}] \neq 0 \quad \rightarrow \quad H_{\text{int}} = H_{\text{int}}(t). \quad (3.25)$$

We remark that if the interaction Hamiltonian would commute with the free one, the free states would evolve trivially in time and hence the whole theory would stay free.

It is left to determine the unitary time evolution operator $U(t, t')$ of the states with the property

$$|f(t)\rangle = U(t, t_0) |f(t_0)\rangle, \quad (3.26)$$

With [\(3.24\)](#) we find

Time evolution of $U(t, t_0)$

$$i\partial_t U(t, t_0) = H_{\text{int}}(t) U(t, t_0). \quad (3.27)$$

The time evolution operator $U(t, t_0)$ defined via the differential equation [\(3.27\)](#) encodes all the scattering processes of the theory. This entails that if we initiate our time evolution within an eigenstate of the free theory at some time t_0 , it will evolve into a fully interacting one via the multi-scattering events encoded in $U(t, t_0)$. The scattering matrix S is defined as the time evolution operator from $t_0 \rightarrow -\infty$ to $t \rightarrow +\infty$,

S-matrix

$$S = \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow +\infty}} U(t, t_0), \quad (3.28)$$

and carries all interactions in the theory. In short, if we know the S -matrix, we have 'solved' the theory: the result of a given experiment is then computed by computing the matrix element $\langle f|S|in\rangle$. Here, $|in\rangle$ is the initial

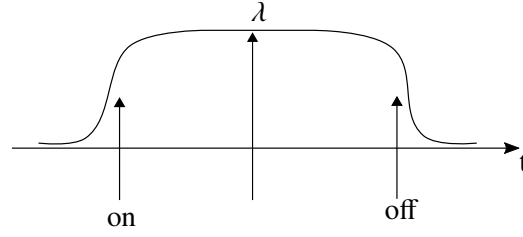


Figure 3.2.: Sketch of the adiabatic switching on and off of the coupling λ of the interaction (3.21).

state of the experiment, say a colliding $p\bar{p}$ pair, two colliding bunches or two colliding beams at LHC, and $|f\rangle$ is the outcome of such a collision measured in the detector (ALICE; ATLAS, CBM, LHCb).

Strictly speaking, the S -matrix is only well-defined if λ is adiabatically switched on and off for asymptotically large times $-t_0$, $t \rightarrow \infty$, for a graphical depiction see Section 3.1. Otherwise it would collect infinitely many scattering events. Thus, the initial state $|i\rangle$ and the final state $|f\rangle$ are free states,

$$\begin{aligned} |\text{state } t \rightarrow -\infty\rangle &= |i\rangle \\ |\text{state } t \rightarrow +\infty\rangle &= |f\rangle, \end{aligned} \quad (3.29)$$

which evolve non-trivially through scattering events transmitted by the interaction Hamiltonian. For a proper treatment of the S -matrix the LSZ-formalism (LSZ: Lehmann, Symanzik, Zimmermann) is used, as the naive treatment of respective scatterings come with infinities, the precursors of which we already discussed in the free theory in Chapter 2. For the discussion of LSZ further preparation is required, and we come back to it in Section 3.5.

It is left to derive an explicit and tractable expression for $U(t, t_0)$ that allows an iterative treatment of scattering events. For this purpose, we take the infinitesimal form of (3.24) and use it to rewrite the state $|f(t)\rangle$ iteratively,

$$\begin{aligned} |f(t + \Delta t)\rangle &= |f(t)\rangle - i \Delta t H_{\text{int}}(t) |f(t)\rangle \\ &= \left(1 - i \Delta t H_{\text{int}}(t)\right) |f(t)\rangle \\ &= \left(1 - i \Delta t H_{\text{int}}(t)\right) \left(1 - i \Delta t H_{\text{int}}(t - \Delta t)\right) |f(t - \Delta t)\rangle \\ &\quad \vdots \\ &= \prod_{n=0}^N \left(1 - i \Delta t H_{\text{int}}(t - n \Delta t)\right) |f(t - N \Delta t)\rangle. \end{aligned} \quad (3.30)$$

Equation (3.30) provides us with a product form of the time evolution in terms of infinitesimal time-steps,

$$U(t + \Delta t, t - N \Delta t) = \prod_{n=0}^N \left(1 - i \Delta t H_{\text{int}}(t - n \Delta t)\right). \quad (3.31)$$

Now we order (3.31) in powers of the infinitesimal time-step Δt ,

$$U(t + \Delta t, t - N \Delta t) = 1 + (-i)\Delta t \sum_{n=0}^N H_{\text{int}}(t - n \Delta t) + (-i)^2(\Delta t)^2 \sum_{n < m} H_{\text{int}}(t - n \Delta t) H_{\text{int}}(t - m \Delta t) + \dots \quad (3.32)$$

Note, that $n < m$ in the second sum corresponds to the time 'on the left' being larger than the time 'on the right' (time ordering). This entails, that for a given index n , the index m is summed from N to n . Put differently, the time $t - m\Delta t$ is summed from $t_0 = t - N\Delta t$ to $t' = t - (n + 1)\Delta t$.

In the limit $\Delta t \rightarrow 0$ with a fixed $N\Delta t = t - t_0$, all sums turn into integrals and (3.32) approaches

$$1 + (-i) \int_{t_0}^t dt' H_{\text{int}}(t') + (-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' H_{\text{int}}(t') H_{\text{int}}(t'') + \dots, \quad (3.33)$$

where the left integral in the last term corresponds to the sum over n and the second integral to that over m . Naturally, the integrals carry the same *time ordering* as the sums. The higher terms in Δt are derived similarly, all carrying the time ordering and the combinatorial factor unity. This leads us to the final expression for the time evolution operator,

Time-evolution operator

$$U(t, t_0) = T \exp \left\{ -i \int_{t_0}^t dt' H_{\text{int}}(t') \right\} \quad \text{for } t > t_0, \quad (3.34)$$

with the time ordering operator T , defined by the property

$$TA(t)B(t') = A(t)B(t')\theta(t - t') + B(t')A(t)\theta(t' - t). \quad (3.35)$$

The series expansion of the exponential in (3.34) without time ordering leads to n th powers of the interaction term $-i \int dt' H_{\text{int}}(t')$ with the prefactor $1/(n!)$. The time ordering of the n th power of the intergal leads to $n!$ identical time-ordered integrals such as in (3.33), which proves the validity of the representation (3.34). As an example for the above we use again the quadratic term, which already reveals the underlying structure.

Example 3-4: Time ordering for the second order term of $U(t, t_0)$. This example shows, how the time ordering operator acts on the second order term in the expansion of (3.34), yielding the second order term of (3.33).

$$\begin{aligned} \frac{1}{2} T \int_{t_0}^t dt' H_{\text{int}}(t') \int_{t_0}^t dt'' H_{\text{int}}(t'') &= \frac{1}{2} \left[\int_{t_0}^t dt' H_{\text{int}}(t') \int_{t_0}^{t'} dt'' H_{\text{int}}(t'') \quad (t'' < t') \right. \\ &\quad \left. + \int_{t_0}^t dt'' H_{\text{int}}(t'') \int_{t_0}^{t''} dt' H_{\text{int}}(t') \right] \quad (t'' > t') \\ &= \int_{t_0}^t dt' H_{\text{int}}(t') \int_{t_0}^{t'} dt'' H_{\text{int}}(t''). \end{aligned} \quad (3.36)$$

As already argued above, this derivation works analogously for the higher order terms in (3.35), and the occurrence of $n!$ identical terms cancels the $1/(n!)$ factor from the series expansion.

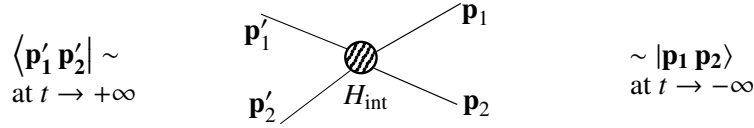


Figure 3.3.: 2-to-2 scattering process. The in-state for $t \rightarrow -\infty$ is proportional to a momentum state of two (free) particles $\text{in} \rangle \propto |\mathbf{p}_1 \mathbf{p}_2 \rangle$, as is the out-state $\text{out} \rangle \propto |\mathbf{p}'_1 \mathbf{p}'_2 \rangle$.

We proceed with the discussion of the interactions or scatterings generated by H_{int} . First we note that all terms in ϕ^4 carries four creation and annihilation operators. Ignoring the operator ordering for the moment, we deal with a^4 , $a^\dagger a^3$, $(a^\dagger)^2 a^2$, $(a^\dagger)^3 a$, $(a^\dagger)^4$. For example, one term reads

$$H_{\text{int}} = \int d^3x \phi^4(x) \sim a^2 (a^\dagger)^2, \quad (3.37)$$

with two creation operators to the right and two annihilation operators to the left. Acting with this specific part of the interaction Hamiltonian on a given state creates two additional particles in this state and then annihilates them again. If we apply H_{int} on the vacuum, this leads to infinite vacuum processes in $\langle 0 | H_{\text{int}} | 0 \rangle$. These processes are not physical (similar to the vacuum energy in the free case) and we shall drop them from now on. This is done by *normal ordering*: all annihilation operators in a product of creation and annihilation operators are pulled to the right. If ordering (3.37) this way, we get $\langle 0 | H_{\text{int}} | 0 \rangle = 0$.

Example 3-5: 2-to-2 scattering.

Now we elucidate this in a more physical example, 2-to-2 scattering: our initial state for $t \rightarrow -\infty$ is a momentum Eigenstate of two (free) particles or fields, $|\mathbf{p}_1 \mathbf{p}_2 \rangle = \sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2}} |\mathbf{p}_1 \mathbf{p}_2 \rangle$, and the final state at $t \rightarrow \infty$ also contains two (free) particles with different momenta, $|\mathbf{p}'_1 \mathbf{p}'_2 \rangle$. This situation or experiment is depicted in Figure 3.3. The S-matrix for this scattering event is given by

$$S = \mathbb{1} + iT, \quad (3.38)$$

where the unit matrix $\mathbb{1}$ represents the part without scattering. For the 2-to-2 scattering event depicted in Figure 3.3 we arrive at

$$iT_{fi} \simeq -i \sqrt{2\omega_{\mathbf{p}'_1} 2\omega_{\mathbf{p}'_2} 2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2}} \langle 0 | a(\mathbf{p}'_1) a(\mathbf{p}'_2) \left[\frac{\lambda}{4!} \int d^4x \phi^4(x) \right] a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle + O(\lambda^2), \quad (3.39)$$

and from now on we drop the higher order terms and only consider the first order linear in λ .

To explicitly compute the scattering part iT_{fi} of the S-matrix, we use the commutation relations (2.95) of the creation and annihilation operators and pull all creation operators in H_{int} to the left and all annihilation operators to the right. We conclude that due to (2.103) and the present situation with the same number of particles in the in and out states, only terms with the same number of creation and annihilation operators contribute. Accordingly, we only have to consider the terms in H_{int} with two creation and two annihilation operators.

Moreover, as discussed above, first we only consider normal ordered terms and hence the normal ordered interaction Hamiltonian. Normal ordered expressions are marked by colons, e.g.,

$$: a(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) : = a^\dagger(\mathbf{p}_2) a(\mathbf{p}_1). \quad (3.40)$$

For instance, with normal ordering the infinite vacuum terms of the free Hamiltonian H_0 in (2.99) disappear,

$$: H_0 := \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(: \frac{1}{2} a^\dagger(\mathbf{p}) a(\mathbf{p}) + \frac{1}{2} a(\mathbf{p}) a^\dagger(\mathbf{p}) : \right) = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a^\dagger(\mathbf{p}) a(\mathbf{p}), \quad (3.41)$$

More importantly, the normal ordered interaction Hamiltonian, integrated over time, $\int dt H_{\text{int}}$, already yields all scattering processes, that connects the incoming fields to the outgoing ones via the interaction term. Concentrating on the operators involved, we get

$$\begin{aligned} \frac{\lambda}{4!} \int d^4x : \phi(x)^4 & \sim \frac{\lambda}{4!} \left[: (a^\dagger)^2 a^2 + a^\dagger a a^\dagger a + a a^\dagger a a^\dagger + a^\dagger a^2 a^\dagger + a (a^\dagger)^2 a + a^2 (a^\dagger)^2 : \right] \\ & = \frac{\lambda}{4} (a^\dagger)^2 a^2, \end{aligned} \quad (3.42)$$

while the respective full expression including the spacial momentum and space-time integrals reads

$$\begin{aligned} \frac{\lambda}{4!} \int d^4x : \phi(x)^4 & := \frac{\lambda}{4} \left(\prod_{i=1}^4 \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}_i}}} \right) \int d^4x e^{-ix(q_3+q_4-q_1-q_2)} \left[a^\dagger(\mathbf{q}_1) a^\dagger(\mathbf{q}_2) a(\mathbf{q}_3) a(\mathbf{q}_4) \right] \\ & = \frac{\lambda}{4} \left(\prod_{i=1}^4 \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}_i}}} \right) \left[a^\dagger(\mathbf{q}_1) a^\dagger(\mathbf{q}_2) a(\mathbf{q}_3) a(\mathbf{q}_4) \right] (2\pi)^4 \delta(q_1 + q_2 - (q_3 + q_4)), \end{aligned} \quad (3.43)$$

where q_3, q_4 are the momenta of the creation operators and q_1, q_2 that of the annihilation operators. The integration over space-time leads to a δ -function that carries total four-momentum conservation in the scattering process. Hence, one of the momentum integrals in (3.43) can be performed, eg. leading to $q_4 = q_1 + q_2 - q_3$. We emphasise that this basic property holds not only for on-shell, but also for off-shell processes.

Acting on the in-state $\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2}} |\mathbf{p}_1 \mathbf{p}_2\rangle$ with the products of annihilation and creation operators in (3.43), annihilates the two incoming fields with on-shell momenta p_1 and p_2 , and generates two with momenta q_3, q_4 with total momentum $q_3 + q_4 = p_1 + p_2$. Then, the projection on the out-state leads to $p_1 + p_2 = p_3 + p_4$. First leaving out the space-time integral that leads to total momentum conservation, we get

$$\begin{aligned} & \frac{\lambda}{4} \left(\prod_i \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}_i}}} \right) e^{-ix(q_3+q_4-q_1-q_2)} \sqrt{2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2} 2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2}} \\ & \quad \times \langle 0 | a(\mathbf{p}'_1) a(\mathbf{p}'_2) \left(a^\dagger(\mathbf{q}_1) a^\dagger(\mathbf{q}_2) a(\mathbf{q}_3) a(\mathbf{q}_4) \right) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \\ & = 4 \frac{\lambda}{4} \left(\prod_i \int \frac{d^3q_i}{(2\pi)^3} \sqrt{\frac{2\omega_{\mathbf{q}_i}}{2\omega_{\mathbf{q}_i}}} \right) e^{-ix(q_3+q_4-q_1-q_2)} \delta(\mathbf{p}'_1 - \mathbf{q}_1) \delta(\mathbf{p}'_2 - \mathbf{q}_2) \delta(\mathbf{p}_1 - \mathbf{q}_3) \delta(\mathbf{p}_2 - \mathbf{q}_4) \\ & = \lambda e^{-ix(p_1+p_2-p'_1-p'_2)}, \end{aligned} \quad (3.44)$$

where we have consecutively pulled the annihilation operators to the right with

$$\begin{aligned} a(\mathbf{q}_4) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle & = \left[a(\mathbf{q}_4), a^\dagger(\mathbf{p}_1) \right] a^\dagger(\mathbf{p}_2) | 0 \rangle + a^\dagger(\mathbf{p}_1) a(\mathbf{q}_4) a^\dagger(\mathbf{p}_2) | 0 \rangle \\ & = \left(2\pi^3 \delta(\mathbf{q}_4 - \mathbf{p}_1) a^\dagger(\mathbf{p}_2) + a^\dagger(\mathbf{p}_1) \left[a(\mathbf{q}_4), a^\dagger(\mathbf{p}_2) \right] + a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) a(\mathbf{q}_4) \right) | 0 \rangle \\ & = (2\pi)^3 \left(\delta(\mathbf{q}_4 - \mathbf{p}_1) a^\dagger(\mathbf{p}_2) + \delta(\mathbf{q}_4 - \mathbf{p}_2) a^\dagger(\mathbf{p}_1) \right), \end{aligned} \quad (3.45)$$

and similar relations. In a final step we perform the space-time integration, leading to total four-momentum conservation

$$\int d^4x e^{-ix(p_1+p_2-p'_1-p'_2)} = (2\pi)^4 \delta(p_1 + p_2 - p'_1 - p'_2). \quad (3.46)$$

In summary this leads us to

$$iT_{fi} = iM(2\pi)^4 \delta(p_1 + p_2 - p'_1 - p'_2) \quad \text{with} \quad M = -\lambda, \quad (3.47)$$

where in general M denotes the matrix element, which here is trivially just the coupling.

We close this example with a qualitative discussion of the remaining terms. We have already argued that these terms, analogously to the infinite vacuum terms in the free Hamiltonian, comprise vacuum processes. To see this explicitly, we start with the full interaction Hamiltonian and rewrite it in terms of the normal ordered one and the remaining terms. The latter are proportional to a single commutator and a pair of creation and annihilation operators, and a product of two commutators. These terms are created, when pulling all creation operators to the left and all annihilation operators to the right. Starting with the sum of terms in (3.42), we arrive at

$$\begin{aligned} H_{\text{int}} = :H_{\text{int}}: & \\ & + \frac{\lambda}{2} \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right) \int d^3x \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \sqrt{\frac{1}{2\omega_{\mathbf{q}_1} 2\omega_{\mathbf{q}_2}}} e^{-ix(q_1 - q_2)} a^\dagger(\mathbf{q}_2) a(\mathbf{q}_1) \\ & + \frac{\lambda}{8} \int d^3x \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2, \end{aligned} \quad (3.48)$$

where the second line collects all (12) terms proportional to $[a, a^\dagger]a^\dagger a$ and the third line collects all (3) terms proportional to $[a, a^\dagger]^2$. The interpretation of this term is discussed in detail in Sections 3.2 and 3.3, at present we only remark that the commutator $[a, a^\dagger]$ is nothing but the vacuum expectation value of two fields, also discussed later around (3.53). The latter expectation value is the propagator (or correlation) of two fields in momentum space. Its Fourier transform is the two-point correlation in space-time, $\langle 0|\phi(y)\phi(x)|0\rangle$ and describes the creation of a field or particle at x , its propagation to y and its annihilation there. In short, it describes the *propagation* of a field from x to y . Hence the vacuum term in (3.48) or rather its integral over time can be rewritten as

$$\frac{\lambda}{8} \int d^4x \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2 = \frac{\lambda}{8} \int d^4x (\langle 0|\phi(x)\phi(x)|0\rangle)^2. \quad (3.49)$$

In view of the above interpretation of propagation, (3.49) describes the creation of two fields at x , their 'virtual' propagation to the same point, and their 'subsequent annihilation there. Clearly such a process is not physical and only arises due to our chosen expansion scheme: perturbation theory. We shall see in the following, that these apparent deficiencies can be treated in a consistent systematic way, leaving us with finite physical results for e.g. cross sections.

Inserting the full Hamiltonian and not just the normal order part in the expansion of the 2-to-2 scattering process, leads to further terms, and their graphical depiction and interpretation as well as the physical term is depicted in Figure 3.4:

The first diagram is the scattering term, we have just discussed. It is proportional to the coupling λ and carries no momentum dependence. This lack of any 'feature' originates in the fact, that we are dealing with a scalar (featureless) field.

The first line of vacuum diagrams includes the trivial process related to the matrix element of the unit matrix $\mathbb{1}$ in the S -matrix, multiplied with a vacuum diagram, that connects two of the four fields with two other ones respectively and is proportional to λ . The full prefactor of the trivial process is given by

$$-i\frac{\lambda}{8} \int d^4x \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2, \quad (3.50)$$

$$\begin{array}{ccc}
 \begin{array}{c} \mathbf{p}_1 \searrow \nearrow \mathbf{p}'_1 \\ \mathbf{p}_2 \nearrow \searrow \mathbf{p}'_2 \end{array} & : & -i\lambda \quad (2\pi)^4 \delta(p_1 + p_2 - p'_1 - p'_2) \\
 & & \text{interaction strength} \quad \text{4-momentum conservation}
 \end{array}$$

Vacuum parts:

$$\begin{aligned}
 & -i \frac{\lambda}{8} \left(\begin{array}{c} \mathbf{p}_1 \quad \mathbf{p}'_1 \\ \hline \mathbf{p}_2 \quad \mathbf{p}'_2 \end{array} + \begin{array}{c} \mathbf{p}_1 \quad \mathbf{p}'_1 \\ \searrow \quad \nearrow \\ \mathbf{p}_2 \quad \mathbf{p}'_2 \end{array} \right) \times \int d^4x \begin{array}{c} \text{two circles} \\ \nearrow \\ \text{circle} \end{array}^2 \\
 & \qquad \qquad \qquad \langle 0 | a(\mathbf{p}'_1) a(\mathbf{p}'_2) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \\
 & -i \frac{\lambda}{2} \left(\begin{array}{c} \mathbf{p}_1 \quad \mathbf{p}'_1 \\ \hline \mathbf{p}_2 \quad \mathbf{p}'_2 \end{array} \times \begin{array}{c} \text{circle} \\ \nearrow \quad \searrow \\ \mathbf{p}_2 \quad \mathbf{p}'_2 \end{array} + (p_1 \leftrightarrow p_2) + (p'_1 \leftrightarrow p'_2) + (p_1 \leftrightarrow p_2, p'_1 \leftrightarrow p'_2) \right).
 \end{aligned}$$

Figure 3.4.: All parts of the linear order of the 2-to-2 scattering process, including the vacuum contributions.

which is precisely the vacuum term in the Hamiltonian. Evidently, this process to the n th power is also generated in the n th power of the interaction term, $1/n! (\int d^4x H_{\text{int}})^n$. Summing up all these terms would lead to a phase multiplying the identity part in the S -matrix,

$$\mathbb{1} \sum_{n=0}^{\infty} \frac{1}{n!} \left[-i \frac{\lambda}{8} \int d^4x \left(\int \frac{d^3}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2 \right]^n = \mathbb{1} \exp \left\{ -i \frac{\lambda}{8} \int d^4x \left(\int \frac{d^3}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2 \right\}, \quad (3.51)$$

the vacuum terms sum up to a phase. We shall see later, that all terms exhibit these phases, which accordingly can and should be dropped. For the present 2-to-2 scattering process and in the first order this entails

$$\begin{aligned}
 & \langle 0 | a(\mathbf{p}'_1) a(\mathbf{p}'_2) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \left[1 - i \frac{\lambda}{8} \int d^4x \left(\int \frac{d^3}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2 \right] \\
 & = \langle 0 | a(\mathbf{p}'_1) a(\mathbf{p}'_2) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \left[\exp \left\{ -i \frac{\lambda}{8} \int d^4x \left(\int \frac{d^3}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \right)^2 \right\} + \mathcal{O}(\lambda^2) \right]. \quad (3.52)
 \end{aligned}$$

As has been indicated above, it can be shown that all the vacuum terms sum up to an infinite *phase*, and this property extends to all vacuum processes. However, these phases and the vacuum loops involve infinities, which calls for an appropriate treatment. This is discussed later in the Chapter about renormalisation.

The above example of the on-shell 2-to-2 scattering process is the building block of perturbation theory: all scattering processes can be built up from on- and of-shell 2-to-2, 1-to-3 and 3-1 scattering processes, all following the above pattern.

At the linear order evaluated above the time ordering did not play any rôle. In all higher orders it does. Accordingly, we will have to deal with time ordered matrix elements or expectation values of powers of fields. Moreover, from the above analysis we saw already that the two point function in the vacuum will play the key rôle in our expansion. In the linear order (and higher order ones) all our final expressions are proportional to

products of commutators of the creation and annihilation operators from the interaction Hamiltonian and the states. In the free theory the commutator is nothing but the vacuum expectation values of a product of fields (2.90) in momentum space,

$$\langle 0 | \tilde{\phi}(p_2) \tilde{\phi}(p_1) | 0 \rangle = \frac{1}{\sqrt{2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2}}} [a(p_2), a^\dagger(p_1)] = \frac{1}{2\omega_{\mathbf{p}_1}} (2\pi)^3 \delta(\mathbf{p}_1 - \mathbf{p}_2). \quad (3.53)$$

With the time ordering, (3.53) has to turn into a time-ordered two-point function. Moreover, the creation and annihilation operators in the initial and final states carry the earliest and latest time respectively, and all our expressions can be reduced to sums of vacuum expectation values of time ordered products of fields, the time-ordered two-point function or propagator playing a special rôle.

For its evaluation we use the fact, that while states and operators and their time evolution depends on the underlying picture, expectation values (matrix elements) do not. In the latter, the respective unitary time evolutions cancel out. Hence, we will start with the two-point function in the Heisenberg picture, and then rewrite it in terms of the interaction picture. The advantage of the latter is that it is tailor made for a perturbative expansion. To begin with, we introduce the vacuum of the full theory $|\Omega\rangle$,

$$i\partial_t |\Omega\rangle = 0. \quad (3.54)$$

As we have discussed in the beginning of this Chapter, in the Heisenberg picture the operators evolve with the full Hamiltonian,

$$i\partial_t \phi_H = [\phi_H, H] \quad \longrightarrow \quad \phi_H(x) = e^{iHt} \phi(0, \mathbf{x}) e^{-iHt}, \quad (3.55)$$

with the solution (3.9) for the field operator $\phi_H(t)$ in the free theory. The two-point function of interest is given by the time-ordered two-point function,

$$\langle \Omega | T \phi_H(x) \phi_H(y) | \Omega \rangle. \quad (3.56)$$

For rewriting (3.56) within the interaction picture we have to disentangle the full time evolution with H and that with H_0 and H_{int} for both, the states and the operators.

In the interaction picture, the states evolve with H_{int} and the operators evolve with H_0 ,

$$\begin{aligned} |f(t)\rangle_I &= U(t, 0) |f(0)\rangle_I \\ i\partial_t \phi_I &= [\phi_I, H_0] \quad \longrightarrow \quad \phi_I(x) = e^{iH_0 t} \phi(0, \mathbf{x}) e^{-iH_0 t}, \end{aligned} \quad (3.57)$$

The time evolution of the operators in the interaction picture is the same as in the free theory, and the solution (3.9) readily carries over to the present case. For the derivation one can use that for an asymptotically early time, $t \rightarrow -\infty$, the field operator obeys the free solution. Then the derivation steps leading to (3.9) are repeated and we arrive at

$$\phi_I(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a(\mathbf{p}) e^{-ipx} + a^\dagger(\mathbf{p}) e^{ipx} \right)_{p_0=\omega_{\mathbf{p}}}. \quad (3.58)$$

For rewriting ϕ_H in (3.56) with ϕ_I defined in (3.58), we use that the operator $\phi(0, \mathbf{x})$ can be obtained from ϕ_H by sandwiching it by the (inverse) time evolution. In the Heisenberg picture we may reverse the time evolution in (3.55), and solve it for $\phi(0, \mathbf{x})$. Doing the same for ϕ_I in (3.57) leads us to

$$\phi(0, \mathbf{x}) = e^{-iHt} \phi_H(x) e^{iHt}, \quad \phi(0, \mathbf{x}) = e^{-iH_0 t} \phi_I(x) e^{iH_0 t}. \quad (3.59)$$

We emphasise that both, H and H_0 are time-independent as their time dependence follows the time evolution equation in the Heisenberg picture (3.55) and interaction picture (3.57) respectively.

In combination (3.59) leads us to

$$\phi_H(x) = U(0, x^0) \phi_I(x) U(x^0, 0), \quad (3.60)$$

with

$$U(t, 0) = e^{iH_0 t} e^{-iHt}. \quad (3.61)$$

With the operator relation between ϕ_I and ϕ_H we can also rewrite $\phi_H|\Omega\rangle$ within the interaction picture. Indeed, it follows for general states, that

$$\begin{aligned} \phi_H(x) |f\rangle_H &= U(0, x^0) \phi_I(x) U(x^0, 0) |f\rangle_H \\ i\partial_t U(x^0, 0) |f\rangle_H &= H_{\text{int}} U(x^0, 0) |f\rangle_H, \end{aligned} \quad (3.62)$$

the state $U(x^0, 0) |f\rangle_H$ satisfies the time evolution equation of states in the interaction picture. As it is a unitary rotation of the state $|f\rangle_H$, it is tempting to identify $U(x^0, 0) |f\rangle_H$ with the normalised interaction picture state $|f(t)\rangle_I$. To check the normalisation, let us discuss its relative normalisation to the free state, which are well-defined:

To that end we consider the states for asymptotically early or late times, $t \rightarrow \pm\infty$, where λ is switched off adiabatically, and $|f\rangle_I$ tend to free in/out states, see Section 3.1. We also use, that the time evolution from $t \rightarrow \infty$ to $t = 0$ is the inverse of that from $t = 0$ to $t \rightarrow \infty$, that is $U(0, \infty) = U(\infty, 0)^{-1}$. This operator includes the time evolution $U(t, 0)$ within an infinite time interval $t \rightarrow \infty$ and has to be regularised. Typically this is done by tilting the time into the complex plane,

$$t \rightarrow t(1 - i\epsilon), \quad (3.63)$$

with $\epsilon \rightarrow 0$. This assumes that the eigenvalues of U are given by that of H , with H_{int} leading to a small perturbation. Then, only the lowest lying Eigenvalues survives the $t \rightarrow \infty$ limit. Moreover, *adiabatically* switching on and off the coupling entails, that the full states approach the free ones with the time evolution operator U for asymptotically early and late times. For example we have

$$|n\text{-particles}\rangle_{\text{full}}(t=0) \xrightarrow{U(t \rightarrow \infty, 0)} |n\text{-particles}\rangle_{\text{free}}. \quad (3.64)$$

In combination this leads us to

$$\begin{aligned} \langle\Omega| U(0, x^0) &= \langle\Omega| U(0, \infty) U(\infty, x^0) \\ &= \sum_n \langle\Omega| U(0, \infty) |n\rangle_I \langle n| U(\infty, x^0) \\ &= \langle\Omega| U(0, \infty) |0\rangle \langle 0| U(\infty, x^0). \end{aligned} \quad (3.65)$$

For the last step we have used (3.64) and the fact, that with (3.63) and asymptotically large times the lowest Eigenvalue of U dominates. Finally, we need a non-vanishing overlap $\langle\Omega| U(0, \infty) |0\rangle \neq 0$, where again we resort to the perturbative setting, that entails that $|\Omega\rangle$ is only a perturbation of the free vacuum and U is only a perturbation of unity. Similarly to (3.65) it follows that

$$U(x^0, 0) |\Omega\rangle = U(x^0, -\infty) |0\rangle \langle 0| U(-\infty, 0) |\Omega\rangle, \quad (3.66)$$

In summary, both left hand sides (states and dual states) in (3.65) and (3.66) evolve from them the free states at $t = \pm\infty$. It also can be shown that the respective factors are phases, i.e. $|\langle\Omega| U(0, \infty) |0\rangle| = 1$. For this purpose we first remark that

$$|\langle\Omega| U(0, x^0)| = 1, \quad |\langle 0| U(-\infty, x^0)|^2 = 1. \quad (3.67)$$

Equation (3.67) follows straightforwardly from U being unitary, and hence

$$|\langle \Omega | U(0, x^0) | \Omega \rangle|^2 = \langle \Omega | U(0, x^0) U^\dagger(0, x^0) | \Omega \rangle = \langle \Omega | \Omega \rangle = 1, \quad (3.68)$$

and similarly for $|\langle 0 | U(-\infty, x^0) | \Omega \rangle|^2$. Combining the two relations in (3.67) yields

$$|\langle \Omega | U(0, \infty) | 0 \rangle|^2 = \langle \Omega | U(0, \infty) | 0 \rangle \langle 0 | U^\dagger(0, \infty) | \Omega \rangle = 1, \quad (3.69)$$

where for the second identity in (3.69) we have used that at $t = \infty$ we can substitute $|0\rangle\langle 0| \rightarrow \sum_n |n\rangle\langle n|$, see also (3.65).

We close this analysis of the link between interaction and Heisenberg pictures with confirming the evolution equation of U as defined in (3.61),

$$\begin{aligned} i\partial_t U(t, 0) &= H_I(t) U(t, 0), \\ H_I(t) &= H_{\text{int}}(t) = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t} = \frac{\lambda}{4!} \int d^3x \phi_I(x)^4, \\ i\partial_t H_I &= [H_I, H_0]. \end{aligned} \quad (3.70)$$

In summary, the evolution operator is that defined in (3.34).

The results above allow us to compute the time ordered propagator (3.56) in terms of interaction picture operators and states. To that end we disentangle the two different parts of the time ordered product,

$$\langle \Omega | T \phi_H(x) \phi_H(y) | \Omega \rangle = \langle \Omega | \phi_H(x) \phi_H(y) | \Omega \rangle \theta(x^0 - y^0) + \langle \Omega | \phi_H(y) \phi_H(x) | \Omega \rangle \theta(y^0 - x^0), \quad (3.71)$$

and proceed with the first part with $x^0 > 0 > y^0$,

$$\begin{aligned} &\langle \Omega | T \phi_H(x) \phi_H(y) | \Omega \rangle \\ (3.60) \rightarrow &= \langle \Omega | U(0, x^0) \phi_I(x) U(x^0, y^0) \phi_I(y) U(y^0, 0) | \Omega \rangle \\ (3.66) \rightarrow &= \frac{\langle 0 | U(\infty, x^0) \phi_I(x) U(x^0, y^0) \phi_I(y) U(y^0, -\infty) | 0 \rangle}{\left(\langle \Omega | U(0, \infty) | 0 \rangle \cdot \langle 0 | U(-\infty, 0) | \Omega \rangle \right)^{-1}}. \end{aligned} \quad (3.72)$$

In (3.72) we have used, that in general

$$U(x^0, z^0) = U(x^0, y^0) U(y^0, z^0) \quad \text{for } x^0 > y^0 > z^0, \quad (3.73)$$

which follows straightforwardly from (3.34). Our result (3.72) shows a very interesting structure. To begin with, the operator in the numerator is a time ordered product of ϕ_I and U . Moreover, the time evolution operators in the denominator entail the evolution from $t = \pm\infty$ to $t = 0$: together this would comprise the time evolution from $t = -\infty$ to $t = +\infty$, the S -matrix. The following final steps aim at making explicit both structures.

First we deal with the denominator in (3.72). We know already from (3.67), that it is a phase. More explicitly we have

$$\langle \Omega | U(0, \infty) | 0 \rangle^{-1} = \langle \Omega | U(0, \infty) | 0 \rangle^* = \langle \Omega | U^\dagger(0, \infty) | 0 \rangle, \quad (3.74)$$

i.e. the normalisation factor in (3.72) is a phase. Now we follow the same arguments as in (3.65) and (3.69) for combining the two matrix elements into one of the S -matrix,

$$\begin{aligned}
 & \langle \Omega | U(0, \infty) | 0 \rangle^{-1} \langle 0 | U(-\infty, 0) | \Omega \rangle^{-1} \\
 \text{using (3.74)} \quad \rightarrow & = \langle 0 | U(\infty, 0) | \Omega \rangle \langle \Omega | U(0, -\infty) | 0 \rangle \\
 & = \langle 0 | U(\infty, 0) U(0, -\infty) | 0 \rangle = \langle 0 | U(-\infty, \infty) | 0 \rangle \\
 & = \langle 0 | S | 0 \rangle = \langle 0 | T \exp \left(-i \int dT H_{\text{int}}(t) \right) | 0 \rangle . \tag{3.75}
 \end{aligned}$$

The normalisation is nothing but the vacuum expectation value of the S -matrix. Accordingly, it has a series expansion in terms of vacuum diagrams, some of which we already have encountered in the 2-to-2 scattering example.

As already mentioned above, the numerator in (3.72) is a time order product, and using the time ordering operator we can combine all the time evolution operators into $U(\infty, -\infty)$. This leads us to

$$\begin{aligned}
 & \langle 0 | U(\infty, x^0) \phi_I(x) U(x^0, y^0) \phi_I(y) U(y^0, -\infty) | 0 \rangle \\
 & = \langle 0 | T \phi_I(x) \phi_I(y) \exp \left(-i \int dT H_{\text{int}}(t) \right) | 0 \rangle . \tag{3.76}
 \end{aligned}$$

The same line of arguments works for the second part in (3.71) with $y^0 > x^0$. In summary we obtain our final expression for the time ordered two-point function, where we drop the subscript I for the fields, $\phi_I = \phi$,

Propagator (two-point function)

$$\langle \Omega | T \phi_H(x) \phi_H(y) | \Omega \rangle = \frac{\langle 0 | T \phi(x) \phi(y) \exp \left(-i \int dT H_{\text{int}}(t) \right) | 0 \rangle}{\langle 0 | T \exp \left(-i \int dT H_{\text{int}}(t) \right) | 0 \rangle} . \tag{3.77}$$

The denominator in (3.77) is a phase, and comprises all possible vacuum diagrams. These diagrams are also present as multiplicative factors in the numerator and cancel out.

We have already discussed this property for the first order terms in our 2-to-2 example, which is related to a time ordered vacuum expectation value of the four fields, the four-point function. The four point function and general higher order correlation functions are straightforward extensions of (3.77). We get

n-point correlation function

$$\langle \Omega | T \phi_H(x_1) \cdots \phi_H(x_n) | \Omega \rangle = \frac{\langle 0 | T \phi(x_1) \cdots \phi(x_n) \exp \left(-i \int dT H_{\text{int}}(t) \right) | 0 \rangle}{\langle 0 | T \exp \left(-i \int dT H_{\text{int}}(t) \right) | 0 \rangle} , \tag{3.78}$$

with the same denominator as in (3.77), that can be expanded in terms of vacuum diagrams, and hence in powers of λ . The respective linear term in λ in the denominator in (3.77) and (3.78) is given by

$$-i \langle 0 | \int dT H_{\text{int}} | 0 \rangle = -i \langle 0 | \frac{\lambda}{4!} \int d^4x \phi(x)^4 | 0 \rangle = -i \frac{\lambda}{8} \int d^4x \left(\int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}}} \right)^2 , \tag{3.79}$$

which precisely cancels the vacuum term in (3.48). We remark, that both, phase factor (denominator) and the vacuum contributions in the numerator, are formally infinite, but cancel.

3.2. Wick's Theorem

The final result (3.78) allows us to compute general scattering processes in terms of time-ordered correlation functions: the given in-states with r fields and out-states with $n - r$ fields with $n > r$ are given by $|i\rangle \propto \phi(x_1) \cdot \phi(x_r)|0\rangle$ and $|f\rangle \propto \phi(x_1) \cdot \phi(x_{n-r})|0\rangle$. The scattering is given by the S -matrix, which leads us to (3.78). An expansion of this correlation function in terms of the number N of single scattering events (an expansion in λ^m) entails, that the general task is the computation of general time-ordered $m + n$ -point functions. This is detail in the next two Chapters: in the present Chapter Section 3.2 we discuss Wick's theorem, that relates time-ordered correlation functions to normal ordered ones, in the next one, Section 3.3, we discuss the Feynman rules, that allow for an efficient book keeping of the computation.

We start with (3.78), and expand it in powers of the coupling. The interaction exponential can be written in terms of the interaction part $S_{\text{int}}[\phi]$ of the action,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)e^{i\int d^4y\mathcal{L}_{\text{int}}(y)}|0\rangle, \quad (3.80)$$

where

$$S_{\text{int}}[\phi] = \int d^4y\mathcal{L}_{\text{int}}(y) = -\int_{-\infty}^{\infty} dt H_{\text{int}}. \quad (3.81)$$

In perturbation theory, we assume the coupling to be small, $\lambda \ll 1$, and expand the exponential in powers of λ . For the m th order we obtain,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)\prod_{i=1}^m\mathcal{L}_{\text{int}}(y_i)|0\rangle = \frac{1}{(4!)^m}\langle 0|\phi(x_1)\cdots\phi(x_n)\phi(x_{n+1})\cdots\phi(x_{n+4m})|0\rangle, \quad (3.82)$$

with $x_{n+1}, \dots, x_{n+4} = y_1; \dots; x_{n+4(m-1)}, \dots, x_{n+4m} = y_m$. In (3.82) we have dropped the m four-dimensional integrals over x_{n+1}, \dots, x_{n+4m} for emphasising the time-ordered correlation functions involved. Hence, the only building block in (3.80) is the general n -point function with $n \in \mathbb{N}$,

Time ordered n-point function

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle. \quad (3.83)$$

Importantly, $\phi = \phi_I$ evolves with the free Hamiltonian, and hence has the form of a free field operator, and (3.83) is a vacuum expectation value with the respective vacuum $|0\rangle$. In combination, this is a crucial technical simplification, we owe to using the interaction picture and it allows for a relatively straightforward computation of (3.83).

It is left to develop efficient computation techniques for evaluating the general time ordered n -point functions (3.83). The respective computation is illustrated at the example of the time-ordered two-point function, which also is important for the computation of the general n -point function. Before we proceed with this specific analysis, we briefly review the general structure: For $x_1^0 > \dots > x_n^0$ (3.83) reduces to $\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$. Now we rewrite the product of fields as its normal ordered version and additional terms, that contain at least one commutator of creation and annihilation operators (2.95), and hence at least two field operators less. In this additional terms we can iteratively normal order the remaining fields. This leaves us finally with normal ordered products of fields, multiplied by products of commutators. Then, the vacuum expectation value annihilates the terms with normal ordered fields and leaves us with the product of commutators. This structure already entails that we only have to concentrate on even n , that is $n = 2m$. In short we write qualitatively

$$\begin{aligned} T\phi(x_1)\cdots\phi(x_n) &= : \phi(x_1)\cdots\phi(x_n) : + \left((n-2) - \text{field operators} \right) \times [a, a^\dagger]\text{-terms} \\ &+ \left((n-4) - \text{field operators} \right) \times [a, a^\dagger]^2\text{-terms} + \dots + [a, a^\dagger]^{\frac{n}{2}}\text{-terms}. \end{aligned} \quad (3.84)$$

As mentioned above (3.84), all the normal ordered parts have a vanishing vacuum expectation value and the remaining last term in (3.84) is a c -number. However, due to the time ordering, the commutator terms will also include products of Heaviside θ -functions $\theta(x_{n_1}^0 - x_{n_2}^0)$ of the time arguments. The vacuum expectation value of (3.84) reads

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = [a, a^\dagger]^{\frac{n}{2}}\text{-terms}, \quad (3.85)$$

Indeed, we have already encountered these terms in our example of 2-to-2 scattering: In hindsight, all terms discussed there, are related to the right hand side of (3.85).

In case of the two-point function we have a single commutator on the right hand side, see also (3.53). For the computation indicated in (3.84), it is beneficial to write the field as a sum of its creation part, ϕ_+ , and its annihilation part, ϕ_- ,

$$\phi(x) = \phi_+(x) + \phi_-(x), \quad (3.86)$$

with

$$\begin{aligned} \phi_+(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} a^\dagger(\mathbf{p}) e^{ipx} \\ \phi_-(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} a(\mathbf{p}) e^{-ipx}. \end{aligned} \quad (3.87)$$

Moreover, we treat the different parts in the time-ordered product separately. First we discuss the case $x^0 > y^0$ and apply the iterative normal ordering procedure in (3.84), to wit,

$$\begin{aligned} T\phi(x)\phi(y) &= \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \phi_-(x)\phi_+(y) + \phi_-(x)\phi_-(y) \\ &= \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \left(\phi_+(y)\phi_-(x) + [\phi_-(x), \phi_+(y)]\right) + \phi_-(x)\phi_-(y), \end{aligned} \quad (3.88)$$

and we arrive at the simple relation

$$T\phi(x)\phi(y)\Big|_{x_0>y_0} = : \phi(x)\phi(y) : + [\phi_-(x), \phi_+(y)]. \quad (3.89)$$

The only non-trivial operator ordering in (3.89) is the normal ordering in $(: \phi_-(x)\phi_+(y) :)$. It reads

$$: \phi_-(x)\phi_+(y) : = \phi_+(y)\phi_-(x), \quad (3.90)$$

valid for all space-time points x, y . This relations follows straightforwardly from

$$: a(\mathbf{p})a^\dagger(\mathbf{q}) : = a^\dagger(\mathbf{q})a(\mathbf{p}). \quad (3.91)$$

Importantly, (3.90) and hence also (3.89) do not distinguish different space-time points (except $y = y$), and the case $x^0 < y^0$ can be treated analogously. In summary this leads us to

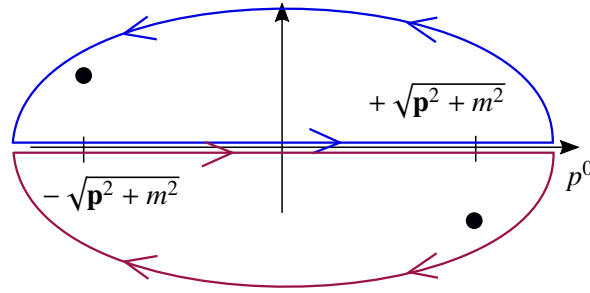
$$T\phi(x)\phi(y) = : \phi(x)\phi(y) : + [\phi_-(x), \phi_+(y)]\theta(x^0 - y^0) + [\phi_-(y), \phi_+(x)]\theta(y^0 - x^0). \quad (3.92)$$

Taking the vacuum expectation value of (3.92), the normal ordered part vanishes. This is the time ordered propagator and is also called the Feynman propagator.

Feynman-propagator

$$\begin{aligned} \mathcal{D}_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle \\ &= [\phi_-(x), \phi_+(y)]\theta(x^0 - y^0) + [\phi_-(y), \phi_+(x)]\theta(y^0 - x^0). \end{aligned} \quad (3.93)$$

Evidently, the Feynman-propagator is the key-ingredient in (time ordered) perturbation theory: Equation (3.84)



$x^0 - y^0 < 0$: close contour in upper half plane

$x^0 - y^0 > 0$: close contour in lower half plane

Figure 3.5.: Sketch of the poles of the integrand of (3.96) and the contour for the Residue theorem.

entails that the right hand side of (3.85) contains products of time ordered commutators, that are nothing but (3.93). Hence, (3.85) is given by sums of products of Feynman propagators.

For the explicit computation of the Feynman propagator we again first consider $x^0 > y^0$ with

$$\begin{aligned} [\phi_-(x), \phi_+(y)] \theta(x^0 - y^0) &= \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}} [a(\mathbf{p}), a^\dagger(\mathbf{q})] e^{-i(p^0 x^0 + \mathbf{p} \cdot \mathbf{x} + q^0 y^0 + \mathbf{q} \cdot \mathbf{y})} \theta(x^0 - y^0) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{-i p(x-y)} \theta(x^0 - y^0). \end{aligned} \quad (3.94)$$

The case $x^0 < y^0$ can be treated analogously, and we are led to

$$\mathcal{D}_F(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(e^{-i p(x-y)} \theta(x^0 - y^0) + e^{i p(x-y)} \theta(y^0 - x^0) \right). \quad (3.95)$$

It follows directly from (3.95) that the Feynman propagator can be written as

Momentum space representation of the Feynman-propagator

$$\mathcal{D}_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-i p(x-y)}, \quad \text{with } \epsilon \rightarrow 0_+. \quad (3.96)$$

To prove (3.96), we use the Residue theorem from complex analysis. We quote it here as a reminder,

Residue theorem

The contour integral of a function $f(z)$ around a closed, counter clockwise path encircling a domain where $f(z)$ has a finite number of isolated singularities (poles at $z = z_i, i = 1, 2, \dots, n$) is

$$\oint dz f(z) = 2\pi i \sum_{i=1}^n \text{Res}(f, z_i), \quad (3.97)$$

where the residue of $f(z)$ at a simple pole z_i is $\text{Res}(f, z_i) = \lim_{z \rightarrow z_i} (z - z_i) f(z)$.

The integrand in (3.96) has poles at $p_\pm^0 = \pm \sqrt{\mathbf{p}^2 + m^2} - i\epsilon$, as shown in figure Figure 3.5. In order to use (3.97) we close the contour either in the upper or lower half plane, depending on the sign of $x^0 - y^0$:

For $x^0 - y^0 > 0$ the exponential $\exp\{-ip_0(x^0 - y^0)\}$ grows exponentially for positive and diverging real part $\text{Re}(ip_0) > 0$ in the upper half plane. Accordingly, we have to close the contour in the lower half plane, where $\text{Re}(ip_0) < 0$ leads to an exponential damping. Then, the integral over the closed (red) path in **Figure 3.5** is finite and the half circle does not contribute. Hence, we can use (3.97), which gives us the result for (3.96). The relevant pole is at $p_+^0 = \sqrt{\mathbf{p}^2 + m^2} - i\epsilon \rightarrow \omega_{\mathbf{p}}$, and thus

$$\begin{aligned} \mathcal{D}_F(x-y) &= - \int \frac{d^3 p}{(2\pi)^3} \frac{2\pi i}{2\pi} \text{res}_{p_+^0} \left(\frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2\omega_{\mathbf{p}}} \Big|_{p^0 = \omega_{\mathbf{p}}}, \end{aligned} \quad (3.98)$$

with the residue $i/\omega_{\mathbf{p}} \exp\{-i\omega_{\mathbf{p}}(x^0 - y^0)\} \exp\{i\mathbf{p}(\mathbf{x} - \mathbf{y})\}$. In turn, for $x^0 - y^0 < 0$ the exponential grows exponentially for $ip_0 < 0$ and we have to close the contour in the upper half plane. The distinction of both cases can be accounted for by θ -functions, and we are led to the equivalence of (3.96) and (3.94).

Let us summarise the result for the two-point function or rather that for the time ordered product of two fields in an efficient way that allows us to extend it straightforwardly to the time order product of n fields: in short, we have parametrised the time ordered two-point function in terms of time ordered commutators, the Feynman propagator. On operator level we have (3.92), which can be written as

$$T\phi(x)\phi(y) = : \phi(x)\phi(y) : + \overline{\phi(x)\phi(y)}, \quad (3.99)$$

with the *contraction*

$$\overline{\phi(x)\phi(y)} = [\phi_-(x), \phi_+(y)] \theta(x^0 - y^0) + [\phi_-(y), \phi_+(x)] \theta(y^0 - x^0) = \mathcal{D}_F(x-y). \quad (3.100)$$

Note, that $\mathcal{D}_F(x-y)$ is a c-number and not an operator. **Equation (3.100)** generalises straightforwardly to the time ordered product of n fields, where $n \in \mathbb{N}$, that is for both even and odd number. This is Wick's theorem,

Wick's theorem

$$T\phi(x_1) \cdots \phi(x_n) = : \phi(x_1) \cdots \phi(x_n) : + \text{all contractions} :, \quad (3.101)$$

where a contraction is defined by

$$\begin{aligned} &\phi(x_1) \cdots \overline{\phi(x_i) \cdots \phi(x_j)} \cdots \phi(x_n) \\ &= \phi(x_1) \cdots \phi(x_{i-1}) \phi(x_{i+1}) \cdots \phi(x_{j-1}) \phi(x_{j+1}) \cdots \overline{\phi(x_n) \phi(x_i) \phi(x_j)}. \end{aligned} \quad (3.102)$$

We illustrate the content of (3.101) and the use of (3.102) at the example of the product of four fields:

Example 3-6: 4-point correlation function. Wick's theorem allows us to rewrite the time ordered product of four fields in terms of normal ordered products and contractions. For the sake of structural simplicity we use

the short hand notation $\phi_i = \phi(x_i)$. We find

$$\begin{aligned}
 T\phi(x_1)\cdots\phi(x_4) &= T\phi_1\phi_2\phi_3\phi_4 \\
 &= : \left[\phi_1\phi_2\phi_3\phi_4 \right. \\
 &\quad + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} \\
 &\quad \left. + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} \right] :, \tag{3.103}
 \end{aligned}$$

where e.g.

$$: \overbrace{\phi_1\phi_2\phi_3\phi_4} : = : \phi_3\phi_4 : \overbrace{\phi_1\phi_2} = : \phi_3\phi_4 : \mathcal{D}_F(x_1 - x_2). \tag{3.104}$$

The term on the right hand side of the first line of (3.103) contains four normal ordered fields, the terms in the second line contain two normal ordered fields, multiplied by one Feynman propagator or contraction, and the terms in the last line are products of two contractions and hence two Feynman propagators.

For the vacuum expectation value all terms but that in the last line vanish due to

$$\langle 0 | : O : | 0 \rangle = 0, \tag{3.105}$$

and we arrive at

$$\begin{aligned}
 \langle 0 | T\phi(x_1)\cdots\phi(x_4) | 0 \rangle \\
 = \mathcal{D}_F(x_1 - x_2)\mathcal{D}_F(x_3 - x_4) + \mathcal{D}_F(x_1 - x_3)\mathcal{D}_F(x_2 - x_4) + \mathcal{D}_F(x_1 - x_4)\mathcal{D}_F(x_2 - x_3). \tag{3.106}
 \end{aligned}$$

Each term in the second line of (3.106) corresponds to one of the terms with two contractions in the last line of (3.103), and they are ordered the same way.

It remains to prove Wick's theorem, which is done by induction: First we show, that (3.101) holds for the one- and two-point function,

$$\begin{aligned}
 T\phi_1 &= : \phi_1 : \\
 T\phi_1\phi_2 &= : \phi_1\phi_2 : + \overbrace{\phi_1\phi_2}, \tag{3.107}
 \end{aligned}$$

where $n = 1$ is trivial and $n = 2$ was our introductory example. Next, we assume that Wick's theorem applies to the n -point function, i.e. $T\phi_2\cdots\phi_{n+1}$. It is left to prove that then also the time-ordered product of $n + 1$ fields, $T\phi_1\cdots\phi_{n+1}$ satisfies Wick's theorem:

Without loss of generality we can assume that $x_1^0 \geq x_i^0$ for all $i = 1, \dots, n$. If this is not the case we can just relabel

the space-time variables x_i such that x_1^0 is the largest time. Then we find

$$\begin{aligned}
 T\phi_1 \cdots \phi_{n+1} &= \phi_1 T\phi_2 \cdots \phi_{n+1} \\
 &= \phi_1 \left(: \phi_2 \cdots \phi_{n+1} + \text{all contractions} : \right) \\
 &= \left(\phi_{1+} + \phi_{1-} \right) \left(: \phi_2 \cdots \phi_{n+1} + \text{all contractions} : \right) \\
 &=: \phi_1 \cdots \phi_{n+1} + [\phi_{1-}, \phi_2] \phi_3 \cdots \phi_{n+1} + \phi_2 [\phi_{1-}, \phi_3] \phi_4 \cdots \phi_{n+1} + \cdots + \phi_2 \cdots [\phi_{1-}, \phi_{n+1}] : \\
 &\quad + \left(\phi_{1+} + \phi_{1-} \right) \left(: \text{all contractions} : \right). \tag{3.108}
 \end{aligned}$$

The commutators in (3.108) are contractions themselves,

$$[\phi_{1-}, \phi_i] = [\phi_{1-}, \phi_{i+}] = \overline{\phi_1 \phi_i}. \tag{3.109}$$

Moreover, the contractions in the last line include smaller powers $m \leq n - 2$ of normal ordered fields, and are treated similarly to the product of n normal ordered fields in (3.108) with

$$\left(\phi_{1+} + \phi_{1-} \right) \left(: \phi_{i_1} \cdots \phi_{i_m} : \right), \tag{3.110}$$

with $i_1, \dots, i_m \in \{1, \dots, n\}$. Accordingly, we are only left with terms of the contraction form in (3.101), and all terms are present. This leads us to

$$T\phi(x_1) \cdots \phi(x_{n+1}) = : \phi(x_1) \cdots \phi(x_{n+1}) + \text{all contractions} :, \tag{3.111}$$

which completes the induction.

3.3. Feynman Rules

With Wick's theorem (3.101) we have turned every time ordered n -point function into a product of Feynman propagators (3.93) and additional normal ordered terms. As mentioned before, scattering amplitudes, as computed in perturbation theory in the interaction picture, only consist of vacuum expectation values of products of n fields. With Wick's theorem all those vacuum expectation values are reduced to the products of Feynman propagators, while the normal ordered terms drop out. As we have already seen in our 2-to-2 scattering example, the propagators either depend on space-time arguments from the in- and out-states or from the powers of the interaction term. We have to integrate over the latter arguments, while the former are kept fixed or are treated with a Fourier transform, if the states are momentum space states. This structure allow for a simple diagrammatic notation with simple composition rules, that also facilitates the computation tremendously: the Feynman diagrams and rules:

To begin with, the propagator is represented by a line with the space-time arguments as the endpoints,

$$\mathcal{D}_F(x_1 - x_2) = \langle 0|T \phi_1 \phi_2|0\rangle = \begin{array}{c} \circ \text{---} \circ \\ 1 \quad 2 \end{array}.$$

The other core ingredient is the interaction term, which is represented as a blob with four lines emanating from it, representing the four propagators, which end at the space-time point of the interaction.

For illustration we again consider the 2-2 scattering. With the simple diagrammatic rules we shall quickly rederive the result obtained there. The zeroth order term proportional to λ^0 , i.e. the term without interaction, is given by the expectation value of the 4-point function and it follows with Wick's theorem that

$$O(\lambda^0) : \quad \langle 0|T \phi_1 \phi_2 \phi_3 \phi_4|0\rangle = \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} .$$

The first order term is given by

$$\begin{aligned} & \frac{-i\lambda}{4!} \int d^4x \langle 0|T \phi_1 \phi_2 \phi_3 \phi_4 \phi \phi \phi \phi|0\rangle \\ &= \frac{-i\lambda}{4!} \int d^4x \left[4! \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\overline{\phi_1 \phi_2 \phi_3 \phi_4}} + 12 \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\overline{\phi_1 \phi_2 \phi_3 \phi_4}} + \text{perm.} \right. \\ & \quad \left. + 3 \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\overline{\phi_1 \phi_2 \phi_3 \phi_4}} + \text{perm.} \right] . \end{aligned} \tag{3.112}$$

Note, that the factor 4! accounts for all possibilities to contract ϕ^4 with $\phi_1 \cdots \phi_4$, and the factors 12 and 3 account for permutations of the contractions, that give an identical expression. This will be discussed further below. Diagrammatically and without the symmetry factors this reads

$$O(\lambda^1) : \quad \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ | \quad | \\ 3 \quad 4 \end{array} + \dots + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ 3 \quad 4 \end{array} \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \dots ,$$

where the vertices correspond to $(-i \lambda \int d^4x)$.

The second order term

$$\frac{1}{2!} \left(\frac{-i\lambda}{4!} \right)^2 \int d^4x \int d^4z \langle 0|T \phi_1 \phi_2 \phi_3 \phi_4 \phi(x)^4 \phi(z)^4|0\rangle , \tag{3.113}$$

comprises

$$\begin{aligned} O(\lambda^2) : \quad & \frac{1}{2} \left(\begin{array}{c} \frac{1}{2!} \cdot 2! \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \\ \frac{1}{4!} \cdot 4! \end{array} + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ | \quad | \\ 3 \quad 4 \end{array} \right) + \dots \\ & \dots + \frac{1}{4!} \cdot 4 \cdot 3 \left(\begin{array}{c} \frac{1}{2!} \cdot 2! \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \\ \frac{1}{4!} \cdot 4! \end{array} + \dots \right) + \frac{1}{8} \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ 3 \quad 4 \end{array} \begin{array}{c} \bigcirc \\ \bigcirc \end{array} + \dots . \end{aligned}$$

To determine the correct prefactor for each diagram, some combinatorics is required: The different possibilities of contracting $\phi\phi\phi\phi$ in H_{int} with the external fields lead to a factor $4!$, which cancels the denominator in $\frac{-i\lambda}{4!}$. Indeed this factor triggered the normalisation in (3.21). When loops are present, we further have to account for the symmetries that result from contracting the ϕ^4 amongst each others in H_{int} . For this purpose we introduce the **symmetry factor** $\frac{1}{S}$, where S corresponds to the number of interchanging components without changing the diagram. This leads us to the *Feynman rules*:

Feynman rules (position space)

$$\begin{aligned}
 \text{i)} \quad & \textcircled{1} \text{---} \textcircled{2} = \mathcal{D}_F(x_1 - x_2) \\
 \text{ii)} \quad & \text{X} = (-i\lambda) \int d^4x \\
 \text{iii)} \quad & \text{multiplication with } \frac{1}{S}.
 \end{aligned} \tag{3.114}$$

With (3.114) we have achieved our main goal of the present Chapter, systematic and computational simple rules for computing n -point functions and hence scattering amplitudes.

Still, this efficient computational way has not resolved the problem of vacuum processes: Beyond the lowest order without interaction we encounter diagrams that are completely disconnected from any of the external space-time points x_1, \dots, x_4 . Instead they close on fields from the interaction terms and are called ‘vacuum bubbles’. As we have discussed before, these terms contain infinities, but we shall see that the infinities can be removed by a proper normalisation of our input parameters, the couplings and masses as well as the normalisation of our field operators, in a systematic way. This is done in the Chapter on renormalisation.

For now we only evaluate the structure of the vacuum bubbles and discuss how the respective infinities emerge. The simplest type one is actually just a closed loop of the propagator. It emerges from a contraction of two fields from the same interaction Lagrangian or action, $\lambda/(4!) \int d^4x \phi(x)^4$. This propagator at the same space-time point x is the time ordered expectation value $\langle 0|T\phi(x)\phi(x)|0\rangle$. With (3.96) we find

$$\mathcal{D}_F(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} = \textcircled{x},$$

where the x in the diagram indicates that this process happens at the space-time point x in the expectation value $\lambda/(4!) \int d^4x \langle 0|\phi^2(x)|0\rangle^2$. The result has no x -dependence as the theory is translation invariant. The momentum integral is divergent, and we *regularise* it with the introduction of a (spatial) momentum cutoff with $\mathbf{p}^2 \leq \Lambda^2$ in the spatial momentum integration. Here, *regularisation* is used in the distributional sense analogously to introducing a regularisation of the tempered distribution such as the Dirac δ -function,

$$\delta(x) \rightarrow \delta_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}, \tag{3.115}$$

with $\epsilon \rightarrow 0_+$. In the present momentum integral case we have $\epsilon \rightarrow 1/\Lambda \rightarrow 0_+$. As the momentum dimension of $\mathcal{D}_F(0)$ is two, the result of the integration has the form

$$\mathcal{D}_F(0) = i \left(\Delta m^2 + \text{const. } \Lambda^2 \right), \tag{3.116}$$

with an additional ‘mass term’ $\text{const. } \Lambda^2$. However, finally we have to remove the cutoff as the physical scattering amplitudes cannot depend on it. We shall see, that terms such as $\text{const. } \Lambda^2$ in (3.116) can be absorbed or rather removed by an appropriate adjustment of the parameters of the theory (renormalisation).

Finally, we remark that the first term of this type we have encountered, (3.49), is even worse than (3.116) in terms of divergences. We have already discussed, that this term carries two singularities,

$$(-i\lambda) \int d^4x [\mathcal{D}_F(0)]^2 \sim i\lambda \mathcal{V}_4 \Lambda^4, \tag{3.117}$$

where \mathcal{V}_4 is the volume of space-time (infrared divergence) and Λ^4 is the volume of four-momentum space. In combination the two infinities are dimensionless as required, they have shown up in the phase factors discussed in (3.79).

There it was already indicated that we do not have to deal with all those divergences directly. The important step is to realise that so far we have only calculated the numerator of the products of Heisenberg operators that we are after. The full expression under investigation is the vacuum expectation value (3.78),

$$\langle \Omega | T \phi_{H,1} \cdots \phi_{H,n} | \Omega \rangle := \frac{\langle 0 | T \phi_1 \cdots \phi_n \exp\left(i \int d^4x \mathcal{L}_{\text{int}}\right) | 0 \rangle}{\langle 0 | T \exp\left(i \int d^4x \mathcal{L}_{\text{int}}\right) | 0 \rangle}. \quad (3.118)$$

For the computation we note, that each term $\langle 0 | T \phi_1 \cdots \phi_n \frac{(\mathcal{L}_{\text{int}})^m}{m!} | 0 \rangle$ can be ordered in terms of contractions between the ϕ_i and the \mathcal{L}_{int} 's,

$$\begin{aligned} \langle 0 | T \phi_1 \cdots \phi_n \frac{(\mathcal{L}_{\text{int}})^m}{m!} | 0 \rangle &= \langle 0 | T \phi_1 \cdots \phi_n | 0 \rangle \frac{1}{m!} \langle 0 | T (\mathcal{L}_{\text{int}})^m | 0 \rangle \\ &+ \langle 0 | T \phi_1 \cdots \phi_n \underbrace{\mathcal{L}_{\text{int}}}_{\text{contractions}} | 0 \rangle \frac{1}{(m-1)!} \langle 0 | T (\mathcal{L}_{\text{int}})^{m-1} | 0 \rangle + \dots, \end{aligned} \quad (3.119)$$

where " $\underbrace{\quad}_{\text{contractions}}$ " denotes all contractions, where internal fields from the interaction Hamiltonian are connected to *external* fields (and not amongst themselves). We use that

$$\begin{aligned} \frac{1}{m!} \langle 0 | T \phi_1 \cdots \phi_n (\mathcal{L}_{\text{int}})^m | 0 \rangle \Big|_{(\mathcal{L}_{\text{int}})^2\text{-contr.}} &= \frac{1}{m!} \langle 0 | T \phi_1 \cdots \phi_n \underbrace{(\mathcal{L}_{\text{int}})^2}_{\text{contractions}} | 0 \rangle \langle 0 | T (\mathcal{L}_{\text{int}})^{m-2} | 0 \rangle \cdot \frac{m \cdot (m-1)}{2} \\ &= \frac{1}{2} \langle 0 | T \phi_1 \cdots \phi_n \underbrace{(\mathcal{L}_{\text{int}})^2}_{\text{contractions}} | 0 \rangle \frac{1}{(m-2)!} \langle 0 | T (\mathcal{L}_{\text{int}})^{m-2} | 0 \rangle, \end{aligned} \quad (3.120)$$

and that in general the combinatorics factor for $\mathcal{L}_{\text{int}}^l$ -contractions is

$$\frac{1}{m!} \binom{m}{l} = \frac{1}{m!} \frac{m!}{(m-l)! l!} = \frac{1}{(m-l)! l!}. \quad (3.121)$$

Then,

$$\begin{aligned} \langle 0 | T \phi_1 \cdots \phi_n \exp\left(i \int d^4x \mathcal{L}_{\text{int}}\right) | 0 \rangle &= \left[\langle 0 | T \phi_1 \cdots \phi_n | 0 \rangle + \int d^4y_1 \langle 0 | T \phi_1 \cdots \phi_n \underbrace{\mathcal{L}_{\text{int}}}_{\text{contractions}} | 0 \rangle \right. \\ &\left. + \int d^4y_1 d^4y_2 \langle 0 | T \phi_1 \cdots \phi_n \underbrace{(\mathcal{L}_{\text{int}})^2}_{\text{contractions}} / 2 | 0 \rangle + \dots \right] \langle 0 | T \exp\left(i \int d^4x \mathcal{L}_{\text{int}}\right) | 0 \rangle. \end{aligned} \quad (3.122)$$

Consequently the denominator cancels all the vacuum terms. Therefore, the vacuum expectation value is given by

$$\frac{\langle 0 | T \phi_1 \cdots \phi_n \exp\left(i \int d^4x \mathcal{L}_{\text{int}}\right) | 0 \rangle}{\langle 0 | T \exp\left(i \int d^4x \mathcal{L}_{\text{int}}\right) | 0 \rangle} = \langle 0 | T \phi_1 \cdots \phi_n \underbrace{e^{i \int d^4x \mathcal{L}_{\text{int}}}}_{\text{contractions}} | 0 \rangle, \quad (3.123)$$

which corresponds to all diagrams without "vacuum bubbles".

Most explicit computations are carried out in **momentum space**, and we have already discussed the singularities in momentum space in terms of vacuum momentum integrals over the propagator. Hence we conclude this section by deriving and discussing the Feynman rules in momentum space. To begin with, the Feynman propagator in momentum space is simply the integrand in (3.96) without the phase,

$$\mathcal{D}_F(x-y) \rightarrow \mathcal{D}_F(p,q) = \mathcal{D}_F(p) (2\pi)^4 \delta^4(p-q), \quad \text{with} \quad \mathcal{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \quad (3.124)$$

where $(2\pi)^4 \delta^4(p - q)$ carries the momentum conservation in the two point function and $\mathcal{D}_F(p)$ is the inverse dispersion in momentum space. Diagrammatically, (3.124) is given by

$$\begin{array}{ccc} \circ & \text{---} & \circ & \rightarrow & \rightarrow & . \\ x_1 & & x_2 & & p & \end{array}$$

where the right hand side represents the Feynman propagator in momentum space, the arrow indicating the momentum direction important for the bookkeeping of momentum conservation at the vertices. The latter are derived from the interaction action term S_{int} in momentum space,

$$-i\lambda \int d^4x \phi(x)^4 = -i\lambda \int \prod_{i=1}^4 \frac{d^4p_i}{(2\pi)^4} \phi(p_i) (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4), \quad (3.125)$$

where the delta-function $(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)$ carries the momentum conservation at the vertex: the sum of all (incoming) momenta vanishes. This leads us to

$$\begin{array}{ccc} -i\lambda & \rightarrow & -i\lambda (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \\ \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} & \rightarrow & \begin{array}{c} p_1 \quad p_4 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ p_2 \quad p_3 \end{array} \quad p_4 = -(p_1 + p_2 + p_3) \end{array}$$

where as in the case for the propagator the arrows indicate the momentum flow.

Before we write down the Feynman rules in momentum space, we illustrate the above diagrammatic notation at the example of the second order correction to the scattering vertex already discussed below (3.113). The respective diagram (fish diagram) in momentum space is given by

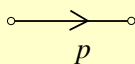
$$\begin{array}{c} p_1 \quad p \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ p_2 \quad p_3 \\ p \quad p_1 - p_2 \quad p_4 \end{array} = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \cdot \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{(p - p_1 - p_2)^2 - m^2 + i\epsilon}$$

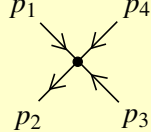
This leads us to the Feynman rules in momentum space for the ϕ^4 theory with the action

$$S[\phi] = - \int d^4x \left[\frac{1}{2} \phi (\partial_\mu \partial^\mu + m^2) \phi + \frac{\lambda}{4!} \phi^4 \right]. \quad (3.126)$$

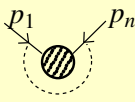
They follow directly from (3.114) and the diagrammatic conversion below (3.124) and (3.125).

Feynman rules (momentum space)

i)  = $\frac{i}{p^2 - m^2 + i\epsilon}$

ii)  = $-i\lambda$ and $p_4 = -(p_1 + p_2 + p_3)$ (momentum conservation)

iii) $\int \frac{d^4p}{(2\pi)^4}$ for each loop

iv) $(2\pi)^4 \delta^4(\sum_i p_i)$ for 

v) multiplication with $\frac{1}{S}$.

(3.127)

We emphasise, that with the developments of the present chapter we can simply read off the Feynman rules from the action (3.126): the propagator is nothing but the (time-ordered) Green function of the kinetic operator $-i(p^2 - m^2)$, and the vertex is simply the coupling $-i\lambda$.

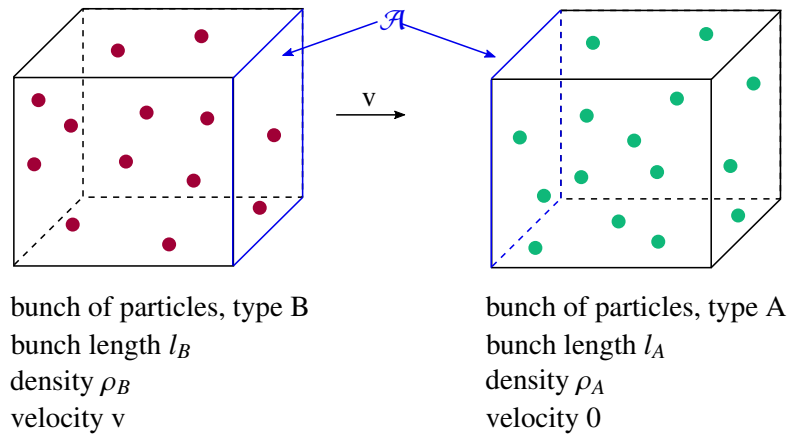


Figure 3.6.: Fixed target collision of a accelerated bunch of particle species B with velocity v with a target that consists out of particles of species A.

3.4. Cross Section

We consider general fixed target (e.g. heavy ion experiments as NA61 at CERN or CBM at FAIR) and collider experiments (e.g. $p\bar{p}$ at LHC or e^+e^- at LEP) with particle species A (fixed target in the first case) and B. In the second case two *bunches* of highly relativistic particles A and B collide 'head on' which increases the interaction energy, in the first case only one of the species (B) is accelerated, while the other is a fixed target. This increases the *luminosity* or *interaction rate* of the experiment.

Consider now as a first instructive example a single particle B colliding with a target of particles A with a constant number density ρ_A (particles / volume). While both particle species A and B are assumed to be pointlike, due to their interaction they can interact/scatter within a given scattering area around their location, effectively giving them a finite size. This scattering area is the *cross-section* σ . A simple example are billiard balls A and B of different size r_A and r_B . Their cross-section simply would be $\pi(r_A + r_B)^2$. Seemingly, the cross-section σ does not comprise that much information. However, as interactions depend on momentum, spins and other properties of the particles, one can indeed learn a lot from its energy and angular dependencies. If the target is asymptotically thick, $l_A \rightarrow \infty$, and we allow for multi-scatterings, the constant particle density ρ_A times the 'scattering volume' σl_A provides the number of scattering events. Solving this relation for the cross-section σ , we are led to

$$\sigma = \frac{N_{\text{events}}}{\rho_A l_A}, \quad (3.130)$$

This example is easily generalised to a more realistic situation where we have a whole bunch of particles B being scattered on the fixed target. Both the target (or bunch in a collider experiment) and the accelerated bunch have a finite transverse area and scatter only in an overlap area \mathcal{A} . For a fixed target experiment this situation is depicted in [Figure 3.6](#). The number of scattering particles of species B is given by

$$N_B = \rho_B (l_B \mathcal{A}), \quad (3.131)$$

with the scattering volume $l_B \mathcal{A}$. [Equation \(3.131\)](#) with $B \rightarrow A$ also holds for the target or bunch A. This preparation allows us to generalise the definition of the cross section in [\(3.130\)](#) for the scattering of a single particle B onto a target to the scattering of a particle bunch B with N_B particles on a target or bunch A: The number of scattering events on the right hand side of [\(3.130\)](#) now has to be divided by N_B , as it has risen by

the same factor. Hence, in the case of constant particle densities in bunch and target, we are led to

$$\sigma = \frac{N_{\text{events}}}{(N_B \cdot N_A)/\mathcal{A}}, \quad (3.132)$$

with the transverse scattering area \mathcal{A} , see [Figure 3.6](#). We have also used that it follows from [\(3.131\)](#), that $\rho_A l_A = N_A/\mathcal{A}$. In terms of the constant particle densities $\rho_{A/B}$, the cross section [\(3.132\)](#) turns into

$$\sigma = \frac{N_{\text{events}}}{(l_A \rho_A)(l_B \rho_B) \mathcal{A}}. \quad (3.133)$$

Finally, if the particle densities are (transverse) space-dependent, the product of the densities has to be integrated over the overlap area

$$(N_B \cdot N_A)/\mathcal{A} \rightarrow \int_{\mathcal{A}} d^2x \rho_A(x) \rho_B(x) l_A l_B, \quad (3.134)$$

and the cross section [\(3.132\)](#) turns into

$$\sigma = \frac{N_{\text{events}}}{l_A l_B \int_{\mathcal{A}} d^2x \rho_A(x) \rho_B(x)}. \quad (3.135)$$

This concludes our discussion of the definition of the cross-section and its interpretation.

Let us now compute the cross section for the above example depicted in [Figure 3.6](#) in terms of the S -matrix element in [\(3.38\)](#). The particles in the bunch are localised in space and (sharply) in momentum. Hence, we consider the wave packet defined in [\(2.109\)](#),

$$|f_{\mathbf{p}}\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} f_{\mathbf{p}}(\mathbf{k}) |\mathbf{k}\rangle, \quad (3.136)$$

where $f_{\mathbf{p}}(\mathbf{k})$ is a Gaussian wave packet with momentum argument \mathbf{k} , centered at \mathbf{p} and with a width $\sqrt{N/2}$,

$$f_{\mathbf{p}}(\mathbf{k}) \sim e^{-(\mathbf{k}-\mathbf{p})^2/N}. \quad (3.137)$$

The prefactor is determined by the normalisation of the states, and it follows

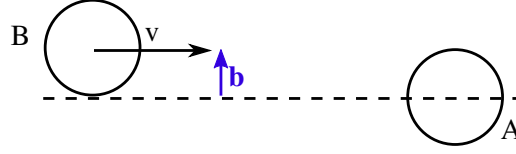
$$\begin{aligned} 1 &= \langle f_{\mathbf{p}} | f_{\mathbf{p}} \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \frac{1}{2\omega_{\mathbf{k}'}} f_{\mathbf{p}}^*(\mathbf{k}') f_{\mathbf{p}}(\mathbf{k}) \langle \mathbf{k}' | \mathbf{k} \rangle \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} |f_{\mathbf{p}}(\mathbf{k})|^2, \end{aligned} \quad (3.138)$$

see also [\(2.110\)](#). The Gaussian in [\(3.137\)](#) is localised in \mathbf{k} and \mathbf{x} as required, since a Fourier transform of a Gaussian remains a Gaussian. In our case, the initial state is given by

$$|i\rangle = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}_A} 2\omega_{\mathbf{k}_B}} f_{\mathbf{p}_A}(\mathbf{k}_A) f_{\mathbf{p}_B}(\mathbf{k}_B) |\mathbf{k}_A \mathbf{k}_B\rangle, \quad (3.139)$$

with

$$|\mathbf{k}_A \mathbf{k}_B\rangle = \sqrt{2\omega_{\mathbf{k}_A} 2\omega_{\mathbf{k}_B}} a^\dagger(\mathbf{k}_A) a^\dagger(\mathbf{k}_B) |0\rangle. \quad (3.140)$$

Figure 3.7.: Sketch of the impact parameter \mathbf{b} .

So far we have described *central* collisions as in our 2-to-2 example. The general scattering event is depicted in Figure 3.7, where we have introduced the impact parameter \mathbf{b} (see figure 3.7). For this purpose, we recall, that the momentum operator \mathbf{P} from (2.61) generates translations. Thus,

$$e^{-i\mathbf{P}\mathbf{b}}|\mathbf{k}\rangle = e^{-i\mathbf{k}\mathbf{b}}|\mathbf{k}\rangle \quad (3.141)$$

for the plain wave momentum state $|\mathbf{k}\rangle$, and the impact parameter only enters as an additional phase shift. This leads us to the initial state for a given impact parameter b :

initial state with impact parameter \mathbf{b}

$$|i_{\mathbf{b}}\rangle = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}_A} 2\omega_{\mathbf{k}_B}} f_{\mathbf{p}_A}(\mathbf{k}_A) f_{\mathbf{p}_B}(\mathbf{k}_B) e^{-i\mathbf{k}_B\mathbf{b}} |\mathbf{k}_A \mathbf{k}_B\rangle . \quad (3.142)$$

For the computation of the cross section we have to compute the probability of the scattering event with an initial state $|i_{\mathbf{b}}\rangle$ and a given final state. For the sake of simplicity we initiated our computation with a final state that is a 2-particle momentum eigenstate. As these states are plain wave normalisable and hence normalised to a δ -function in momentum space, this computation only leads us to the momentum density of the cross-section and finally we have to integrate over a given momentum region that follows from the geometry of the detector. We leave the respective evaluation to the end and proceed with the momentum Eigenstates. To begin with, we prepare our initial state such that the wave packets have an infinite distance from each other for $t_0 = -\infty$. Then we compute the time-evolution and the scattering event within perturbation theory, and finally, the measurement of the scattering products takes place at $t = +\infty$. Accordingly the transition amplitude is given by

$$\lim_{t_0 \rightarrow -\infty, t \rightarrow +\infty} \langle \mathbf{p}_1 \mathbf{p}_2 | U(t, t_0) | i_{\mathbf{b}} \rangle = \langle \mathbf{p}_1 \mathbf{p}_2 | S | i_{\mathbf{b}} \rangle , \quad (3.143)$$

where we have used the time evolution operator $U(t, t_0)$ and the definition of the S-matrix from Eq. (3.28). The probability density in (final) momentum space is then given by $|\langle \mathbf{p}_1 \mathbf{p}_2 | S | i_{\mathbf{b}} \rangle|^2$. In a final step one has to integrate over a momentum region of final momenta, v_f , which we postpone for now as discussed above.

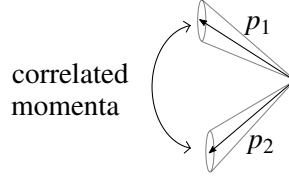
In the following we restrict ourselves to a collision of the bunch with a *single target*, i.e. $N_A = 1$. Let us now distribute N_B particles homogeneously over an area \mathcal{A} . Then, the integration over the impact area \mathcal{A} is equal to integration over the impact parameter \mathbf{b} and the number of events in a dense beam is

$$N_{\text{events}} \propto \frac{N_B}{\mathcal{A}} \int_{\mathcal{A}} d^2b |\langle \mathbf{p}_1 \mathbf{p}_2 | S | i_{\mathbf{b}} \rangle|^2 , \quad (3.144)$$

where the right hand side is rather an event density in momentum space. With (3.132) it follows

$$\sigma(\mathbf{p}_1, \mathbf{p}_2) = \frac{N_{\text{events}}}{(N_B \cdot 1)/\mathcal{A}} \propto \int_{\mathcal{A}} d^2b |\langle \mathbf{p}_1 \mathbf{p}_2 | S | i_{\mathbf{b}} \rangle|^2 . \quad (3.145)$$

Equation (3.145) is the result for a sharp momentum measurement of the final state with momenta \mathbf{p}_1 , \mathbf{p}_2 for the two final particles. A realistic scenery or rather a detector measures a maybe small but finite momentum region v_f ,



This cross section in the momentum regime v_f is obtained from (3.145) by integrating over v_f , to wit,

$$\sigma(v_f) = \int_{v_f} \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}_1}} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}_2}} \int d^2 b |\langle \mathbf{p}_1 \mathbf{p}_2 | S | i_{\mathbf{b}} \rangle|^2, \quad (3.146)$$

where (2.24) implies that all final momenta are on-shell, i.e. that $p_i^2 = m^2$ for $i = 1, 2$. Accordingly, the core of the cross section (3.146) is the differential one, $d\sigma$:

differential cross section (for n particles)

$$d\sigma = \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}_i}} \int d^2 b |\langle \mathbf{p}_1 \cdots \mathbf{p}_n | S | i_{\mathbf{b}} \rangle|^2. \quad (3.147)$$

Equation (3.147) relates the differential cross section to the S -matrix element between the initial state and a momentum Eigenstate. The S -matrix has a trivial forward scattering part, where the particles simply traverse the target without interaction and the scattering part, see (3.38). For the following computation we assume, that the \mathbf{p}_i are not parallel to $\mathbf{p}_{\mathbf{B}}$, so we do not want to consider the forward scattering part. Using

$$S_{fi} = \mathbb{1}_{fi} + i T_{fi}, \quad i T_{fi} = i M_{fi} (2\pi)^4 \delta\left(\sum p_i - \sum k_i\right), \quad (3.148)$$

we can represent the interaction part of the S -matrix element squared in (3.147) as

$$|\langle \mathbf{p}_1 \cdots \mathbf{p}_n | S | i_{\mathbf{b}} \rangle|^2 = \langle \mathbf{p}_1 \cdots \mathbf{p}_n | S | i_{\mathbf{b}} \rangle \langle \mathbf{p}_1 \cdots \mathbf{p}_n | S | i_{\mathbf{b}} \rangle^*, \quad (3.149)$$

and insert all the expressions for this matrix element into that for $d\sigma$. This leads us to

$$\begin{aligned} d\sigma &= \prod_i \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}_i}} \int d^2 b \int \frac{d^3 k_A}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}_A}} \frac{d^3 k_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}_B}} f_{\mathbf{p}_A}(\mathbf{k}_A) f_{\mathbf{p}_B}(\mathbf{k}_B) \\ &\quad \times \int \frac{d^3 k'_A}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}'_A}} \frac{d^3 k'_B}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}'_B}} f_{\mathbf{p}_A}^*(\mathbf{k}'_A) f_{\mathbf{p}_B}^*(\mathbf{k}'_B) \\ &\quad \times e^{i\mathbf{b}(\mathbf{k}'_B - \mathbf{k}_B)} |M_{fi}|^2 (2\pi)^4 \delta\left(\sum p_i - \sum k_i\right) (2\pi)^4 \delta\left(\sum p_i - \sum k'_i\right), \end{aligned} \quad (3.150)$$

with $k_1 = k_A$, $k_2 = k_B$. For the explicit computation of (3.150) we first perform the integral over the impact parameter, as the only \mathbf{b} is that in the phase factor. This leads us to yet another momentum δ -function with

$$\int d^2 b e^{i\mathbf{b}(\mathbf{k}'_B - \mathbf{k}_B)} = (2\pi)^2 \delta(\mathbf{k}'_{B\perp} - \mathbf{k}_{B\perp}), \quad (3.151)$$

with $\mathbf{q}_{\perp} = (q_1, q_2)$ is the two-dimensional transverse momentum vector of a spatial momentum $\mathbf{q} = (q_1, q_2, q_3)$. Transverse here refers to vectors perpendicular to the beam axis.

Next we examine the integral over the primed momenta,

$$\begin{aligned}
& \int d^3 k'_A d^3 k'_B \delta\left(\sum p_i - \sum k'_i\right) \delta(\mathbf{k}'_{B\perp} - \mathbf{k}_{B\perp}) \\
&= \int d(k'_A)^3 d(k'_B)^3 \delta\left(\sum p_i^3 - \sum (k'_i)^3\right) \delta\left(\sum p_i^0 - \sum (k'_i)^0\right) \\
&= \int d(k'_A)^3 \delta\left(\sum p_i^0 - \sum (k'_i)^0\right), \tag{3.152}
\end{aligned}$$

with $k'_{B\perp} = k_{B\perp}$, $k'_{A\perp} = k_{A\perp}$, $(k'_B)^3 = \sum p_i^3 - (k'_A)^3$. Here, the integral over the transverse δ -function for the $\mathbf{k}'_{B\perp}$ is trivial. The equality $k'_{A\perp} = k_{A\perp}$ is due to the four dimensional δ -function. For example in the 1-direction it enforced,

$$p_1^1 + p_2^1 - k_A^1 - k_B^1 = p_1^1 + p_2^1 - k_A^1 - k_B^1, \tag{3.153}$$

Using the equality of the transverse $k'_{B\perp} = k_{B\perp}$ we readily obtain $k_{A\perp} = k'_{A\perp}$. This allows us to perform the integration over $(k'_A)^3$, where we use that the arguments in the δ -function in the last line of (3.152) are functions of $(k'_A)^3$ with $(\mathbf{k}'_A)^2 = k_{A\perp}^2 + ((k'_A)^3)^2$ and $(\mathbf{k}'_B)^2 = k_{B\perp}^2 + (\sum p_i^3 - (k'_A)^3)^2$. This leads us to

$$\begin{aligned}
& \int d^3 k'_A d^3 k'_B \delta^4\left(\sum p_i - \sum k'_i\right) \delta^2(\mathbf{k}'_{B\perp} - \mathbf{k}_{B\perp}) \\
&= \int d(k'_A)^3 \delta\left(\sum p_i^0 - \sqrt{(\mathbf{k}'_A)^2 + m_A^2} - \sqrt{(\mathbf{k}'_B)^2 + m_B^2}\right) \Bigg|_{\substack{k_{A/B\perp} = k'_{A/B\perp} \\ \sum k_i^3 = \sum (k'_i)^3 \\ \sum k_i^0 = \sum (k'_i)^0}} \\
&= \frac{1}{\left| \frac{(k'_A)^3}{(k'_A)^0} - \frac{(k'_B)^3}{(k'_B)^0} \right|} = \frac{1}{|v_A - v_B|}, \tag{3.154}
\end{aligned}$$

the term in the denominator is just the relative velocity of the beams in the laboratory frame. Now we use that the wave packages $f_{\mathbf{p}_{A/B}}$ are located around $\mathbf{p}_{A/B}$. Accordingly we can substitute $k'_{A/B} \rightarrow p_{A/B}$ in all prefactors, and arrive at

$$\begin{aligned}
d\sigma = & \prod_i \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}_i}} \frac{1}{4p_A^0 p_B^0 |v_A - v_B|} \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} \frac{1}{2k_A^0 2k_B^0} |f_{\mathbf{p}_A}(\mathbf{k}_A)|^2 |f_{\mathbf{p}_B}(\mathbf{k}_B)|^2 \\
& \times |M_{fi}|^2 (2\pi)^4 \delta^4\left(\sum p_i - \sum k_i\right). \tag{3.155}
\end{aligned}$$

Again, we use the localisation to replace $\sum k_i \rightarrow \sum p_i = p_A + p_B$. The remaining $\mathbf{k}_{A/B}$ -integral are simply the normalisations of the wave packets, $\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} |f_{\mathbf{p}}(\mathbf{k})|^2 = 1$, and drop out. Moreover, we can use $\sum p_i = \sum p_f - p_i$. This leads us to the final expression for the differential cross section,

differential cross section

$$d\sigma = \prod_{j=1}^n \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2p_j^0} \frac{1}{4p_A^0 p_B^0 |v_A - v_B|} |M_{fi}|^2 (2\pi)^4 \delta\left(\sum p_f - \sum p_j\right). \tag{3.156}$$

Note, that except for the first fraction all expressions are Lorentz invariant. The first fraction is invariant under boosts along the beam axis. This is to be expected, as $d\sigma$ is a differential transverse area and has to be **invariant**

under boosts along the beam axis. We define the n -particle phase space factor as

$$d\Pi_n := \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2p_i^0} (2\pi)^4 \delta(p_f - p_i). \quad (3.157)$$

Let us now consider the highly relativistic case. Then

$$|s| = (p_A + p_B)^2 = (p_A^0)^2 - \mathbf{p}_A^2 + (p_B^0)^2 - \mathbf{p}_B^2 + 2p_A^0 p_B^0 - 2\mathbf{p}_A \mathbf{p}_B \quad (3.158)$$

$$= m_A^2 + m_B^2 + 2p_A^0 p_B^0 - 2\mathbf{p}_A \mathbf{p}_B \gg m_A^2 + m_B^2. \quad (3.159)$$

and it follows

$$4p_A^0 p_B^0 |v_A - v_B| \rightarrow 2s, \quad (3.160)$$

and

$$d\sigma = \frac{1}{2s} |M_{fi}|^2 d\Pi_n. \quad (3.161)$$

Example 3-8: Differential cross section of 2-to-2 scattering.

Let us exemplarily discuss the 2-to-2 scattering in ϕ^4 -theory in the highly relativistic case. Then, we have $n = 2$ in (3.161). It follows

$$\begin{aligned} \int d\Pi_2 &= \int (2\pi)^4 \delta^4(p_1 + p_2 - (p_A + p_B)) \cdot \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2p_1^0} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2p_2^0} \\ &\simeq \frac{1}{(2\pi)^2 4p_1^0 p_2^0} \int d^3 p_2 \delta(p_1^0 + p_2^0 - \sqrt{s}) \quad \text{for } (p_A + p_B)^2 \gg m_A^2, m_B^2. \end{aligned} \quad (3.162)$$

We compute this in the center of mass system (CMS). Therefore, we have $\mathbf{p}_1 = -\mathbf{p}_2 \rightarrow p_1^0 = p_2^0$, i.e. equal masses. We also use

$$d^3 p_2 = d\Omega |\mathbf{p}_2|^2 d|\mathbf{p}_2|, \quad (3.163)$$

with the solid angle $d\Omega = d\varphi \sin\theta d\theta$.

It follows, $p_1^0 + p_2^0 - \sqrt{s} = 2p_2^0 - \sqrt{s} = 2|\mathbf{p}_2|^2 - \sqrt{s}$, $p_i^0 = \sqrt{s}/2$

$$\int d\Pi_2 = \frac{1}{2} \frac{s/4}{(2\pi)^2 4p_1^0 p_2^0} d\Omega = \frac{1}{32\pi^2} d\Omega. \quad (3.164)$$

Now we use (3.47), i.e. that for classical scattering it is

$$|M_{fi}|^2 = \lambda^2. \quad (3.165)$$

With this, we obtain the

differential cross section (2-2 scattering)

$$\frac{d\sigma}{d\Omega} = \frac{1}{2s} |M_{fi}|^2 \int_{d\Omega(p_2)\text{fixed}} d\Pi_2 = \frac{\lambda^2}{64\pi^2} \frac{1}{s}. \quad (3.166)$$

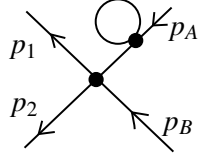
Lastly, we discuss the **computation of the S-matrix elements**. In the 2-2 scattering example we used, that

$$|M_{fi}|^2 = \lambda^2 + \mathcal{O}(\lambda^3). \quad (3.167)$$

We make an expansion in the Feynman diagrams:

$$\langle \mathbf{p}_1 \mathbf{p}_2 | i T | \mathbf{p}_A \mathbf{p}_B \rangle = \begin{array}{c} O(\lambda) \\ \text{diagram} \end{array} + \left(\begin{array}{c} O(\lambda^2) \\ \text{diagram} + \text{perm.} \end{array} \right) + \begin{array}{c} O(\lambda^3) \\ \text{diagram} \end{array} + \dots$$

Computing



with

$$\frac{1}{2} \text{diagram} = -i\Pi$$

gives

$$\begin{aligned} & \frac{i}{p_A^2 - m^2 + i\epsilon} (-i\Pi)(-i\lambda) \cdot (2\pi)^4 \delta(p_A + p_B - p_1 - p_2) \\ &= \left(\frac{i}{p_A^2 - m^2 + i\epsilon} \right)^{-1} \frac{i}{p_A^2 - m^2 + i\epsilon} (-i\Pi) \frac{i}{p_A^2 - m^2 + i\epsilon} (-i\lambda) \cdot (2\pi)^4 \delta(p_A + p_B - p_1 - p_2) \\ &\Rightarrow \frac{1}{2} \left(\text{diagram} + \text{diagram} \right) \\ &= -i\lambda \left(\frac{i}{p_A^2 - m^2 + i\epsilon} \right)^{-1} \left(\frac{i}{p_A^2 - m^2 + i\epsilon} + \frac{i}{p_A^2 - m^2 + i\epsilon} (-i\Pi) \frac{i}{p_A^2 - m^2 + i\epsilon} \right) \cdot (2\pi)^4 \delta(p_A + p_B - p_1 - p_2) \\ &= -i\lambda \left(\frac{i}{p_A^2 - m^2 + i\epsilon} \right)^{-1} \frac{i}{p_A^2 - (m^2 + \Pi) + i\epsilon} \cdot (2\pi)^4 \delta(p_A + p_B - p_1 - p_2) \end{aligned} \tag{3.168}$$

$$= \left[-i\lambda \delta^4(p_A + p_B - p_1 - p_2) \right] \begin{array}{c} \xrightarrow{p_A}^{-1} \\ \text{(bare) free inverse} \\ \text{propagator with } p_A \end{array} \cdot \begin{array}{c} \xrightarrow{p_A} \\ \text{full propagator} \\ \text{with } p_A \end{array} + O(\lambda^3) .$$

We remark, that the free inverse propagator is related to the fact, that the particle A in the initial state was prepared as a free state, which is only true for $t \rightarrow -\infty$. The correct state should relate to full (inverse) propagation, i.e.

$$\xrightarrow{p_A}^{-1} \rightarrow \xrightarrow{p_A}$$

This leads to

$$1 = \xrightarrow{p_A} \cdot \xrightarrow{p_A}$$

in the above equation. Thus, we conclude, that M_{fi} is obtained by computing amputated, connected scattering diagrams. This will be discussed further in the subsequent section.

3.5. LSZ-Formalism

In the last section we have seen that the crucial QFT input in the calculation of scattering cross sections are the S-matrix elements. As has been apparent from all the computations before, the S-matrix elements have a representation in terms of Feynman diagrams. In our examples so far we have seen that the overall phase factors related to vacuum diagrams cancelled. However, we have also seen in our discussion of the one loop contribution (tadpole) to the propagator, that the measurable observables such as pole masses of particles or fields are not identical with the respective input parameters that enter the classical action. Put differently, while the *in* and *out* states are related to the states in the free theory, they are not identical and the transformation between them is also part of the S-matrix. This can be dealt with the LSZ-reduction formula, named after the three German physicists Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann.

3.5.1. The spectral function and the Källén-Lehmann representation of the propagator

The crucial ingredient in perturbation theory is the propagator, and the proportionality problem discussed above is all about the propagation from $t \rightarrow -\infty$ to the scattering time and then to $t \rightarrow +\infty$. In the previous section we have seen, that the naive preparation of our in-state lead to a product of the free inverse propagator with the full propagator in our scattering amplitudes (3.168). We have encountered a similar problem with vacuum bubbles before. In this section we shall see that

$$\phi_H(t \rightarrow \mp\infty) \rightarrow Z^{1/2} \phi_{\text{in/out}} \quad (\text{weak op. equivalence}), \quad (3.169)$$

with $Z \leq 1$. So far, we have implicitly assumed $Z = 1$. In the following we determine Z by computing the two-point function and subsequent generalisation to the n-point function. We shall see that the factor Z is related to the probability of an interaction free propagation while $(1 - Z)$ takes care of scattering events.

For its computation as well as its interpretation we begin with the vacuum expectation value of the two-point function, which is expanded in terms of Eigenstates of the Hamiltonian, which are given by $|\lambda\rangle$ with the energy E_λ , the *spectral index* λ and vanishing (total) momentum $\mathbf{P}|\lambda\rangle = 0$. If boosting these states with a momentum \mathbf{p} , we get a complete set of states $|\lambda, \mathbf{p}\rangle$. This leads us to

$$\langle \phi(x) \phi(y) \rangle = \langle \Omega | \phi_H(x) \phi_H(y) | \Omega \rangle = \sum_{\lambda, \mathbf{p}} \langle \Omega | \phi_H(x) | \lambda, \mathbf{p} \rangle \langle \lambda, \mathbf{p} | \phi_H(y) | \Omega \rangle . \quad (3.170)$$

In (3.170) we have introduced the short hand notation $\langle \phi(x) \phi(y) \rangle$ for the vacuum expectation value of two fields. In a second step in (3.170) we have inserted the identity operator in the Hilbert space, $\mathbb{1}$, in terms of a sum/integral over a complete set of states $|\lambda, \mathbf{p}\rangle$. As discussed above (3.170), these states are eigenstates of the four momentum operator

$$(\hat{P}^\mu) = (H, \mathbf{P}) . \quad (3.171)$$

The $|\lambda, \mathbf{p}\rangle_H$ are Eigenstates of the Hamiltonian H with the given energy E_λ , and the spatial momentum operator \mathbf{P} with a given spatial momentum \mathbf{p}_λ and the (on-shell) mass m_λ^2 with $E_\lambda^2 - \mathbf{p}_\lambda^2 = m_\lambda^2$. While the integration over spatial momentum for a given on-shell condition originates in Poincaré invariance, the *spectral index* carries the sum or integral over the mass spectrum of the theory. In summary this basis obeys the relations

$$H |\lambda, \mathbf{p}\rangle = E_\lambda |\lambda, \mathbf{p}\rangle , \quad \mathbf{P} |\lambda, \mathbf{p}\rangle = \mathbf{p}_\lambda |\lambda, \mathbf{p}\rangle , \quad (3.172a)$$

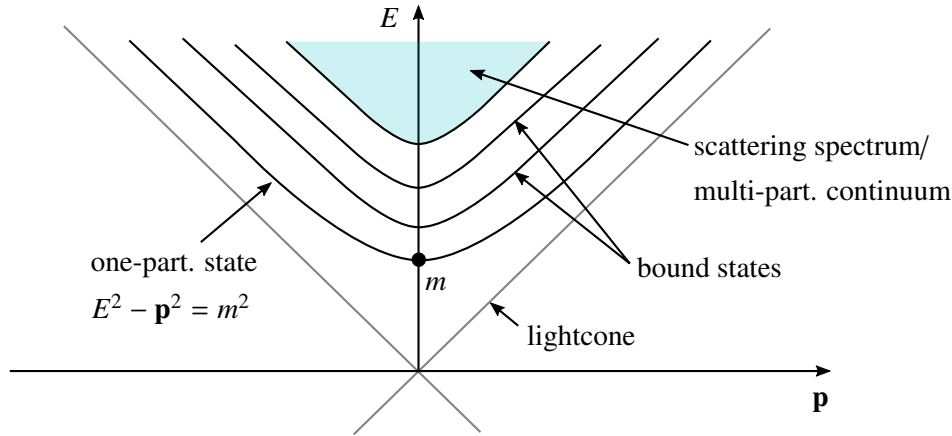


Figure 3.8.: On-shell mass spectrum

with the relativistic normalisation (2.107), and hence

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\lambda}(\mathbf{p})} |\lambda, \mathbf{p}\rangle\langle\lambda, \mathbf{p}|. \quad (3.172b)$$

Equation (3.170) represents the two-point function as an integral/sum of the overlap of the field state $\phi_H(x)|\Omega\rangle$ with mass eigenstate with mass m_{λ} . The expression $|\langle\Omega|\phi_H(x)|\lambda, \mathbf{p}\rangle|^2$ is the respective probability density. It depends on the spectral parameter λ , spatial momentum \mathbf{p} and the space-time coordinate. In order to eliminate the latter dependence we transport the field to $x = 0$ with

$$\phi_H(x) = e^{i\hat{P}x} \phi_H(0) e^{-i\hat{P}x}. \quad (3.173)$$

With $\langle\Omega|e^{i\hat{P}x} = \langle\Omega|$ and $e^{-i\hat{P}x}|\lambda, \mathbf{p}\rangle = |\lambda, \mathbf{p}\rangle e^{-ipx}$ we convert the matrix elements in (3.170) to that at $x = 0$ and $\mathbf{p} = 0$,

$$\langle\Omega|\phi_H(x)|\lambda, \mathbf{p}\rangle = \langle\Omega|\phi_H(0)|\lambda, \mathbf{p}\rangle e^{-ipx} = \langle\Omega|\phi_H(0)|\lambda\rangle e^{-ipx}, \quad (3.174)$$

where the last relation follows from the boost invariance of the vacuum. In summary we arrive at

$$\langle\phi(x)\phi(y)\rangle = \sum_{\lambda} \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\lambda}(\mathbf{p})} e^{-ip_{\lambda}(x-y)} \right] |\langle\Omega|\phi_H(0)|\lambda\rangle|^2, \quad (3.175)$$

for $x^0 > y^0$. The expression in the square bracket is nothing but the forward part ($x_0 - y_0 > 0$) of the Feynman propagator (3.95). Repeating the same steps for $x_0 - y_0 < 0$ leads us to

$$\langle T\phi(x)\phi(y)\rangle = \sum_{\lambda} \left[\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m_{\lambda}^2 + i\epsilon} e^{-ip(x-y)} \right] |\langle\Omega|\phi_H(0)|\lambda\rangle|^2, \quad (3.176)$$

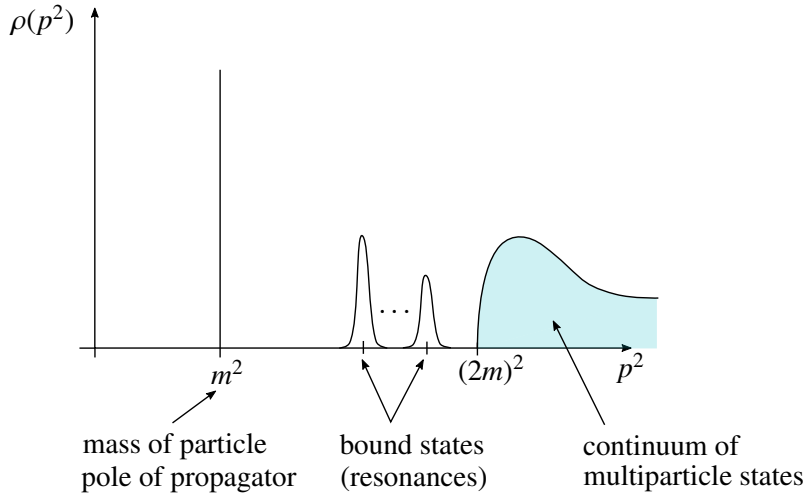


Figure 3.9.: Spectral function of a particle with mass m , bound states below the threshold $m_{\text{scat}} = 2m$ that originates from $1 \rightarrow 2$ scattering.

where the $i\epsilon$ with $\epsilon \rightarrow 0_+$ takes care of the time ordering. This leads us to the *Källén-Lehmann spectral representation* of the two-point function,

Källén-Lehmann spectral representation

$$\langle T\phi(x)\phi(y)\rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \mathcal{D}_F(x-y; M^2), \quad (3.177)$$

with the *spectral function* $\rho(M^2)$ with

Spectral function

$$\rho(p^2) = \sum_\lambda (2\pi) \delta(p^2 - m_\lambda^2) |\langle \Omega | \phi_H(0) | \lambda \rangle|^2. \quad (3.178)$$

The spectral function is depicted in figure 3.9. In the absence of excited states/resonances it can be parametrised as

$$\rho(p^2) = Z 2\pi \delta(p^2 - m^2) + \theta(p^2 - m_{\text{scat}}^2) \tilde{\rho}(p^2), \quad (3.179)$$

with the pole mass m , and the scattering spectrum starts at the threshold m_{scat} . In the exemplary ϕ^4 -theory we have $1 \rightarrow 3$ scattering and hence $m_{\text{scat}} = 3m$, the total energy has to be larger than three times the rest mass of the particle. In the presence of $1 \rightarrow 2$ scattering processes such as in a ϕ^3 theory (or ϕ^4 -theory in the broken phase) we have $m_{\text{scat}} = 2m$, see Figure 3.9, which also features bound states. Inserting (3.179) in (3.177) leads us to

$$\langle T\phi(x)\phi(y)\rangle = Z \mathcal{D}_F(x-y; m^2) + \int_{m_{\text{scat}}^2}^\infty \frac{dM^2}{2\pi} \rho(M^2) \mathcal{D}_F(x-y; M^2). \quad (3.180)$$

where the first term proportional to $\mathcal{D}_F(x-y; M^2)$ carries the one-particle pole of ϕ . This consolidates our previous interpretation of Z as the probability of the simple propagation of the field without scattering events. In the beginning we have already indicated, that the initial state field ϕ_{in} is related to the full field by (3.169), and hence simply by multiplication with $z^{1/2}$. We now assume that the proportionality constant is $\tilde{Z}^{1/2}$ and prove that $\tilde{Z} = Z$. To that end we consider only one-particle states $|\lambda_1\rangle$ in $|\lambda\rangle\langle\lambda|$ instead of the full spectral sum. This projects the spectral function on the first term in (3.180),

$$\rho_{\text{pole}} = \sum_{\lambda_1} e^{-ip_\lambda(x-y)} \left| \langle \Omega | \phi(0) | \lambda_1 \rangle \right|^2, \quad (3.181)$$

and propagates the field from $t = 0$ to $t \rightarrow -\infty$ with the time evolution operator, $U = U(-\infty, 0)$,

$$\begin{aligned} Z &= \left| \langle \Omega | \phi(0) | \lambda_1 \rangle \right|^2 = \left| \langle \Omega | U^{-1} U \phi U^{-1} U | \lambda_1 \rangle \right|^2 = \left| {}_I \langle 0 | \phi_H(-\infty) | \lambda_1 \rangle_I \right|^2 = \left| {}_I \langle 0 | \tilde{Z}^{1/2} \phi_{\text{in}} | \lambda_1 \rangle_I \right|^2 \\ &= \tilde{Z} \end{aligned} \quad (3.182)$$

Let us now determine Z . For this purpose, we consider the expectation value $\langle \phi(x) \phi(y) \rangle$ *without time ordering*. Then \mathcal{D}_F in (3.180) is substituted by the propagator \mathcal{D} without time ordering. The expectation value of the commutator is then given by $\mathcal{D}(x-y) - \mathcal{D}(y-x)$. A time derivative with respect to y_0 converts this into a commutator of the field and the field momentum, and for $y_0 = x_0$ we can use the canonical commutation relations. We find

$$-i \left[\frac{\partial}{\partial y^0} (D(x-y) - D(y-x)) \right]_{x^0=y^0} = -i \left[\frac{\partial}{\partial y^0} \langle [\phi(x), \phi(y)] \rangle \right]_{x^0=y^0} = -i \langle [\phi(x), \Pi(y)]_{x^0=y^0} \rangle = \delta^3(\mathbf{x} - \mathbf{y}), \quad (3.183)$$

An integration over position eliminates the spatial δ -function and (3.183) reduces to unity. We also can use the representation (3.175) for the left hand side of (3.183). There the y_0 -derivative pulls down $\pm iE_\lambda$, which cancels the denominator $1/(2E_\lambda)$. This leads us with the *spectral sum rule*,

Spectral sum rule

$$1 = Z + \int_{m_1^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2), \quad (3.184)$$

where $m_1 > m$ is small enough to include possible resonances. In their absence we can choose $m_1 = m_{\text{scat}}$ and $\rho = \tilde{\rho}$ in the integral in (3.184). The spectral sum rule entails probability conservation: the square of the overlap of the field state with a complete set of states is normalised to unity. In consequences this also proves that $Z > 0$ is bounded from above by one.

$$0 < Z \leq 1. \quad (3.185)$$

In the free theory and hence in the absence of any scattering we have $Z = 1$. In turn, in the interacting theory scattering is present, leading to $Z < 1$. As already indicated before, $1 - Z$ accounts for the overlap of $\phi |\Omega\rangle$ with multi-particle states, and (3.169) holds for $t \rightarrow -\infty$.

As most computations in QFT are done in momentum space, both analytically and numerically, we close this discussion with some properties of the momentum space representation of the spectral function. The Fourier transform of the full propagator leads us to

$$\langle T\phi(p)\phi(-p) \rangle = \int d^4x \langle T\phi(x)\phi(0) \rangle e^{ipx} = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}. \quad (3.186)$$

Close to the mass shell, i.e. $p^2 - m^2 \rightarrow 0$, the propagator is dominated by the one particle state,

$$\langle T\phi(p)\phi(-p) \rangle \rightarrow \frac{iZ}{p^2 - m^2 + i\epsilon} + \text{finite}. \quad (3.187)$$

Equation (3.187) is crucial for bound state computations such as Bethe-Salpeter equations as these typically inhomogeneous integral equations turn into homogeneous ones which are far simpler to solve. It is readily derived within a split of the integral in (3.177) into a part from 0 to m_1^2 and one from m_1^2 to ∞ ,

$$\langle T\phi(p)\phi(-p) \rangle = \int_0^{m_1^2} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon} + \int_{m_1^2}^\infty \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}. \quad (3.188)$$

The first part gives (3.187) whereas the second is finite in the limit $p^2 \rightarrow m^2$.

As a final example for the power of the spectral representation we derive an interesting general result for the decay of the propagator at large space-like momenta with $p^2 = p_0^2 - \mathbf{p}^2 \rightarrow -\infty$. For space-like momenta the spectral kernel $i/(p^2 - M^2 + i\epsilon)$ in (3.186) has no pole, and we arrive at

$$\lim_{p^2 \rightarrow -\infty} |p^2 \langle T\phi(p)\phi(-p) \rangle| = \lim_{p^2 \rightarrow -\infty} \left| \int_0^\infty \frac{dM^2}{2\pi} \frac{p^2}{M^2 - p^2} \rho(M^2) \right| = 1. \quad (3.189)$$

Here, we have used the spectral sum rule (3.184): the M^2 -integral of $\rho(M^2)$ exists and hence the p^2 -limit can be commuted with the integration. Furthermore, with the sum rule the integral is one and hence the propagator turns into the classical propagator for large spatial momenta (without a non-trivial factor).

We have used the limit of space-like momenta as then the integrand of (3.186) exhibits no pole. For large time-like momenta, $p^2 \rightarrow +\infty$, the integrand has a pole at $p^2 = M^2 - i\epsilon$ and we use

$$\frac{1}{p^2 - M^2 + i\epsilon} = P \left[\frac{1}{p^2 - M^2} \right] - i\pi \delta(p^2 - M^2), \quad (3.190)$$

where $P[1/x]$ is the principal value. The principal value part is treated as in (3.189) and the integral over the δ -function is readily performed. We are led to

$$\lim_{p^2 \rightarrow \infty} |p^2 \langle T\phi(p)\phi(-p) \rangle| = \lim_{p^2 \rightarrow \infty} \sqrt{1 + \frac{1}{2} p^2 \rho(p^2)} = 1, \quad (3.191)$$

where we have used that the spectral sum rule (3.184) implies that $\rho(p^2)$ decays more rapidly than $1/p^2$.

Equations (3.189) and (3.191) imply that the full propagator of a physical field decays precisely with $1/p^2$ without a non-trivial prefactor,

$$\lim_{p^2 \rightarrow \infty} \langle T\phi(p)\phi(-p) \rangle = \frac{i}{p^2}, \quad (3.192)$$

that is the classical propagator. In turn, for $p^2 \rightarrow m^2$ the propagator also takes a classical form, but with a non-trivial factor Z , see (3.187).

We close the investigation of the Källén-Lehmann spectral representation with the remark, that the derivation of the sum rule with the commutator relation related the trivial limit (3.192) to the canonical commutation relations of the field. One either can see this by 'inverting' the derivation of the sum rule with (3.183) or applying time derivatives on the Feynman propagator itself: We note that

$$\partial_{y_0}^2 \langle T\phi(x)\phi(y) \rangle = 2 \left\langle \left[\phi(x), \partial_{y_0}\phi(y) \right] \delta(x_0 - y_0) \right\rangle + \left\langle \left[\phi(x), \phi(y) \right] \partial_{y_0}\delta(x_0 - y_0) \right\rangle + \left\langle T\phi(x)\partial_{y_0}^2\phi(y) \right\rangle, \quad (3.193)$$

Taking the Fourier transform of (3.193) with respect to $x_0 - y_0$ leads us to

$$p_0^2 \langle T\phi(p_0, \mathbf{x})\phi(-p_0, \mathbf{y}) \rangle = \langle [\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] \rangle + \langle T\phi(-p_0, \mathbf{x})\dot{\pi}(-p_0, \mathbf{y}) \rangle. \quad (3.194)$$

In the limit $p_0 \rightarrow \infty$ the second term decays as it is a correlation function of local operators. Accordingly this limit leaves us with the equal-time commutator of the field ϕ and its momentum π . This procedure is called the Bjorken-Johnson-Low (BJL) limit and is a simple way of computing commutators (or rather their expectation values) in an interacting quantum field theory. Taking also the Fourier transform with respect to spatial momentum leads us to

$$\lim_{p_0^2 \rightarrow \infty} p_0^2 \langle T\phi(p)\phi(-p) \rangle = i \quad (3.195)$$

where we have used (3.191). Hence, the large momentum limit of the propagator encodes the canonical commutation relations. This is a very strong result as the only presupposition of the derivation of the Källén-Lehmann spectral representation was the completeness of the set of states of the Hilbert space and the property, that the field state $\phi(x)|\Omega\rangle$ has an expansion in these states: in short, the field generates a state in the Hilbert space. We shall however see that this seemingly trivial property is not present in many QFTs.

3.5.2. The LSZ reduction formula

The results concerning the spectral representation of the propagator are pivotal for that of the n -point function relevant for the computation of the cross section, which is proportional to or can be obtained from S-matrix elements of momentum eigenstates, see (3.147). Accordingly, we start with a general n -point function

$$\langle T\phi(x_1)\phi(x_2)\cdots\phi(x_n) \rangle, \quad (3.196)$$

and evaluate its Fourier transform on-shell. Analogously to the derivation of the Källén-Lehmann representation of the two-point function we then insert the identity in form of a sum of energy and total momentum Eigenstates (3.172). This leads to the wanted relation, which is called the *LSZ-reduction formula*. From our $2 \rightarrow 2$ scattering example at the end of Section 3.4 we already expect the residues Z of the on-shell propagators (3.187) to emerge.

The analysis described above is done iteratively for each field, and we first perform the Fourier transform for the field $\phi(x)$ with $x = x_1$,

$$\int d^4x e^{ipx} \langle T\phi(x)\phi(x_2)\cdots\phi(x_n) \rangle = \langle T\phi(p)\phi(x_2)\cdots\phi(x_n) \rangle. \quad (3.197)$$

In anticipation of the emerging structure we split the time integration in asymptotic ones, where the time x^0 is either larger or smaller than all the other times x_i^0 with $i = 2, \dots, n$ and an intermediate one which includes all the times of the other fields. With $T_+ > x_2^0, \dots, x_n^0$ and $T_- < x_2^0, \dots, x_n^0$ this split of the time intergration reads

$$\int dx^0 e^{ip^0x^0} = \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+} + \int_{T_+}^{+\infty} \right) dx^0 e^{ip^0x^0}. \quad (3.198)$$

We expect that the two asymptotic regimes carry the poles of the asymptotic on-shell propagation, while the integral over the intermediate times is finite. We get

$$\begin{aligned} \int d^4x e^{ipx} \langle T\phi(x)\phi(x_2)\cdots\phi(x_n)\rangle &= \left(\int_{-\infty}^{T_-} + \int_{T_-}^{T_+} \right) \langle T\phi(x)\phi(x_2)\cdots\phi(x_n)\rangle \\ &+ \int_{T_+}^{\infty} d^4x e^{ipx} \sum_{\lambda} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}}} \langle \Omega|\phi(x)|\lambda, \mathbf{q}\rangle \langle \lambda, \mathbf{q}|T\phi_2\cdots\phi_n\rangle, \end{aligned} \quad (3.199)$$

where, additionally to the split, we also have inserted the identity operator in terms of the complete set of Eigenstates of the Hamiltonian as in the case of the two-point function. With (3.173) and (3.174) we transport the field to $x = 0$ and convert the momentum dependence of the state into a phase, $\langle \Omega|\phi(x)|\lambda, \mathbf{q}\rangle = \langle \Omega|\phi(0)|\lambda\rangle e^{-iqx}$. Inserting this relation in the second line of (3.199) we can perform the x -integration. The full spatial \mathbf{x} dependence is given by the phase and the respective integral leads to a spatial momentum δ -function $(2\pi)^3\delta(\mathbf{p} - \mathbf{q})$. The remaining x^0 integral leads us to

$$\begin{aligned} \sum_{\lambda} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}}} \int_{T_+}^{\infty} dx^0 e^{i(p^0 - q^0 + i\epsilon)x^0} \langle \Omega|\phi(0)|\lambda\rangle \langle \lambda, \mathbf{q}|T\phi_2\cdots\phi_n\rangle \cdot (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \\ = \sum_{\lambda} \frac{1}{2\omega_{\mathbf{p}}} \frac{i e^{i(p^0 - \omega_{\mathbf{p}} + i\epsilon)T_+}}{p^0 - \omega_{\mathbf{p}} + i\epsilon} \langle \Omega|\phi(0)|\lambda\rangle \langle \lambda, \mathbf{p}|T\phi_2\cdots\phi_n\rangle. \end{aligned} \quad (3.200)$$

Now we use that the in- and out-states are on-shell and hence $p^0 \rightarrow \omega_{\mathbf{p}}$. Accordingly, on the pole the spectral sum in (3.199) reduces to one over one-particle states as already discussed for the two-point function around (3.181). Hence, the overlap is nothing but $Z^{1/2}$ and we arrive at

$$\lim_{p^0 \rightarrow \omega_{\mathbf{p}}} \int_{T_+}^{+\infty} d^4x e^{ipx} \langle T\phi(x)\phi(x_2)\cdots\phi(x_n)\rangle = \frac{iZ^{1/2}}{p^2 - m^2 + i\epsilon} \langle \mathbf{p}|T\phi(x_2)\cdots\phi(x_n)\rangle + \text{finite}. \quad (3.201)$$

Equation (3.201) is the anticipated result, the correlation function with an on-shell momentum state $|\mathbf{p}\rangle$ is related to the Fourier transform of the correlation function with a field state $\phi(x)|\Omega\rangle$ by the full on-shell propagator. Similarly to the derivation of (3.201) we find for the $\int_{-\infty}^{T_-}$ -term,

$$\lim_{p^0 \rightarrow -\omega_{\mathbf{p}}} \int_{-\infty}^{T_-} d^4x e^{ipx} \left\langle \left(T\phi(x_2)\cdots\phi(x_n) \right) \phi(x) \right\rangle = \frac{iZ^{1/2}}{p^2 - m^2 + i\epsilon} \langle T\phi(x_2)\cdots\phi(x_n)|-\mathbf{p}\rangle + \text{finite}. \quad (3.202)$$

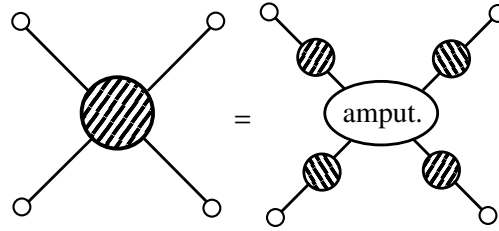
As mentioned before, the last term $\int_{T_-}^{T_+} \cdots$ is finite as the integration interval has a finite length. Now we repeat the above analysis iteratively for the remaining $\phi(x_i)$ with $i = 2, \dots, n$. Importantly, states $|\mathbf{p}\rangle$ are at time $t \rightarrow -\infty$ and states $\langle \mathbf{p}|$ are at time $t \rightarrow +\infty$, and that after iteration we arrive at the matrix element

$$-\infty \langle \mathbf{p}_1 \cdots \mathbf{p}_n | \mathbf{k}_1 \cdots \mathbf{k}_m \rangle_{+\infty} = \langle \mathbf{p}_1 \cdots \mathbf{p}_n | S | \mathbf{k}_1 \cdots \mathbf{k}_m \rangle, \quad (3.203)$$

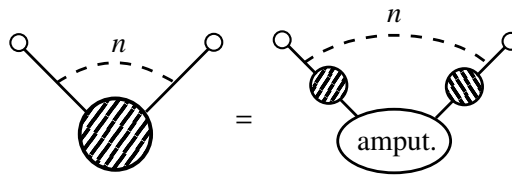
with

$$\frac{i}{p^2 - (m_0^2 + \Pi(p)) + i\epsilon} \xrightarrow{p^2 \rightarrow m^2} \frac{iZ}{p^2 - m^2 + i\epsilon} \quad (3.208)$$

Example 3-10: n=4.



And in general:



Evidently, the right hand side of the LSZ-reduction formula (3.204) can be rewritten in terms of the amputated correlation function, multiplied by $Z^{(n+m)/2}$. Moreover, in terms of the renormalised fields the amputated correlation function is defined by a multiplication of classical inverse propagators,

$$\begin{aligned} Z^{(n+m)/2} \langle T\phi(p_1) \cdots \phi(k_m) \rangle_{\text{amp}} \\ \simeq Z^{(n+m)/2} \prod_i \frac{p_i^2 + m^2}{Z^{1/2}} \prod_j \frac{k_j^2 + m^2}{Z^{1/2}} \langle T\phi(p_1) \cdots \phi(k_m) \rangle \\ = \prod_i (p_i^2 + m^2) \prod_j (k_j^2 + m^2) \langle TZ^{-1/2} \phi(p_1) \cdots Z^{-1/2} \phi(k_m) \rangle, \end{aligned} \quad (3.209)$$

where $\langle TZ^{-1/2} \phi(p_1) \cdots Z^{-1/2} \phi(k_m) \rangle$ is just the expectation value of the renormalised fields.

In conclusion this leads us to a simple relation of S -matrix elements in terms of *amputated* correlation functions, This entails for the S -matrix elements with (3.204)

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_n | S | \mathbf{k}_1 \cdots \mathbf{k}_m \rangle \Big|_{\text{on-shell}} = Z^{(n+m)/2} \text{amput.}$$

which concludes our discussion of the LSZ formalism.

4. Fermions

With the last chapter, [Section 3.5](#), we have completed the discussion of the scalar quantum field theory with the computation of S-matrix elements. Still, some computation details and the systematic treatment of the singularities within a regularisation and renormalisation procedure is postponed to the chapter on renormalisation.

In fundamental particle physics the only scalar field is the Higgs boson, all the other matter fields are fermions, namely the leptons and quarks, and the Standard Model is completed with the gauge fields of the SM gauge group $U(1) \times SU(2) \times SU(3)$. In turn, in condensed matter and statistical physics systems scalar particles are composite fields, the fundamental ones are bosons with spin $n \neq 0$ and fermions with spin $n + 1/2$.

With the results in scalar quantum field theories we are prepared to discuss general field theories, and in particular fermions (this chapter) and gauge fields (next chapter). Fermions play a pivotal rôle in the dynamics of the Standard Model (SM) of particle physics and beyond SM physics; in low energy QCD with Confinement and strong chiral symmetry breaking, the latter being responsible for most of the mass of visible matter; as well as in many condensed matter and statistical physics systems. Fermionic systems show some of the most intricate phase structures, their understanding being essential for high T_c super conductivity, graphene, fermionic ultracold gases and many other exciting systems.

4.1. Fields and Lorentz Invariance

As mentioned in the introduction above, fundamental fermions carry a spin 1/2 representations of the Lorentz group, while scalar fields carry the trivial representation: the scalar field is invariant under Lorentz transformations,

$$\phi(x) \xrightarrow{\Lambda} \phi'(x') = \phi(x) \quad \text{with} \quad x^\mu \rightarrow (x')^\mu = \Lambda^\mu_\nu x^\nu \quad (4.1)$$

and $\phi'(x) = \phi(\Lambda^{-1} x)$. This entails that the full transformation of the field simply undoes that of the coordinates, the field itself is invariant. In other words, all Lorentz transformations are *represented* as a multiplication with one on the field. This defines a map R or *representation* from the Lorentz group to the scalar field space with $R(\Lambda) = 1$ for all Λ .

Similarly, a vector field A_μ transforms as a covariant four-vector under Lorentz transformations such as the position or momentum vectors x_μ and p_μ , see [Section 2.1](#),

$$A_\mu(x) \rightarrow \Lambda_\mu^\nu A_\nu(x), \quad (4.2)$$

and $A'_\mu(x) = \Lambda_\mu^\nu A_\nu(\Lambda^{-1} x)$. The transformation law (4.2) entails that a Lorentz transformation is represented by the transformation matrix Λ with the map $R(\Lambda) = \Lambda$.

The discussion of transformation properties of different field is extended straightforwardly to tensor fields (e.g. tensor fields of rank two as the fieldstrength in QED, QCD, weak interactions),

$$F_{\mu\nu}(x) \rightarrow \Lambda_\mu^\rho \Lambda_\nu^\sigma F_{\rho\sigma}(x), \quad (4.3)$$

and an example for a fundamental tensor field is the graviton $g_{\mu\nu}$, which carries spin 2 degrees of freedom. In four dimensions quantum field theories with spin $n > 1$ are typically perturbatively non-renormalisable, the perturbative renormalisation programme discussed in the renormalisation chapter leads to infinitely many fundamental parameters or couplings. Consequently, the (perturbative) theory lacks predictive power. As for

the scalar field and the vector field we can define the representation map R from the Lorentz group to the tensor field space.

For a general field ϕ^i we define the transformation law under Lorentz transformations with

$$\phi^i(x) \rightarrow R(\Lambda)^i_j \phi^j(x), \quad (4.4)$$

with the general index i , e.g. $i = \{, \mu, \mu\nu, \dots$ and the representation R in field space. The representation is chosen accordingly to the field, i.e.

$$\begin{aligned} \text{scalar: } R(\Lambda) &= 1 \quad \text{trivial representation} \\ \text{vector: } R(\Lambda) &= \Lambda \quad \text{fundamental representation} \\ \text{(2nd rank) tensor: } R(\Lambda) &= \left(\Lambda^\mu_\rho \Lambda^\nu_\sigma \right) \quad \text{tensor representation.} \end{aligned} \quad (4.5)$$

While we have introduced the concept of a representation at the example of the Lorentz group, it applies to general groups G . Then, the representation $R : G \rightarrow R(G)$ has the properties,

$$\begin{aligned} R(\mathbb{1}) &= \mathbb{1}, \\ R(g \cdot h) &= R(g) \cdot R(h). \end{aligned} \quad (4.6)$$

For instance, for rotations in \mathbb{R}^3 , i.e. the Lie group $\text{SO}(3)$ group, we find

$$\begin{aligned} \text{trivial rep: } R(\Lambda) &= 1 \quad \Lambda \in \text{SO}(3) \\ \text{fundamental rep: } R(\Lambda) &= \Lambda \quad \text{Lie group.} \end{aligned} \quad (4.7)$$

A useful property of Lie groups is, that we can write every element of the connected component of unity in the group in terms of an exponential

$$\Lambda = e^{i\omega J}, \quad (4.8)$$

where $\mathbf{J} = (J^1, \dots, J^n)$ are the generators of the Lie group, and \mathbf{J} is contracted with $\boldsymbol{\omega} = (\omega^1, \dots, \omega^n)$. In our example Lie group $\text{SO}(3)$, the generators are $\mathbf{J} = (J^1, J^2, J^3)$. The three-dimensional vector $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)$ has real entries, $\omega^j \in \mathbb{R}$ and the scalar product $\boldsymbol{\omega} \mathbf{J}$ is an element in the Lie algebra $\text{so}(3)$.

We proceed by discussing different representations of the rotation group as simple and relevant examples. To begin with, the trivial representation has the generators $J^i = 0$ for $i = 1, 2, 3$. The three generators in the fundamental representation read

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad J^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.9)$$

with eigenvalues ± 1 . As discussed before, the vector field carries the fundamental representation of the Lorentz group that includes the rotation group. Accordingly, the eigenvalues of the generators are linked to the spin eigenvalues of a vector fields, which is a spin 1 field.

Example 4-11: Rotation about the x_3 axis.

Let us now consider a rotation about the x_3 axis with angle $\omega_3 = \alpha$, and hence the rotation vector

$$\boldsymbol{\omega} = (0, 0, \alpha). \quad (4.10)$$

Inserting this in (4.8) we are led to

$$\begin{aligned}
 \exp(i\omega\mathbf{J}) &= \exp(i\alpha J^3) = \exp\left(\alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\alpha^2}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 + \dots \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\alpha^2}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dots \\
 &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned} \tag{4.11}$$

Evidently this corresponds to a rotation about the three-axis.

An important property of the generators J are their commutation relations, the Lie algebra. They entail the composition law of the group as well as its non-Abelian nature. In the case of our SO(3) example the Lie algebra reads

$$[J^i, J^j] = i\epsilon^{ijk} J^k. \tag{4.12}$$

with the structure constants ϵ^{ijk} that encode the *local* structure of the Lie group. Indeed, the Lie algebra (4.12) is that of SO(3), but also that of SU(2), and we have the map $\text{SO}(3) \simeq \text{SU}(2)/\mathbb{Z}_2$, where the latter is given by $\pm\mathbb{1}$. Equation (4.12) holds true in any representation. For the trivial representation this is evident. Let us now discuss a infinite-dimensional representation in terms of space-derivatives and coordinates, already known from Quantum Mechanics.

$$J^i = -i\epsilon^{ijk} x^j \partial^k = -\frac{1}{2}\epsilon^{ijk} J^{jk}, \quad \text{with} \quad J^{jk} = i(x^j \partial^k - x^k \partial^j). \tag{4.13}$$

For instance, (4.13) is used in quantum mechanics in the n -dimensional representation of spins with $n = 2s + 1$. Finally we consider the spin 1/2 representation of the rotation group. To that end we recall, that the Lie algebra (4.12) is also that of SU(2) with the generators

$$t^i = \frac{\sigma^i}{2}, \quad i = 1, 2, 3, \tag{4.14}$$

with the Eigenvalues $\pm 1/2$. The generators are half of the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.15}$$

Using the commutation relations of the Pauli matrices or their explicit form we get,

$$[t^i, t^j] = i\epsilon^{ijk} t^k, \tag{4.16}$$

which agrees with (4.12). This analysis has also elucidated the fact that the Lie algebra provides the *local* information about the Lie group (tangential space).

For the preparation of the similar discussion of the Lorentz group we wrap up this information in the following brief discussion of the respective manifolds.

Example 4-12: SO(3) and SU(2) $\simeq S^3$. The Lie algebra of SO(3) and SU(2) is given by (4.16). As the Lie group is a differentiable manifold, SU(2) $\simeq S^3$ is the double covering of SO(3) $\simeq \mathbb{RP}^3$, which is visualised in figure 4.1

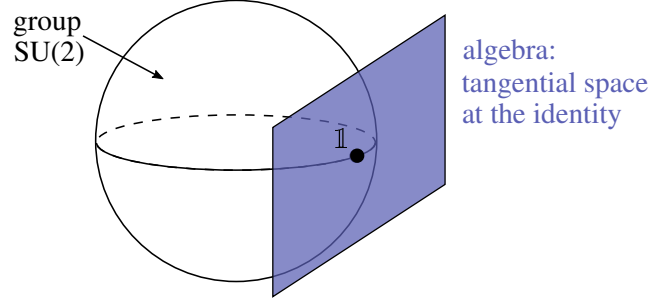


Figure 4.1.: Schematic representation of the SU(2) and the Lie algebra.

This concludes our analysis of the rotation group, and we readily proceed to the Lorentz group.

As for the rotation group we can embed the Lorentz group SO(3) in a universal covering group (simply connected), SL(2, \mathbb{C}). Before we discuss this aspect, we derive the Lie algebra of the Lorentz group. To that end we consider an infinitesimal Lorentz transformation $\Lambda \in \text{SO}(1,3)$,

$$\Lambda_\mu{}^\nu = (e^{iT})_\mu{}^\nu = \delta_\mu{}^\nu + iT_\mu{}^\nu, \quad (4.17)$$

where T is in the Lie algebra of the Lorentz group. We recall the invariance property of the scalar product (2.2),

$$\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}, \quad (4.18)$$

which reads in terms of the traceless Lie algebra element T ,

$$(\delta_\mu{}^\nu + iT_\mu{}^\nu)(\delta_\rho{}^\sigma + iT_\rho{}^\sigma)\eta_{\nu\sigma} = \eta_{\mu\rho} + \mathcal{O}(T^2) \quad \Rightarrow \quad T_{\mu\rho} + T_{\rho\mu} = 0. \quad (4.19)$$

This entails that T is antisymmetric and traceless, and hence has $(16-4)/2 = 6$ independent components. Three of them generate rotations about the axes x_1, x_2, x_3 and three of them generate boosts parallel to these axes. The Lie algebra is written in terms of antisymmetric components $\omega^{\rho\sigma}$ and the generators $M_{\rho\sigma}$ of the Lorentz group.

Generators M of SO(1,3)

$$T_\mu{}^\nu = \frac{\omega^{\rho\sigma}}{2} (M_{\rho\sigma})_\mu{}^\nu, \quad (4.20)$$

The generators M satisfy the Lie algebra of the Lorentz group,

Lie Algebra of SO(1,3)

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho}). \quad (4.21)$$

Equation (4.21) is the Lie algebra of SO(1,3) rotations. To see this more explicitly, we extend the SO(3)-generators of rotations, J^{ij} in Equation (4.13) to boosts J^{0i} and find

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu), \quad (4.22)$$

which satisfy (4.21). To find general representations, we also look for M , that satisfy (4.21). For instance, for the fundamental representation we obtain

$$(M^{\mu\nu})_{\rho\sigma} = i(\delta_{\rho}^{\mu}\delta_{\sigma}^{\nu} - \delta_{\sigma}^{\mu}\delta_{\rho}^{\nu}). \quad (4.23)$$

Thus, boosts and rotations are given by

$$\begin{aligned} \text{Rotations: } J_i &= \frac{1}{2}\epsilon_{ijk} M_{jk} \\ \text{Boosts: } K_i &= M_{0i}. \end{aligned} \quad (4.24)$$

Example 4-13: Boosts along x_1 -axis. Analogously to our example of a rotation about the x_3 axis we briefly discuss a boost along the x_1 axis.

$$\Lambda_{\nu}^{\mu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (e^{i w K_1})_{\nu}^{\mu}, \quad \text{with } \gamma = \frac{1}{\sqrt{1-v^2}}, \quad (4.25)$$

with the rapidity $w = \text{artanh} \frac{v}{2}$ and the generator

$$i(K_1^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad i(K_2^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad i(K_3^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (4.26)$$

where we have also provides the generators of boosts along the x_2 and x_3 axes. Removing the empty rows and columns, the generators $\mathbf{K} = (K_1, K_2, K_3)$ resembles the generators \mathbf{J} of the rotations. This is an expected property, as the boosts have the structure of 'imaginary' rotation in i.e. x_0, x_1 in our example due to the relative minus sign in the Minkowski metric.

The generators in (4.24) make the structure of the Lorentz group more apparent. To that end we formulate the Lie-algebra in terms of \mathbf{J} and \mathbf{K} ,

$$\begin{aligned} [J^i, J^j] &= i\epsilon^{ijk} J^k \\ [K^i, K^j] &= -i\epsilon^{ijk} J^k \\ [J^i, K^j] &= i\epsilon^{ijk} K^k. \end{aligned} \quad (4.27)$$

Equation (4.27) entails that the rotation group is a subgroup in the Lorentz group as the commutator of two generators of rotations is a third rotation. In turn, the commutator of two boosts is a rotation, hence boosts are not a subgroup of the Lorentz group. Finally, a rotation of a boost is a boost.

We close this discussion with the remark, that $SU(2)$ and $SL(2, \mathbb{C})$ with the generators $(J^i + iK^i, J^i - iK^i)$ are *universal covering groups* of $SO(3)$ and $SO(1,3)$ respectively.

4.2. Spinor Fields

In [Section 4.1](#) we have discussed the mathematical structure of the Lorentz group. It contains the rotation group with its covering group $SU(2)$ and hence spin 1/2 representations. Moreover, [\(4.27\)](#) already suggests the existence of a further $SU(2)$ subgroup in the Lorentz group we want to uncover now. This leads us to the definition of right- and left-handed spinors respectively. In short, we use different combinations of the generators \mathbf{J} , \mathbf{K} that both satisfy the Lie algebra relation of $SU(2)$. We already have mentioned in the discussion of the example of the boost in x_1 direction that the structure of the generators of boosts and rotations are related, the former is an 'imaginary version' of the latter. This suggests to consider the following combination of boosts K^i and rotations J^i ,

$$N_{\pm}^i = \frac{1}{2} (J^i \pm i K^i), \quad (4.28)$$

that both satisfy the $SO(3)$, $SU(2)$ Lie-algebra relation,

$$[N_{\pm}^i, N_{\pm}^j] = i \epsilon_{ijk} N_{\pm}^k, \quad [N_{+}^i, N_{-}^j] = 0, \quad (4.29)$$

and hence can be represented by $SU(2)$ matrices and carry spin 1/2 eigenvalues. These two-dimensional spin 1/2 representations $R_{L/R}(\Lambda) = \Lambda_{L/R}$ are that of left-handed and right-handed spinors

$$\begin{aligned} \text{left-handed spinors: } \Lambda_L &= \exp \left\{ \frac{i}{2} \sigma^i (w_i - i v_i) \right\} \\ \text{right-handed spinors: } \Lambda_R &= \exp \left\{ \frac{i}{2} \sigma^i (w_i + i v_i) \right\}, \end{aligned} \quad (4.30)$$

where w_i and v_i denote the rotation parameters and boost parameters respectively and $\Lambda_L, \Lambda_R \in SL(2, \mathbb{C})$, the universal covering group of the Lorentz group. The two spin representations in [\(4.30\)](#) are linked with a parity transformation P with

$$(x^0, \mathbf{x}) \xrightarrow{P} (x^0, -\mathbf{x}), \quad (4.31)$$

that maps *left-handed* \leftrightarrow *right-handed*. Moreover, it leaves spacial pseudo-vectors such as e.g. $\mathbf{x} \times \mathbf{p}$ invariant. It follows

$$\begin{aligned} \mathbf{J} &\xrightarrow{P} \mathbf{J} \quad \text{pseudo-vector} \\ \mathbf{K} &\xrightarrow{P} -\mathbf{K} \quad \text{vector}. \end{aligned} \quad (4.32)$$

The spinor representations [\(4.30\)](#) also encode the Lorentz transformations of coordinates x_{μ} or more generally four-vectors. For this purpose we define

$$\hat{x} = x_{\mu} \sigma^{\mu}, \quad \text{with} \quad (\sigma^{\mu}) = (\sigma^0, \boldsymbol{\sigma}), \quad \sigma^0 = \mathbb{1}_{2 \times 2}, \quad (4.33)$$

and the spatial vector $\boldsymbol{\sigma}$ of the Pauli matrices $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ with the σ^i provided in [\(4.15\)](#). The explicit form of this matrix is given by

$$\hat{x} = \begin{pmatrix} x_0 - x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 + x_3 \end{pmatrix}, \quad \text{with} \quad \det \hat{x} = x_{\mu} x^{\mu}, \quad (4.34)$$

and Lorentz transformations of x_{μ} are implemented via

$$\hat{x}' = \Lambda_L \hat{x} \Lambda_L^{\dagger} \quad \text{with} \quad \det \hat{x}' = \det \hat{x}, \quad (4.35)$$

as $\det \Lambda_L^{(\dagger)} = 1$. For the right-handed spinor representation we have to contract x_μ with

$$\bar{\sigma} = (\sigma^0, -\boldsymbol{\sigma}), \quad (4.36)$$

and $\sigma, \bar{\sigma}$ defined in (4.33) and (4.36) transform as vectors under Lorentz transformations. We also remark, that Λ_L and $-\Lambda_L$ lead to the same \hat{x}' (double covering). Furthermore we have

$$\Lambda_{L/R}^\dagger = \Lambda_{R/L}^{-1}. \quad (4.37)$$

These relations are discussed below in more details. We close this discussion with putting the representation to work within a simple example of a boost in x_3 -direction.

Example 4-14: Boosts along x_3 -direction. Let us consider the transformation where only v_3 is non-vanishing. The respective spinor representation of $\Lambda(v_3)$ is given by

$$\Lambda_L = \exp\left[\frac{i}{2}\sigma^3(-iv_3)\right] = \exp\left[\frac{v_3\sigma^3}{2}\right] \quad (4.38)$$

$$\begin{aligned} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{v_3}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \left(\frac{v_3}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots \\ &= \begin{pmatrix} \exp(v_3/2) & 0 \\ 0 & \exp(-v_3/2) \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \end{aligned} \quad (4.39)$$

Now we apply (4.39) to \hat{x} , leading to

$$\begin{aligned} \hat{x}' &= \Lambda_L \hat{x} \Lambda_L^\dagger = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a(x_0 - x_3) & a^{-1}(x_1 + ix_2) \\ a(x_1 + ix_2) & a^{-1}(x_0 + x_3) \end{pmatrix} \\ &= \begin{pmatrix} a^2(x_0 - x_3) & x_1 + ix_2 \\ x_1 - ix_2 & a^{-2}(x_0 + x_3) \end{pmatrix}. \end{aligned} \quad (4.40)$$

Evidently, the coordinates x_1 and x_2 remain unchanged under this transformation. Moreover, x'_0 and x'_1 can be read off from (4.40) as

$$x'_0 = \gamma x_0 - \gamma v x_3, \quad x'_3 = \gamma x_3 - \gamma v x_0, \quad (4.41)$$

with

$$\gamma = \frac{1}{2} \left(a^2 + \frac{1}{a^2} \right), \quad \gamma v = \frac{1}{2} \left(a^2 - \frac{1}{a^2} \right), \quad (4.42)$$

which is the standard form for a Lorentz boost along the x_3 -axis.

This concludes our introduction of spinor representations of the Lorentz group.

4.2.1. Dirac equation

It is left to construct the classical Lorentz invariant action of a spinor field. We start this derivation by considering the field equations for a two-component left handed spinor,

$$\mathcal{D}_L \psi_L = 0, \quad (4.43)$$

where ψ_L is the left-handed *Weyl spinor* and \mathcal{D}_L is the kinetic operator of the theory, the analogue of the Klein-Gordon operator $\partial_\mu \partial^\mu + m^2$. Equation (4.43) or rather its solutions have to be left unchanged under Lorentz transformations. The field carries a left-handed spinor representation of the Lorentz group,

$$\psi_L(x) \xrightarrow{\Lambda} \Lambda_L \psi_L(x). \quad (4.44)$$

It is left to determine the transformation property of \mathcal{D}_L . As an operator in spinor space its transformation form is that of \hat{x} , however it has to transform from the right with $\Lambda_L^{-1} = \Lambda_R^\dagger$ in order to annihilate the transformation of the spinor ψ_L . These considerations leads us to

$$\mathcal{D}_L \psi_L(x) \xrightarrow{\Lambda} \mathcal{D}'_L \Lambda_L \psi_L(x) = \Lambda_R \mathcal{D}_L \psi_L(x), \quad (4.45)$$

with

$$\mathcal{D}'_L = \Lambda_R \mathcal{D}_L \Lambda_R^\dagger, \quad \text{as} \quad \Lambda_R^\dagger = \Lambda_L^{-1}, \quad (4.46)$$

where we have used (4.37). Equation (4.45) entails that the Weyl equation of the left-handed spinor transforms as a right-handed spinor. Hence, the operator \mathcal{D}_L maps left-handed-spinors into right-handed ones. Moreover, the minimal form of the left-handed kinetic operator is given by $\mathcal{D}_L = i \bar{\sigma}^\mu \partial_\mu$ and $\bar{\sigma} = (\sigma^0, -\vec{\sigma})$. The latter follows from the fact that \mathcal{D}_L transforms with the right-handed spinor representation. The analogous derivation can also be done for the right-handed Weyl spinor, with $\mathcal{D}_R = i \sigma^\mu \partial_\mu$, which yields the Weyl equations.

Weyl equations

$$\begin{aligned} i \bar{\sigma}^\mu \partial_\mu \psi_L &= 0 \\ i \sigma^\mu \partial_\mu \psi_R &= 0, \end{aligned} \quad (4.47)$$

with $\sigma^\mu, \bar{\sigma}^\mu$ defined in (4.33) and (4.36). The Weyl equations (4.47) are the equations of motion of two-component spinors. We emphasise, that they do not have parity invariance, but a parity transformation leads us from the left- to the right-handed Weyl equations and vice versa. Let us now connect (4.47) to the Klein-Gordon equation (2.12). To that end we multiply the left-handed Weyl operator with the right-handed one, and the product maps left-handed spinors to left-handed spinors. We arrive at

$$\sigma^\mu \partial_\mu \bar{\sigma}^\nu \partial_\nu \psi_L = \frac{1}{2} (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) \partial_\mu \partial_\nu \psi_L = \eta^{\mu\nu} \partial_\mu \partial_\nu \psi_L, \quad (4.48)$$

For the derivation of (4.48) we used that

$$\frac{1}{2} (\sigma^0 \sigma^0 + \sigma^0 \sigma^0) = 1, \quad \frac{1}{2} (\sigma^0 \bar{\sigma}^j + \sigma^j \bar{\sigma}^0) = 0, \quad \frac{1}{2} (\sigma^i \bar{\sigma}^j + \sigma^j \bar{\sigma}^i) = -\delta^{ij}. \quad (4.49)$$

Equation (4.48) is nothing but the Klein-Gordon equation for a massless field, $\partial_\mu \partial^\mu \psi_L = \partial^2 \psi_L = 0$. Similarly, one shows $\partial^2 \psi_R = 0$, which implies, that the Weyl spinors also satisfy the *massless* Klein-Gordon equation.

While Weyl fermions carry the spinor representation of the Lorentz group, they are not invariant under parity transformations, that map left-handed spinors into right-handed ones and $P\sigma = \bar{\sigma}$, and hence

$$P\mathcal{D}_L = \mathcal{D}_R, \quad \text{with} \quad \mathcal{D}_L = i\bar{\sigma}^\mu \partial_\mu, \quad \mathcal{D}_R = i\sigma^\mu \partial_\mu. \quad (4.50)$$

This suggests to combine left- and right-handed spinors into a four-dimensional spinor, the *Dirac spinor*

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad (4.51)$$

We have already discussed, that $\mathcal{D}_L\psi_L$ transforms as a right-handed spinor, and $\mathcal{D}_R\psi_R$ transforms as a left-handed one. This is reflected in the following representation of the two Weyl equations in terms of the massless *Dirac equation* for the Dirac spinor,

$$\begin{pmatrix} 0 & \mathcal{D}_R \\ \mathcal{D}_L & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathcal{D}\psi_D = i\gamma^\mu \partial_\mu \psi_D, \quad \text{with} \quad \mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_R \\ \mathcal{D}_L & 0 \end{pmatrix}, \quad (4.52)$$

with the Dirac operator \mathcal{D} and the γ -matrices γ^μ

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (4.53)$$

with $\sigma^\mu, \bar{\sigma}^\mu$ defined in (4.33) and (4.36). The γ -matrices satisfy the Clifford algebra,

Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (4.54)$$

which is their defining property. Note that there are different representations of (4.54) related by unitary rotations, that better suited for different application. For instance, the representation (4.53) is the *chiral representation*, and emphasises the chiral structure of left- and right-handed spinors.

The transformation properties under Lorentz transformation of the Dirac spinor ψ_D follows from that its left- and right-handed components $\psi_{L/R}$ with

$$\psi_D \xrightarrow{\Lambda} \Lambda_{1/2} \psi_D = \begin{pmatrix} \Lambda_L \psi_L \\ \Lambda_R \psi_R \end{pmatrix}, \quad \text{with} \quad \Lambda_{1/2} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \quad (4.55)$$

where $\Lambda_{1/2}$ is in the four-dimensional spin 1/2 representation of the Lorentz group. Similarly, the transformation properties of the Dirac operator \mathcal{D} follow from that of $\mathcal{D}_{L/R}$ in (4.46),

$$\mathcal{D} \xrightarrow{\Lambda} \Lambda_{1/2} \mathcal{D} \Lambda_{1/2}^{-1}. \quad (4.56)$$

As the above transformations were derived from that of the left- and right-handed components, the massless Dirac equation (4.52) has to transform as a Dirac spinor under Lorentz transformations. With (4.55) and (4.56) this properties follows straightforwardly,

$$i\gamma^\mu \partial_\mu \psi_D \xrightarrow{\Lambda} \Lambda_{1/2} i\gamma^\mu \partial_\mu \Lambda_{1/2}^{-1} \Lambda_{1/2} \psi_D = \Lambda_{1/2} i\gamma^\mu \partial_\mu \psi_D. \quad (4.57)$$

With the above analysis we are in the position to formulate the general equation of motion for spin 1/2 fields. So far, the spinors have been massless, as the Weyl equations do not admit a mass term: the Weyl operators $\mathcal{D}_{L/R}$ map left- to right-handed spinors and vice versa, while a mass term is diagonal in spinor space,

$$m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \psi_D = m \psi_D. \quad (4.58)$$

with the 2×2 identity matrices $\mathbb{1}_{2 \times 2}$ in the left- and right-handed spaces. In consequence, the massive Dirac equation will mix left- and right-handed spinors. In the following we drop the subscript D for the Dirac spinor, $\psi = \psi_D$, and the massive Dirac equation follows from combining the massless Dirac equation (4.52) and the Dirac mass term (4.58),

Dirac equation

$$(i \not{\partial} - m) \psi = 0, \quad (4.59)$$

with the shorthand notation

$$\not{p} := \gamma^\mu p_\mu. \quad (4.60)$$

for a given four-vector p . Similarly to the Weyl spinors, that satisfied the massless Klein-Gordon equation, the Dirac spinor satisfies the *massive* Klein-Gordon-equation (2.12),

$$(-i \gamma^\mu \partial_\mu - m)(i \gamma^\nu \partial_\nu - m) \psi = (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2) \psi = \left(\frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right) \psi = (\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2) \psi, \quad (4.61)$$

where we have used the Clifford algebra (4.54). We close our discussion of the equation of motion for spin 1/2 spinors with a final remark: while the two-dimensional Weyl spinors are necessarily massless, it is possible to construct massive two-dimensional spinor representations, the Majorana spinors, that play a rôle in the Standard Model, if massive Neutrinos are taken into account.

In the remainder of this section we collect some results on the spin 1/2 representation of the Lorentz group that are relevant for the derivation of the Dirac action in Section 4.2.2 as well as for the discussion of general symmetry properties of spinor fields. To begin with, the spin 1/2 representations of the generators M , (4.20), is given by,

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}, \quad (4.62)$$

with $\sigma \bar{\sigma} : L \rightarrow L$ and $\bar{\sigma} \sigma : R \rightarrow R$ and hence

$$\Lambda_{1/2} = e^{i \frac{\omega_{\mu\nu}}{2} S^{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \quad (4.63)$$

with the block diagonal components $\Lambda_{L/R}$ in (4.30): We can confirm straightforwardly that the $S^{\mu\nu}$ are a representation of the generators of the Lorentz group as they satisfy the Lie algebra (4.21). Moreover, (4.62) makes the block-diagonal structure of left-handed and right-handed transformations explicit. The structure is even more apparent when discussing the boosts and rotations \mathbf{K}, \mathbf{J} in (4.24) in the spin 1/2 representation. First concentrating on the left-handed part with K_L, J_L , we are led to

$$\begin{aligned} K_L^i &= S_L^{0i} = -i \frac{\sigma^i}{2} = i \frac{\bar{\sigma}^i}{2} \\ J_L^i &= \frac{1}{2} \epsilon^{ijk} S^{jk} = -\frac{i}{2} \epsilon^{ijk} \left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2} \right] = -\frac{i}{2} \epsilon^{ijk} \left(i \epsilon^{jkl} \frac{\sigma^l}{2} \right) = \frac{\sigma^i}{2}. \end{aligned} \quad (4.64a)$$

The relations for the right handed transformation follows analogously,

$$K_{iR} = i \frac{\sigma_i}{2}, \quad J_{iR} = \frac{\sigma_i}{2}. \quad (4.64b)$$

With (4.64) we can convert (4.63) into the form (4.30) with

$$\Lambda_{1/2} = e^{i \frac{w_{\mu\nu}}{2} S^{\mu\nu}} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \quad \text{with} \quad w_{0i} = v_i, \quad w_{ij} = \epsilon_{ijk} w_k. \quad (4.65)$$

In summary, we have worked out the representation of the Lorentz group in the four-dimensional spin representation in terms of its generators $M_{\mu\nu} = S_{\mu\nu}$. The final ingredient of our derivation and consistency checks of the Dirac equation was its transformation properties under Lorentz transformations that required the inverse $\Lambda_{1/2}^{-1}$. We have already seen, that the inverse of the right- and left-handed transformations where the adjoint of the left- and right-handed transformations, (4.46). With this relation we can show, that the inverse of $\Lambda_{1/2}$ is obtained by a 'rotation' or conjugate with γ^0 : To begin with, the Clifford algebra (4.54) entails that γ^0 is Hermitian, while the γ^i are anti-Hermitian,

$$(\gamma^0)^2 = \mathbb{1}, \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^2 = -\mathbb{1}, \quad (\gamma^i)^\dagger = -\gamma^i. \quad (4.66)$$

As (4.66) follows only with the Clifford algebra, these properties hold in any representation. From (4.53), that is in the chiral representation, it also follows

$$\gamma^0 (\gamma^i)^\dagger \gamma^0 = \gamma^i, \quad (4.67)$$

As (4.66), the relation (4.67) is independent of the representation of the γ -matrices: rotating the γ -matrices into another representation does not change (4.66). With (4.67) we derive the wanted relation, the adjoint of the generators $S^{\mu\nu}$ is given by the conjugation with γ^0 ,

$$\gamma^0 (S^{\mu\nu})^\dagger \gamma^0 = S^{\mu\nu}. \quad (4.68)$$

Thus, the inverse Lorentz transformation for the four-dimensional spin 1/2 representation is given by

$$\gamma^0 \Lambda_{1/2}^\dagger \gamma^0 = \Lambda_{1/2}^{-1}. \quad (4.69)$$

4.2.2. Dirac action, invariants and symmetry properties

In the remainder of this section we discuss the Dirac Lagrangian, that leads to the Dirac equation as well as the transformation properties and generators in the spin 1/2 representation. The Dirac equation is linear in the Dirac spinor ψ , and hence the Dirac Lagrangian is a bi-linear in ψ , and we simply multiply the Dirac equation from the left with $\bar{\psi}$. This leads us to the Dirac action S_D with

$$S_D[\psi, \bar{\psi}] = \int d^4x \mathcal{L}_D, \quad \text{with} \quad \mathcal{L}_D = \bar{\psi} (i\partial - m) \psi. \quad (4.70)$$

The field $\bar{\psi}$ is linearly related to ψ or rather ψ^\dagger , the latter being seemingly obvious choice. However, the Lagrangian is a Lorentz scalar and we use this property to determine the transformation properties of $\bar{\psi}$ which turn out not to be that of ψ^\dagger . A Lorentz transformation of the Dirac Lagrangian (4.70) is given by

$$\mathcal{L}_D \xrightarrow{\Lambda} \bar{\psi}' \Lambda_{1/2} (i\partial - m) \psi \quad \longrightarrow \quad \bar{\psi}' = \bar{\psi} \Lambda_{1/2}^{-1}. \quad (4.71)$$

The adjoint of the Dirac spinor transforms with $\psi^\dagger \rightarrow \psi^\dagger \Lambda_{1/2}^\dagger$, and this factor does not cancel $\Lambda_{1/2}^{-1}$, but the transformation of the *Dirac conjugate* does. The Dirac conjugate is defined with

$$\bar{\psi} = \psi^\dagger \gamma^0, \quad (4.72)$$

and with (4.69) it follows

$$\bar{\psi}' = \psi^\dagger \Lambda_{1/2}^\dagger \gamma^0 = \psi^\dagger \gamma^0 \gamma^0 \Lambda_{1/2}^\dagger \gamma^0 = \bar{\psi} \Lambda_{1/2}^{-1}. \quad (4.73)$$

With the definition (4.72) for $\bar{\psi}$ the Dirac Lagrangian (4.70) is invariant under Lorentz transformations. The Dirac equation (4.59) is the equation of motion, that follows from (4.70),

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 = (i\cancel{\partial} - m) \psi, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0 = \bar{\psi} (i\overleftarrow{\cancel{\partial}} - m), \quad (4.74)$$

with the notation $f \overleftarrow{\partial}_\mu = -\partial_\mu f$. In Section 4.3 we discuss the quantisation of fermionic theories. In view of this application we also provide the Hamiltonian density of the Dirac field, that is obtained from the Dirac Lagrangian (4.70) with a Legendre transform. With $\pi_\psi = i\bar{\psi}\gamma^0$ we derive

$$\mathcal{H} = \pi_\psi \dot{\psi} - \mathcal{L} = i\bar{\psi}\gamma^0 \dot{\psi} - \mathcal{L} = \psi^\dagger \gamma^0 (i\boldsymbol{\gamma} \boldsymbol{\partial} + m) \psi, \quad (4.75)$$

where $\boldsymbol{\gamma} \boldsymbol{\partial} = \gamma^i \partial^i$. As the kinetic operator is linear and not quadratic as for the scalar field theory, the Hamiltonian is seemingly not bounded from below. This particularity of fermionic theories will be discussed in detail in the chapter about the quantisation of the theory, Section 4.3.

We close the present chapter with a discussion of (bilinear) invariants and general properties of the spinor space. First we note that some of the derivations above were performed within a specific representation of our spinors in left- and right-handed Weyl spinors, the chiral representation. While we have argued that the respective derivations hold true under unitarity rotations of the γ -matrices, the chiral representation may not always be the best-suited one for the application at hand. In particular for massive Dirac fermions this is certainly not the case.

Hence, we discuss different representations more systematically. First we note that unitary transformations U of γ 's and ψ 's with

$$\gamma \rightarrow U \gamma U^\dagger, \quad \psi \rightarrow U \psi. \quad (4.76)$$

leave the Dirac Lagrangian (4.71) invariant. Moreover, (4.76) leaves the Clifford algebra unchanged. However, after such a unitary rotation of the γ -matrices and the generators $S_{\mu\nu}$ of the Lorentz group, the projection $P_{L/R}$ onto the left- and right-handed eigenspaces is not obvious any more. Obviously, a definition of the projection operators in terms of the γ -matrices provides us with an representation-invariant definition of the $P_{L/R}$. For this purpose we define

$$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (4.77)$$

In the chiral representation, the matrix γ_5 is given by

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \quad (4.78)$$

Hence it assigns in any representation the Eigenvalue -1 to left-handed spinors, $\gamma_5\psi_L = \psi_L$ and the Eigenvalue $+1$ to right-handed spinors, $\gamma_5\psi_R = \psi_R$. While the diagonal form (4.78) only holds true in specific representation, i.e. the chiral one, γ_5 exhibits remarkable representation-independent properties,

$$\begin{aligned}\gamma_5^2 &= \mathbb{1} \quad \rightarrow \text{eigenvalues } \pm 1, \\ \{\gamma_5, \gamma^\mu\} &= 0, \\ [S_{\mu\nu}, \gamma_5] &= 0 \quad \rightarrow S_{\mu\nu}, \gamma_5 \text{ can be diagonalised at the same time.}\end{aligned}\tag{4.79}$$

In particular we can define the projection operators $P_{L/R}$ with

Projection operators on the spaces of left- and right-handed spinors

$$P_{L/R} = \frac{\mathbb{1} \mp \gamma_5}{2},\tag{4.80}$$

with $P_{L/R}^2 = P_{L/R}$ and $P_L + P_R = \mathbb{1}$, and

$$P_{L/R}\psi = \psi_{L/R}.\tag{4.81}$$

With γ_μ and γ_5 we can also generate the full space of 4×4 matrices in terms of Lorentz scalars, vectors and tensors. With these matrices we can also span the space of all Dirac field bilinears. Amongst other applications, they are relevant for the construction of interacting fermionic theories. For instance, with the Dirac equation or rather the Dirac Lagrangian we have already introduced two of these bi-linears, the Lorentz scalar $\bar{\psi}\psi$ (mass term) and the Lorenz vector $\bar{\psi}\gamma_\mu\psi$ (kinetic term). Their squares give us Lorentz invariant four-Fermi interactions. The scalar and vector property origin in the transformation properties of the γ -matrices with $\gamma^\mu \rightarrow \Lambda^\mu_\nu \gamma^\nu$, and hence products of the γ -matrices carry the respective tensor representation. This leads us to a basis of sixteen 4×4 matrices, defined as antisymmetric combinations of γ -matrices,

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\gamma^{[\mu}\gamma^{\nu]}$	tensor	6
$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$	pseudo-vector	4
γ_5	pseudo-scalar	1
		16 total

where $\gamma^{[\mu_1 \dots \mu_n]}$ are the (completely) antisymmetric part of the product of n γ matrices. In particular we have

$$\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2} [\gamma^\mu, \gamma^\nu], \quad \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \frac{1}{6} \left(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu \right).\tag{4.83}$$

The prefix *pseudo* in (4.82) entails, that the respective quantities transform as pseudo-scalars / vectors / tensors under the Lorentz transformations (odd under Parity); a simple example for a pseudo-vector is the angular momentum.

Evidently, γ_5 is a pseudo-scalar as the temporal γ^0 does not change under parity, while the product of three spatial matrices in the definition (4.77) lead to $(-1)^3$ under a parity transformation. Moreover, any of the anti-symmetric products $\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$ can multiplied by $(\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma)^2 = \epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}$, and hence is proportional to $i\gamma\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma$, the latter being a pseudo-vector.

Finally, we derive the conserved Noether charges for the Dirac action. To begin with, the Dirac action (4.70) is invariant under unitary phase rotations, similarly to the complex scalar field,

$$\psi \rightarrow e^{i\alpha} \psi \quad \longrightarrow \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha}, \quad (4.84)$$

and with (2.34) (second equation) the respective Noether current is given by

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad \text{with} \quad \partial_\mu (\bar{\psi} \gamma^\mu \psi) \Big|_{\text{EOM}} = i m \bar{\psi} \psi - i m \bar{\psi} \psi = 0. \quad (4.85)$$

The Noether charge Q is the fermion number or electric charge with

$$Q = \int d^3x j^0 = \int d^3x \psi^\dagger \psi, \quad \text{with} \quad j^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi, \quad (4.86)$$

and plays a pivotal rôle in relativistic and non-relativistic theories.

We know from the derivation, that the massless Dirac action simply is a sum of the left- and right-handed Weyl actions, as only the mass term connects the left- and right-handed spinors. Hence, for $m = 0$, the Dirac action enjoys a further phase symmetry, best captured with $\psi_{L/R} \rightarrow \exp\{\mp i\alpha\} \psi_{L/R}$. These are axial (pseudo-scalar) transformations summarised with

$$\psi \rightarrow e^{i\gamma_5 \alpha} \psi \quad \longrightarrow \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\gamma_5 \alpha}. \quad (4.87)$$

For the derivation of the transformation law for $\bar{\psi}$ from that of ψ we have used that $\{\gamma_5, \gamma^0\} = 0$ implies

$$e^{-i\gamma_5 \alpha} \gamma^0 = \gamma^0 e^{+i\gamma_5 \alpha}. \quad (4.88)$$

Similarly to the derivation of (4.85), the axial Noether current j_5^μ for massless Dirac fermions is given by

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi, \quad \text{with} \quad \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) \Big|_{\text{EOM}} = 0, \quad (4.89)$$

and the Noether charge Q is the axial charge

$$Q_5 = \int d^3x j_5^0 = \int d^3x \psi^\dagger \gamma_5 \psi, \quad \text{with} \quad j_5^0 = \bar{\psi} \gamma^0 \gamma_5 \psi = \psi^\dagger \gamma_5 \psi. \quad (4.90)$$

We emphasise again, that the axial current j_5^μ is only conserved for $m = 0$ with chiral symmetry. In the presence of a mass term we find

$$\partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) = 2i m \bar{\psi} \gamma_5 \psi \quad (4.91)$$

In quantum field theories the axial current conservation is not only violated by mass terms, but also by the axial anomaly, related to topological configurations (instantons). It manifests itself already on the level of perturbation theory, where it originates in the incompatibility of a *local* regularisation that respects both, the vector symmetry carried by the bilinear $\bar{\psi} \gamma^\mu \psi$, and the axial vector symmetry carried by the bilinear, $\bar{\psi} \gamma^\mu \gamma_5 \psi$.

4.2.3. Solutions of the Dirac equation

Before we turn to the quantisation of the fermion field in the next Chapter, Section 4.3, we discuss the general solutions of the Dirac equation. As in the scalar field theory, the general solution of the Dirac equation can be written as a spatial momentum integral over all solutions with general coefficients, labelled by the on-shell condition $p^2 = m^2$. This follows already from the fact, that a solution of the Dirac equation also satisfies the Klein-Gordon equation, see (4.61). Moreover, due to the spin degrees of freedom we expect additional spin sums. As in the scalar QFT we derive the quantum field by the canonical quantisation relations, elevating

the coefficients of the general classical solution to creation and annihilation operators, whose commutation relations follow from that of the fermionic field.

In view of the fact, that the Dirac fermion also satisfies the Klein-Gordon equation, we parametrise the solutions with

$$\psi(x) = u(p) e^{-ipx}, \quad \psi(x) = v(p) e^{ipx} \quad (4.92)$$

with a spinor vectors $u(p), v(p)$ and the on-shell momentum $p^2 = m^2$. Inserting (4.51) into the Dirac equation (4.59) leads to an Eigenvector equations for $u(p), v(p)$ with

$$e^{ipx} (i\not{\partial} - m) \psi(x) = (\not{p} - m)u(p) = 0, \quad \text{and} \quad (\not{p} + m)v(p) = 0, \quad (4.93)$$

where it is understood that m stands for $m\mathbb{1}_{4\times 4}$. For the solution of the two matrix equations in (4.93) we first simplify them by going to the rest frame of the field with vanishing spatial momentum $\mathbf{p} = 0$. There, the equation is solved readily and the general solution is then obtained by applying a general boost to the rest frame solution.

In the rest frame, $p = (p_0, 0)$ with $p_0 = m$, the Dirac equation simplifies significantly and reads

$$m(\gamma^0 - \mathbb{1})u(p) = 0, \quad \text{and} \quad m(\gamma^0 + \mathbb{1})v(p) = 0. \quad (4.94)$$

In the chiral representation used so far for explicit computation, we find

$$(\gamma^0 \pm \mathbb{1}) = \begin{pmatrix} \pm \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \pm \mathbb{1} \end{pmatrix}. \quad (4.95)$$

where the $\mathbb{1}$ are 2×2 matrices. With (4.95), the vectors $u(p), v(p)$ with vanishing Eigenvalues take a complicated form. Clearly, a diagonal form of $\gamma^0 \pm \mathbb{1}$ is preferred both for finding a solution as well as discussing its properties, and in particular discussing the non-relativistic limit. Therefore, we change the representation with a unitary rotation U defined in (4.76) such that $\gamma^0 \pm \mathbb{1}$ is diagonal. This is the *Dirac representation*, which is obtained from the chiral representation with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}. \quad (4.96)$$

The γ -matrices in the Dirac representation are derived from that in the chiral one, (4.53), by applying the rotation (4.76) with (4.96),

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (4.97)$$

In particular, the Dirac representation contains a diagonal γ^0 , while γ_5 is off-diagonal. The diagonal γ^0 leads to a simple diagonal form of (4.95) with

$$(\gamma^0 - \mathbb{1}) = 2 \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad (\gamma^0 + \mathbb{1}) = 2 \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad (4.98)$$

This leads to simple solutions to the eigenvector equations (4.94),

$$u_s(p^0) = \sqrt{2m} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix}, \quad v_s(p^0) = \sqrt{2m} \begin{pmatrix} 0 \\ \epsilon \chi_s \end{pmatrix}, \quad (4.99)$$

with the spin eigenvalues $s = \pm 1/2$ and the respective eigenvectors $\chi_{\pm 1/2}$ with

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2. \quad (4.100)$$

In (4.100) we have also defined ϵ , the symplectic metric in spinor space. The solutions (4.51) with the vectors (4.99) constitute the general solutions of the Dirac equation in the rest frame. Now we apply a general Lorentz boost to these solutions for obtaining the solutions in a general frame. As the solutions (4.51) have been obtained in the Dirac representation, we have to transform our Lorentz transformations $\Lambda_{1/2}$ from the chiral representation Λ_{chiral} in (4.55), to the Dirac representation Λ_D . With (4.96) we are led to

$$\Lambda_D = U \Lambda_{\text{chiral}} U^\dagger = \frac{1}{2} \begin{pmatrix} \Lambda_L + \Lambda_R & \Lambda_R - \Lambda_L \\ \Lambda_R - \Lambda_L & \Lambda_L + \Lambda_R \end{pmatrix}. \quad (4.101)$$

Finally, boosting our solution with (4.101), we arrive at,

Solutions of the Dirac equation

$$u_s(p) = \frac{1}{\sqrt{2m}} \frac{\not{p} + m}{\sqrt{p^0 + m}} u_s(p^0) = \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \frac{\sigma \mathbf{p}}{p^0 + m} \chi_s \end{pmatrix}$$

$$v_s(p) = \frac{1}{\sqrt{2m}} \frac{\not{p} - m}{\sqrt{p^0 + m}} v_s(p^0) = -\sqrt{p^0 + m} \begin{pmatrix} \frac{\sigma \mathbf{p}}{p^0 + m} \epsilon \chi_s \\ \epsilon \chi_s \end{pmatrix}. \quad (4.102)$$

We close this Chapter with some important identities, such as orthogonality and completeness relations of the general solutions of the Dirac equations. We start with the normalisation and orthogonality of the spinors $u_s(p), v_s$,

$$\bar{u}_r(p) u_s(p) = 2m \delta_{rs}, \quad \bar{v}_r(p) v_s(p) = -2m \delta_{rs}, \quad \bar{u}_r(p) v_s(p) = 0 = \bar{v}_r(p) u_s(p). \quad (4.103)$$

They also satisfy completeness relations with

$$\sum_s u_s(p)_\xi \bar{u}_s(p)_{\bar{\xi}} = (\not{p} + m)_{\xi\bar{\xi}}, \quad \sum_s v_s(p)_\xi \bar{v}_s(p)_{\bar{\xi}} = (\not{p} - m)_{\xi\bar{\xi}}. \quad (4.104)$$

The above relations are specifically important within the computation of S-matrix elements but find applications everywhere in fermionic QFTs. Their proof is deferred to [Appendix B](#).

4.3. Quantisation

With the preparations in the last Chapter [Section 4.2](#) we proceed with the quantisation of the fermionic field, where we follow, as much as possible, the steps of the quantisation of the scalar field, as done in [Section 2.3](#).

4.3.1. Canonical anti-commutation relations

In analogy to (2.145), the fermionic field operator is given by

General solution to the Dirac equation

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \sum_s \left[e^{-ipx} a_s(\mathbf{p}) u_s(\mathbf{p}) + e^{+ipx} b_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) \right], \quad \text{with} \quad p^0 = \sqrt{\mathbf{p}^2 + m^2}. \quad (4.105)$$

The creation and annihilation operators a_s, b_s^\dagger have undetermined commutation properties. We proceed with the Hamiltonian operator, whose spectrum should be positive or rather bounded from below. It follows with the Hamiltonian density in (4.75) as

$$\begin{aligned} H &= \int d^3 x \mathcal{H} = \int d^3 x \psi^\dagger(\mathbf{x}) \gamma^0 (i \boldsymbol{\gamma} \boldsymbol{\partial} + m) \psi(\mathbf{x}) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{2p^0}{2p^0} p^0 \sum_s \left[a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) - b_s(\mathbf{p}) b_s^\dagger(\mathbf{p}) \right]. \end{aligned} \quad (4.106)$$

where the $2p^0$ in the numerator originates in the completeness relation (4.104). We emphasise again, that $\partial^i x^i = -3$ and hence the $i \boldsymbol{\gamma} \boldsymbol{\partial} e^{ipx} = \boldsymbol{\gamma} \mathbf{p} e^{ipx}$. Then, the relative sign between the two terms in (4.106) is derived from (4.103) and

$$\begin{aligned} (\boldsymbol{\gamma} \mathbf{p} + m) u(p) &= \left(-(\boldsymbol{p} - m) + \gamma^0 p^0 \right) u(p) = \gamma^0 p^0 u(p) \\ (-\boldsymbol{\gamma} \mathbf{p} + m) v(p) &= -\gamma^0 p^0 v(p), \end{aligned} \quad (4.107)$$

Due to the relative minus sign between the two terms in the Hamiltonian operator we cannot impose canonical commutation relations for the creation and annihilation operators as for the scalar field. There, the relative minus sign was missing. The incompatibility of the canonical commutation relations is easily seen by using commuting operators with

$$b_s b_s^\dagger = b_s^\dagger b_s + \text{c-number}, \quad (4.108)$$

leading to

$$H \simeq \int \frac{d^3 p}{(2\pi)^3} p^0 \sum_s \left[a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) - b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) \right].$$

Evidently, such the Hamiltonian operator is unbounded from below and the theory would be unstable. Note also that canonical commutation relations for the fermionic fields

$$\left[\psi(\mathbf{x}), i \psi^\dagger(\mathbf{y}) \right] = i \delta(\mathbf{x} - \mathbf{y}), \quad (4.109)$$

implies

$$\left[a_s(\mathbf{p}), a_r^\dagger(\mathbf{q}) \right] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) \delta_{sr} = - \left[b_s(\mathbf{p}), b_r^\dagger(\mathbf{q}) \right]. \quad (4.110)$$

The relative minus sign in the last line rescues causality but does not cure the unboundedness due to the minus sign between the two terms in the Hamiltonian.

This leaves us with anti-commutation relations as the only potentially consistent option,

$$b_s b_s^\dagger = -b_s^\dagger b_s + \text{c-number} . \quad (4.111)$$

The fermionic creation and annihilation operators have satisfy canonical anti-commutation relations,

Anti-commutation relations of creation and annihilation operator

$$\begin{aligned} \{a_s(\mathbf{p}), a_r^\dagger(\mathbf{q})\} &= (2\pi)^3 \delta_{sr} \delta(\mathbf{p} - \mathbf{q}) \\ \{b_s(\mathbf{p}), b_r^\dagger(\mathbf{q})\} &= (2\pi)^3 \delta_{sr} \delta(\mathbf{p} - \mathbf{q}), \end{aligned} \quad (4.112)$$

while all the other anti-commutators vanish,

$$\{a_s(\mathbf{p}), a_r(\mathbf{q})\} = \{a_s(\mathbf{p}), b_r(\mathbf{q})\} = \{a_s^\dagger(\mathbf{p}), b_r(\mathbf{q})\} = \{a_s(\mathbf{p}), b_r^\dagger(\mathbf{q})\} = \{b_s(\mathbf{p}), b_r(\mathbf{q})\} = 0. \quad (4.113)$$

This implies in particular $a_s(\mathbf{p}) a_s(\mathbf{p}) = a_s^2 = 0$, and hence the annihilation and creation operators are Grassmann 'numbers'. It follows, that the fermionic fields satisfy canonical equal time anti-commutation relations,

Anti-commutation relations of field operators

$$\{\psi_\xi(\mathbf{x}), \psi_{\xi'}^\dagger(\mathbf{y})\} = \delta_{\xi\xi'} \delta(\mathbf{x} - \mathbf{y}), \quad \{\psi_\xi(\mathbf{x}), \psi_{\xi'}(\mathbf{y})\} = 0 = \{\psi_\xi^\dagger(\mathbf{x}), \psi_{\xi'}^\dagger(\mathbf{y})\}, \quad (4.114)$$

with $\psi(\mathbf{x}) = \psi(x^0 = 0, \mathbf{x})$. The relations (4.114) hold true for any $\psi(x), \psi^\dagger(y)$ with $x^0 = y^0$ as the temporal phase in (4.105) drops out. They can also be rewritten in terms of ψ and the Dirac conjugate $\bar{\psi}$ that occur in the action, i.e.

$$\{\psi_\xi, \bar{\psi}_{\bar{\xi}}\} = \gamma_{\xi\bar{\xi}}^0 \delta(\mathbf{x} - \mathbf{y}), \quad (4.115)$$

where we indicated the Dirac index of $\bar{\psi}$ with $\bar{\xi}$. The anti-commutation relations (4.114) also have consequences for the classical fermionic field theory: for the scalar quantum field theory the classical field theory is obtained by setting the canonical commutators to zero, which is also the classical limit in quantum mechanics. However, setting the right hand side of the anti-commutation relations to zero leads to anti-commuting spinors. Hence, the classical analogue anti-commute, and are represented by Grassmann numbers. The basic properties of the latter are discussed in [Appendix C](#).

4.3.2. Fock space of the Dirac field and conserved charges

Similarly to the construction of the Fock space of the scalar field in [Section 2.3.3](#), we construct the Fock space of the Dirac field. We define a vacuum state $|0\rangle$ with

$$\sqrt{2w_p} a_s(\mathbf{p}) |0\rangle = 0 = \sqrt{2w_p} b_s(\mathbf{p}) |0\rangle, \quad (4.116)$$

for the analogue in the scalar field theory see (2.103). Then, the one-particle momentum states are obtained by applying the creation operators $a^\dagger(\mathbf{p})$ and $b^\dagger(\mathbf{p})$ to the vacuum state,

$$|\mathbf{p}, s\rangle = \sqrt{2w_p} a_s^\dagger(\mathbf{p}) |0\rangle, \quad \sqrt{2w_p} b_s^\dagger(\mathbf{p}) |0\rangle. \quad (4.117)$$

Equation (4.117) comprises the one-particle and one-anti-particle states respectively. These states are normalised similarly to the one-particle states in the scalar theory, including a Kronecker δ (orthogonality) in their spins $s = \pm 1/2$,

$$\langle \mathbf{q}, r | \mathbf{p}, s \rangle = (2\pi)^3 2p^0 \delta_{rs} \delta(\mathbf{p} - \mathbf{q}). \quad (4.118)$$

Iteratively applying creation operators $a^\dagger(\mathbf{p})$ and $b^\dagger(\mathbf{p})$ generates all (N-particle, M-anti-particle) states of the fermionic quantum field theory,

N-particle state

$$|\mathbf{q}_1, r_1 \cdots \mathbf{q}_M, r_M; \mathbf{p}_1, s_1 \cdots \mathbf{p}_N, s_N\rangle = \prod_{j=1}^M \sqrt{2\omega_{\mathbf{q}_j}} b^\dagger(\mathbf{q}_j) \prod_{i=1}^N \sqrt{2\omega_{\mathbf{p}_i}} a^\dagger(\mathbf{p}_i) |0\rangle, \quad (4.119)$$

Note, that these states are *anti-symmetric* under interchanging any of the particles /anti-particles or fields due to the anti-commutation relations of the creation operators, (4.113). For instance, for two-particle states we find,

$$a_s^\dagger(\mathbf{p}) a_r^\dagger(\mathbf{q}) |0\rangle = -a_r^\dagger(\mathbf{q}) a_s^\dagger(\mathbf{p}) |0\rangle, \quad (4.120)$$

which implies in particular

$$a_r^\dagger(\mathbf{p}) a_r^\dagger(\mathbf{p}) |0\rangle = 0, \quad (4.121)$$

the Pauli exclusion principle.

We conclude this Chapter with the discussion of continuous symmetries and the respective conserved charges in the fermionic quantum field theory. Poincaré invariance leads to a conserved 4-momentum, and the corresponding operator is given by

$$P^0 \simeq \int \frac{d^3 p}{(2\pi)^3} p^0 \sum_s [a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p}) b_s(\mathbf{p})] = H, \quad \text{with} \quad p^0 = E > 0$$

$$P^i \simeq \int \frac{d^3 p}{(2\pi)^3} p^i \sum_s [a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p}) b_s(\mathbf{p})], \quad (4.122)$$

where the anti-commutation relation are required to obtain the relative plus sign in (4.122).

We have already discussed the U(1) invariance of the Dirac theory under phase transformations (4.84). For later use we augment the phase transformation with the electric charge: $\psi \rightarrow e^{ie\alpha} \psi$. The respective Noether charge (4.86) is the fermion number or electric charge and the operator of the Noether charge follows from (4.86) as

Noether charge

$$Q = \int d^3 x j^0 = e \int d^3 x \psi^\dagger(x) \psi(x) \simeq e \int \frac{d^3 p}{(2\pi)^3} \sum_s [a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) - b_s^\dagger(\mathbf{p}) b_s(\mathbf{p})], \quad (4.123)$$

where e is the elementary charge, and $(a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}))$, $(b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}))$ correspond to a fermion with charge e and an anti-fermion with charge $-e$, respectively.

We close this Chapter with a remark on the interpretation of the Noether charge (4.123): the charge Q can be positive and negative, depending on the number of fermions and anti-fermions in a given state. This is in seeming contradiction to the definition of the charge density $\bar{\psi} \gamma^0 \psi = \psi^\dagger \psi$, that would be positive for \mathbb{C} -valued functions ψ and ψ^\dagger . However, the fermionic fields are Grassmann-valued, leading to (4.123).

This concludes our discussion of the quantisation of the free Dirac field.

4.3.3. Perturbation theory and Feynman rules

We proceed with setting up perturbation theory for interacting fermionic theories. A simple example is a combination of the two theories discussed so far, the Dirac fermion and the scalar field theory, a *Yukawa* theory. The respective action is simply a combination of the two separate actions, connected by an interaction term,

$$S[\psi, \bar{\psi}, \phi] = \int d^4x \mathcal{L}, \quad \text{with} \quad \mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_I, \quad (4.124)$$

with the scalar Lagrangian (2.7) with or without the interaction term, the Dirac Lagrangian (4.70), and the *Yukawa* interaction term,

$$\mathcal{L}_I = -h \bar{\psi} \phi \psi, \quad (4.125)$$

with the Yukawa coupling h . Then, perturbation theory and the corresponding Feynman rules are obtained with the scalar propagator, the Dirac propagator, and the Yukawa vertex derived from (4.125). A more elaborate variant of this theory is the Higgs sector of the Standard Model, where the fermions are the matter fields (quarks and leptons) and the scalar field is the Higgs field which is an electroweak doublet.

A further example are gauge theories, and the simplest one is quantum electrodynamics. This is discussed in details later and we only quote here, that the interaction term is given by $e \bar{\psi} \gamma_\mu A^\mu \psi$ with the photon gauge field A_μ and the electric coupling e .

In all these cases perturbation theory and the Feynman rules are based on the propagators of all the fields as well as the vertices. While the latter can be obtained readily from the interaction terms, we still have to derive the Feynman propagator of the Dirac field. The respective computation follows the same line of arguments as in the scalar case, but special care has to be taken with the ordering of operators due to the anti-commuting properties of the fermion fields.

Before we consider the time-ordered Feynman propagator, we consider the two-point function without ordering,

$$\begin{aligned} \langle 0 | \psi_\xi(x) \bar{\psi}_{\xi'}(y) | 0 \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \left[\sum_s (u_s)_\xi (\bar{u}_s)_{\xi'} \right] e^{-ip(x-y)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} (\not{p} + m)_{\xi\xi'} e^{-ip(x-y)} \\ &= (i\not{\partial}_x + m)_{\xi\xi'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ip(x-y)} \end{aligned} \quad (4.126)$$

where we have used (4.104) in the first line. Similarly we have

$$\langle 0 | \bar{\psi}_{\xi'}(y) \psi_\xi(x) | 0 \rangle = - (i\not{\partial}_x + m)_{\xi\xi'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ip(y-x)}. \quad (4.127)$$

We remark that both integrals in the final expressions correspond to the scalar propagator in (3.95) without the θ -function from time ordering. Also note the relative global minus sign in (4.127) in comparison to (4.126), that originates in the anti-commuting properties of the fermionic fields. This has to be taken into account in the time ordering and we define

$$T \psi(x) \bar{\psi}(y) = \theta(x^0 - y^0) \psi(x) \bar{\psi}(y) - \theta(y^0 - x^0) \bar{\psi}(y) \psi(x) = -T \bar{\psi}(y) \psi(x). \quad (4.128)$$

In (4.128) it is implicitly understood that the Dirac indices of the fermionic fields ψ and $\bar{\psi}$ are open and not contracted. Hence, (4.128) is a matrix in Dirac space. With the fermionic time ordering (4.128) we find in analogy to Eq. (3.96) the Feynman propagator,

Feynman-propagator

$$S_F(x-y) = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \quad (4.129)$$

In (4.129) we have used the short hand notation introduced in (4.128), and the propagator S_F is a matrix in

Dirac space with the components $S_{F,\xi\bar{\xi}}(x-y) = \langle 0|T \psi_\xi(x) \bar{\psi}_{\bar{\xi}}(y)|0\rangle$, and hence

$$S_{F,\xi\bar{\xi}}(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)_{\xi\bar{\xi}}}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}, \quad (4.130)$$

Now we are in the position to formulate the Feynman rules for fermions. We can directly transfer the results for the scalar theory [Section 2.3](#) – [Section 3.5](#) to the fermionic theory, the only novel ingredient or difference is the anti-symmetry of fermions. That showed already in the time ordering of the fermions [\(4.128\)](#) with the structure

$$T \psi \bar{\psi} = -T \bar{\psi} \psi. \quad (4.131)$$

Accordingly, contractions of fields in the fermionic theory are the reduction of a product of n field operators to a product of $n - 2$ field operators, multiplied with the propagator of the contracted fields. For the respective ordering of fields we have to take into account that the fields anti-commute. First we note

$$\overbrace{\psi(x) \bar{\psi}(y)} = \langle 0|T \psi(x) \bar{\psi}(y)|0\rangle = S_F(x-y) = -\overbrace{\bar{\psi}(y) \psi(x)}, \quad (4.132)$$

without any contraction of Dirac indices. For a general Wick contraction of two fields, between which we have a product of n fields ψ and m fields $\bar{\psi}$ we find

$$\dots \overbrace{\psi \psi^n \bar{\psi}^m \bar{\psi}} \dots = (-1)^{n+m} \dots \overbrace{\bar{\psi} \bar{\psi} \psi^n \psi^m} \dots. \quad (4.133)$$

Evidently, this anti-commuting rules also hold true for normal ordering,

$$\begin{aligned} : a a^\dagger : &= - : a^\dagger a : = -a^\dagger a \\ : \psi_1 \dots \psi_n \psi_{n+1} \dots : &= - : \psi_1 \dots \psi_{n+1} \psi_n \dots : \\ : \psi_1 \dots \psi_n \bar{\psi}_{n+1} \dots : &= - : \psi_1 \dots \bar{\psi}_{n+1} \psi_n \dots : \end{aligned} \quad (4.134)$$

Collecting all the anti-commuting rules discussed above, we obtain the Wick theorem for fermions,

Wick's theorem

$$T \psi(x_1) \dots \psi(x_n) \bar{\psi}(x_{n+1}) \dots \bar{\psi}(x_{n+m}) = : \psi(x_1) \dots \psi(x_n) \bar{\psi}(x_{n+1}) \dots \bar{\psi}(x_{n+m}) + \text{all contractions} :, \quad (4.135)$$

similarly to the Wick theorem [\(3.101\)](#) for scalars. Note that all Dirac indices in [\(4.135\)](#) are open.

With these preparations we can derive the Feynman rules for interacting fermionic theories. Here we concentrate on the Yukawa theory defined in [\(4.124\)](#) with the interaction term, [\(4.125\)](#). The Feynman rules require the propagators and vertex, and we readily the scalar and fermionic propagators and the Yukawa vertex,

Propagators:

$$\begin{aligned} \phi : \quad \overbrace{\phi \phi} &= \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad = \frac{i}{p^2 - m_\phi^2 + i\epsilon} \\ \psi : \quad \overbrace{\psi \bar{\psi}} &= \quad \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad = \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \end{aligned}$$

Vertex:

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} = -i h \quad (4.136)$$

The second ingredient in the Feynman rules are the contractions of the external legs with the external states. For scalars the contraction of the external legs is taken into account with

$$\phi |p\rangle, \langle p| \phi \longrightarrow 1. \quad (4.137)$$

For fermions we derive

$$\begin{aligned} \psi(x) \Big|_{\text{annihilation}} |p, s\rangle &= \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{2p^0}{2q^0}} \sum_r \left[e^{-iqx} u_r(\mathbf{q}) a_r(\mathbf{q}) a_s^\dagger(\mathbf{p}) |0\rangle \right] \\ &= \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{2p^0}{2q^0}} \sum_r \left[e^{-iqx} u_r(\mathbf{q}) \{a_r(\mathbf{q}), a_s^\dagger(\mathbf{p})\} |0\rangle \right] \\ &= e^{-ipx} u_s(p). \end{aligned} \quad (4.138)$$

We drop the phase in (4.138) and are led to

$$\psi |p, s\rangle := u_s(p) = \begin{array}{c} \diagup \\ \diagdown \\ \text{---} \\ p \end{array}$$

and

$$\langle p, s| \bar{\psi} := \bar{u}_s(p) = \begin{array}{c} \text{---} \\ \diagup \\ \diagdown \\ p \end{array}$$

Anti-fermions:

$$\begin{aligned} \begin{array}{c} \diagup \\ \text{---} \\ \diagdown \\ k \end{array} &= \bar{\psi} |k, s\rangle := \bar{v}^s(k) \\ &\parallel \\ &= \langle 0| b_s^\dagger(k) \end{aligned} \\ \begin{array}{c} \text{---} \\ \diagup \\ \diagdown \\ k \end{array} &= \langle k, s| \psi := v^s(k) \end{aligned} \quad (4.139)$$

Finally, we have to augment the Feynman rules for the loops with the anti-commutation properties. This is exemplified with the vacuum polarisation diagram depicted in (4.140), that is a quantum correction to the propagation of the scalar field.

$$\begin{array}{c} k+p \\ \text{---} \xrightarrow{p} \text{---} \xrightarrow{p} \text{---} \\ \text{---} \xrightarrow{p} \text{---} \xrightarrow{p} \text{---} \end{array} \quad (4.140)$$

This diagram comes from considering the scalar two-point function in first order perturbation theory, which requires the insertion of two Yukawa interaction terms. The two Dirac propagators depicted above come from the contraction of the fermions in the two interaction terms involved.

$$\langle q| \overbrace{(\phi \bar{\psi} \psi)} (\phi \bar{\psi} \psi) |p\rangle = \langle q| \overbrace{(\phi \bar{\psi} \psi)} (\phi \bar{\psi} \psi) |p\rangle = - \langle q| \overbrace{(\psi \bar{\psi} \psi \bar{\psi})} \phi |p\rangle. \quad (4.141)$$

Consequently, closed fermionic loops lead to minus signs, and the final expression for the vacuum polarisation is given by

$$-(-i h)^2 \int d^4x \int d^4y \langle q| \phi(x) \overbrace{\psi_{x\xi} \bar{\psi}_{y\eta}} \overbrace{\psi_{y\eta} \bar{\psi}_{x\xi}} \phi(y) |p\rangle. \quad (4.142)$$

Evidently, (4.142) contains a Dirac trace,

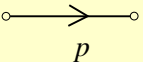
$$\overline{\psi_{x\xi}} \overline{\psi_{y\eta}} \overline{\psi_{y\eta}} \overline{\psi_{x\xi}} = S_{F\xi\eta}(x-y) S_{F\eta\xi}(y-x) = \text{tr}_{\text{Dirac}} \left[S_F(x-y) S_F(y-x) \right]. \quad (4.143)$$

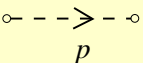
In summary, the loop in (4.140) is represented by the loop integral

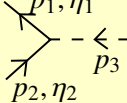
$$\begin{aligned} & -(-i h)^2 \int d^4 x \int d^4 y \langle q | \phi_x \overline{\psi_{x\xi}} \overline{\psi_{y\eta}} \overline{\psi_{y\eta}} \overline{\psi_{x\xi}} \phi_y | p \rangle \\ & = -h^2 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left(\frac{\not{k} + m_\psi}{k^2 - m_\psi^2 + i\epsilon} \frac{\not{k} + \not{p} + m_\psi}{(k+p)^2 - m_\psi^2 + i\epsilon} \right), \end{aligned} \quad (4.144)$$

which is analogous to the same expression for a scalar loop but with an overall minus sign that originates in the Grassmann nature of fermions. In combination this leads us to the Feynman rules of the Yukawa theory in momentum space,

Feynman rules for Yukawa theory

i)  = $\frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon}$

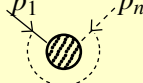
 = $\frac{i}{p^2 - m_\phi^2 + i\epsilon}$

ii)  = $-i h \delta_{\eta_1 \eta_2}$ and $p_2 = -(p_1 + p_3)$ (momentum conservation)

iii) $\int \frac{d^4 p}{(2\pi)^4}$ for each loop

(-)

for each fermion loop

iv) $(2\pi)^4 \delta^4(\sum_i p_i)$ for . (4.145)

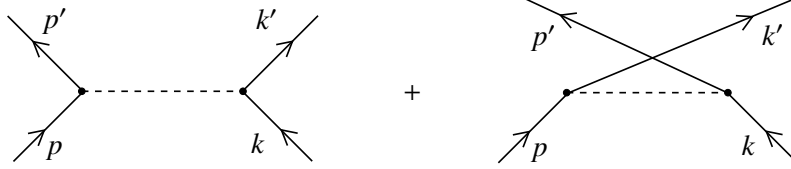
In comparison to (3.127), there is *no symmetry factor* in (4.145): the interaction Lagrangian \mathcal{L}_I is built-up from 3 different fields. Also, the momentum flow of the fermion line is important. Finally, we note that along fermion lines Dirac indices are contracted, e.g.

$$\begin{aligned} & \left(\overline{\psi} \phi \left[\overline{\psi} \overline{\psi} \right] \phi \left[\psi \overline{\psi} \right] \phi \psi \right)_{\xi\xi'} \simeq (-i h)^3 \left(\frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \right)_{\xi\eta} \left(\frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \right)_{\eta\xi'} \\ & \quad \parallel \\ & \overline{\psi} \phi \left[(\psi \overline{\psi}) \phi (\psi \overline{\psi}) \right]_{\xi\xi'} \phi \psi \end{aligned}$$

We close this Chapter on fermionic QFTs with two illustrative examples for the application of Feynman rules in fermionic theories:

A first example concerns the (tree-level) scattering of fermions via a boson (scalar) exchange.

Example 4-15: scattering process.



$$\Rightarrow i M = (-i h)^2 \left[\bar{u}(\mathbf{p}') u(\mathbf{p}) \frac{1}{(p - p')^2 - m_\phi^2} \bar{u}(\mathbf{k}') u(\mathbf{k}) - \bar{u}(\mathbf{p}') u(\mathbf{k}) \frac{1}{(p - k)^2 - m_\phi^2} \bar{u}(\mathbf{k}') u(\mathbf{p}) \right]. \quad (4.146)$$

The second example is a sneak preview on quantum electrodynamics, to be discussed in the next two chapters. Here we only derive the electron-photon vertex.

Example 4-16: QED: Electron-photon vertex. The QED action is given by the Dirac action and an interaction term, which couples the vector current to the vector (gauge) field A_μ , the photon field. The full action is given by

$$S_{\text{QED}}[\psi, \bar{\psi}, A] = \int d^4x \mathcal{L}_{\text{QED}}, \quad \text{with} \quad \mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{photon}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_I. \quad (4.147)$$

The photon part of the action with the Lagrangian $\mathcal{L}_{\text{photon}}$ is derived in the next Chapter. The Dirac action and the interaction term can be combined in a closed form,

$$\mathcal{L}_{\text{Dirac}} + \mathcal{L}_I = \bar{\psi} (i \not{D} - m) \psi, \quad (4.148)$$

with the covariant derivative

$$D_\mu = \partial_\mu - i e A_\mu, \quad (4.149)$$

known from classical electrodynamics. In this form the gauge invariance of the action is apparent. In any case, the interaction Lagrangian is given by

$$\mathcal{L}_I = e \bar{\psi} A_\mu \gamma^\mu \psi, \quad (4.150)$$

and we can read off the electron-photon vertex with

$$= i e \quad (4.151)$$

For explicit computation of diagrams with inner photon lines we need the photon propagator to be derived in the next Chapter. However, the electron-photon vertex can be already used for the computation of diagrams with fermion lines only such as the *vacuum polarisation* of the photon. This computation is analogous to that of the vacuum polarisation of the scalar field, (4.140), substituting the Yukawa vertex by (4.151).

5. Gauge Fields

Gauge theories play an important part in particle physics and condensed matter physics. In the Standard Model all interactions are transmitted by means of gauge bosons, the photon as a carrier of the electromagnetic force with the gauge group U(1) (Quantum Electrodynamics (QED)), the intermediate vector bosons Z, W^\pm for the weak interactions with the gauge group SU(2) with three generators, and the gluons $A_\mu = A_\mu^a t^a$ (or sometimes G_μ) with the gauge group SU(3) with eight generators (the Gell-Mann matrices) of the strong force (Quantum Chromodynamics (QCD)). Moreover, also (quantum) gravity is a gauge theory, the gauge group being the diffeomorphism group. In condensed matter systems, QED is pivotal for superconducting systems, and there are also systems with 'emergent' gauge symmetries.

The best-known example is electrodynamics with the vector field A_μ that satisfies the classical Maxwell equations. In the present Chapter we discuss its quantisation, and basically we only have to discuss one intricacy: the redundancy of the gauge field A_μ that obstructs the construction of the field operator with canonical commutation relations. This can be seen already on the classical level as the canonical momentum π^0 is vanishing, the derivative with respect to $\partial^0 A^0$ vanishes. Consequently we expect the standard introduction of the quantum gauge field A_μ with creation and annihilation operators for all four vector components to be modified by constraints. In QED these constraints are linear, and can be worked in within the *Gupta-Bleuler* formalism, essentially dividing the Fock space with the gauge redundancies into the physical Hilbert space and the unphysical complement. In non-Abelian gauge theories the constraints are non-linear and this separation is more complicated, typically done with the *Faddeev Popov quantisation* discussed in the second part of the lecture course.

5.1. Gauge Symmetry

We consider two different U(1) gauge theories, the first is Quantum ElectroDynamics (QED) already introduced in the last Chapter in (6.1a) at the example of the electron-photon vertex. The free fermionic action of a theory with electrons and positrons, e^-, e^+ is given by

$$S_D[\psi, \bar{\psi}] = \int d^4x \mathcal{L}_D, \quad \text{with} \quad \mathcal{L}_D = \bar{\psi}(x) (i \not{\partial} - m) \psi(x), \quad (5.1)$$

The second theory we consider is scalar electrodynamics based on a complex scalar theory with the action

$$S_\phi[\phi] = \int d^4x \mathcal{L}_\phi, \quad \text{with} \quad \mathcal{L}_\phi = \partial_\mu \phi \partial_\mu \phi^* - V(\phi \phi^*), \quad (5.2)$$

where we have absorbed a potential mass term in the potential $V = m^2 \phi^* \phi + \lambda/2 (\phi^* \phi)^2$, see also (2.26) with (2.29) in Section 2.1. This theory serves as a toy model for the Higgs sector of the Standard Model and is also called the Abelian Higgs model used for the description of superconductivity.

We have already discussed before about (2.70) in Section 2.2, and in Section 4.2.2 below (4.84), that the actions in (5.1) and (5.2) are invariant under global U(1)-rotations. The respective Noether charge operators are (2.152) and (4.123), and constitute the electric charges in the theories. We sum up the respective transformations,

$$\psi \rightarrow e^{ie\alpha} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-ie\alpha} \quad \phi \rightarrow e^{ie\alpha} \phi, \quad \phi^* \rightarrow \phi^* e^{-ie\alpha}, \quad (5.3)$$

Let us require the invariance of the theories under **local rotations** (*gauge symmetry*). We first discuss the QED example with the local transformations

$$\psi(x) \rightarrow e^{ie\alpha(x)} \psi(x). \quad (5.4)$$

This transformation does not leave the QED action invariant, indeed it can be used to derive the Noether current. We find

$$\mathcal{L}_D \rightarrow \mathcal{L}_D - e \bar{\psi} (\not{\partial} \alpha) \psi = \mathcal{L}_D - e \partial_\mu \alpha j^\mu, \quad \text{with} \quad j^\mu = \bar{\psi} \gamma^\mu \psi, \quad (5.5)$$

with the gauge invariant current j^μ . Accordingly, we may add a term $A_\mu j^\mu$ to \mathcal{L}_D with the vector field A_μ and fix the transformation property of the vector field under local U(1) transformations such, that the total action is invariant. First we note that A_μ has to transform as a vector under Lorentz transformations in order to keep Lorentz invariance of the action,

$$A_\mu \xrightarrow{\Lambda} \Lambda^\mu_\nu A_\nu \quad \longleftarrow \quad A_\mu j^\mu \xrightarrow{\Lambda} A_\mu j^\mu. \quad (5.6)$$

The requirement of local U(1) invariance of the action leads us to

$$\mathcal{L}_D + e A_\mu j^\mu \rightarrow \mathcal{L}_D - \partial_\mu \alpha j^\mu + e A'_\mu j^\mu \stackrel{!}{=} \mathcal{L}_D + e A_\mu j^\mu \quad \longrightarrow \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x). \quad (5.7)$$

The Lagrangian $\mathcal{L}_D + A_\mu j^\mu$ is nothing but that of the Dirac action (4.148) with the covariant derivative (4.149), and we define

$$S_D[\psi, \bar{\psi}, A] = \int d^4x \mathcal{L}_D(\psi, \bar{\psi}, A), \quad \text{with} \quad \mathcal{L}_D(\psi, \bar{\psi}, A) = \bar{\psi} (i \not{D} - m) \psi, \quad (5.8)$$

with the covariant derivative

$$D_\mu = \partial_\mu - ie A_\mu. \quad (5.9)$$

A_μ is also called a connection (German: "Zusammenhang"). Its transformation under *gauge* transformations,

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha, \quad (5.10)$$

induces covariant transformation properties for D_μ ,

$$D_\mu \rightarrow \partial_\mu - ie A_\mu - ie \partial_\mu \alpha = e^{ie\alpha(x)} D_\mu e^{-ie\alpha(x)}, \quad (5.11)$$

the covariant derivative transforms as a tensor under gauge transformations. Hence $D_\mu \psi$ transforms as the field itself,

$$D_\mu \psi \rightarrow e^{ie\alpha(x)} D_\mu e^{-ie\alpha(x)} e^{ie\alpha(x)} \psi = e^{ie\alpha(x)} D_\mu \psi. \quad (5.12)$$

Accordingly, the covariant derivative transports the gauge transformation to the left. Geometrically, it generates a parallel transport of the phase.

A similar analysis also leads to the gauge invariant Abelian Higgs model. With the transformation property (5.11) also the covariant derivative of the scalar field transforms as the field itself,

$$D_\mu \phi \rightarrow e^{ie\alpha(x)} D_\mu \phi. \quad (5.13)$$

With (5.13), the covariant version of (5.2),

$$S_\phi[\phi, A] = \int d^4x \mathcal{L}_\phi(\phi, A) \quad \text{with} \quad \mathcal{L}_\phi(\phi, A) = D_\mu \phi (D_\mu \phi)^* - m^2 \phi \phi^* - V(\phi \phi^*), \quad (5.14)$$

is invariant under the combined gauge transformations of scalar and gauge field,

$$\phi(x) \rightarrow e^{ie\alpha(x)} \phi(x) \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha. \quad (5.15)$$

This finalises our analysis of gauge invariant extensions of fermionic and scalar theories. In short, we promote the partial derivatives of the field to covariant ones and the latter transform as fields.

It is left to derive the pure gauge field action and to quantise the gauge field A_μ . We start this endeavour by constructing general gauge-invariant scalar quantities from A_μ . This is easily done from general functions of D_μ , using its covariant transformation properties. The simplest of these functions is the commutator of the covariant derivative,

$$F_{\mu\nu} = \frac{i}{e} [D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu). \quad (5.16)$$

This is nothing but the fieldstrength tensor in electromagnetism. In terms of differential geometry it is the curvature tensor, as it 'measures' the difference of infinitesimal parallel transports first in the ν -direction and then in the μ -direction and the other way around. The fieldstrength $F_{\mu\nu}$ is a covariant rank two Lorentz tensor, which follows from the transformation law of the derivative as a vector and that of the gauge field A_μ as a vector, (5.6). Combining these transformations we obtain explicitly,

$$F_{\mu\nu} \rightarrow \Lambda_\mu^\rho \Lambda_\nu^\sigma F_{\rho\sigma}. \quad (5.17)$$

Moreover, as the covariant derivatives transform as tensors, so does $F_{\mu\nu}$ as a product of two covariant derivatives. However, since it is a function, we can commute the phases through and hence $F_{\mu\nu}$ is gauge invariant,

$$F_{\mu\nu} \rightarrow \frac{i}{e} [e^{-ie\alpha} D_\mu e^{ie\alpha}, e^{-ie\alpha} D_\nu e^{ie\alpha}] = \frac{i}{e} e^{-ie\alpha} [D_\mu, D_\nu] e^{ie\alpha} = \frac{i}{e} e^{-ie\alpha} (-ie F_{\mu\nu}) e^{ie\alpha} = F_{\mu\nu}. \quad (5.18)$$

In summary we find that $F_{\mu\nu}$ is a gauge invariant Lorentz tensor. We can construct a simple action with a kinetic term from it with contracting $F_{\mu\nu}$ with itself, leading to a gauge field (kinetic) action,

$$S_A[A] = \int d^4x \mathcal{L}_A, \quad \text{with} \quad \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (5.19)$$

Equation (5.19) is the (minimal) searched-for action of the gauge field. Note that from the construction principle we may have used even higher orders of the field strength such as $(F_{\mu\nu}^2)^2$ or sandwiching the fieldstrength with powers of the covariant derivative such as $F_{\mu\nu} D_\rho^2 F^{\mu\nu}$. The latter kinetic term does not lead to a $1/r$ -potential observed in electrodynamics while the $F_{\mu\nu}^2$ -term does. Instead, $F_{\mu\nu} D_\rho^2 F^{\mu\nu}$ leads to a linearly rising potential. In turn, the higher order term in the field strength, $(F_{\mu\nu}^2)^2$ is simply an interaction term which results in a four-photon scattering (non-linear effects in QED). This scattering is not observed on the classical level but only on the quantum level. Finally, all these terms do not lead to a perturbatively renormalisable quantum field theory. In short, experimental observations together with conceptual ones leave us with a unique choice: **Equation (5.19)** with the Dirac action (5.8) leads us to the classical action of QED,

$$S_{\text{QED}}[\psi, \bar{\psi}, A] = \int d^4x \mathcal{L}_{\text{QED}}, \quad \text{with} \quad \mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{D} - m) \psi. \quad (5.20)$$

with the invariance under the gauge transformation

$$\psi(x) \rightarrow e^{ie\alpha(x)} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi} e^{-ie\alpha(x)}, \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x) \quad (5.21)$$

This concludes our construction of the classical QED action. That of the Abelian Higgs model or scalar electrodynamics follows as a combination of (5.14) and the pure gauge field action,

$$S_\phi[\phi, A] = \int d^4x \mathcal{L}_\phi(\phi, A), \quad \mathcal{L}_\phi(\phi, A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi (D_\mu \phi)^* - m^2 \phi \phi^* - V(\phi \phi^*). \quad (5.22)$$

We close this Chapter with a short discussion of generalisations of the construction and the ensuing actions above in view of the non-Abelian gauge theories in the Standard Model and beyond. To begin with, the construction also goes through for $SU(N)$ gauge theories or more generally non-Abelian gauge theories. To that end we consider fermions ψ^A that also carry the fundamental representation of $SU(N)$ with group index A apart from the four-dimensional spinor representation with indices ξ which we suppressed here. For instance, for $SU(2)$ (weak gauge group) the indices A run over $A = 1, 2$ and for $SU(3)$ (strong interactions) the indices A run over $A = 1, 2, 3$. The gauge transformation now are elements of $SU(N)$. This leads us to the classical action of a non-Abelian gauge theory coupled to fermions in the fundamental representation, and for QCD we find

$$S_{\text{QCD}}[\psi, \bar{\psi}, A] = \int d^4x \mathcal{L}_{\text{QCD}}, \quad \text{with} \quad \mathcal{L}_{\text{QCD}} = -\frac{1}{2} \text{tr}_f F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{D} - m) \psi, \quad (5.23)$$

where the trace tr_f is in the fundamental representation. The covariant derivative D in the fundamental representation and the fieldstrength $F_{\mu\nu}$ read,

$$D_\mu = \partial_\mu - igA_\mu, \quad F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu], \quad \text{with} \quad A_\mu = A_\mu^a t^a, \quad (5.24)$$

where g is the $SU(N)$ gauge coupling. The t^a are the generators of $SU(N)$ with $a = 1, \dots, N^2 - 1$, satisfying the $SU(N)$ Lie algebra,

$$[t^a, t^b] = if^{abc} t^c, \quad \text{tr}_f t^a t^b = \frac{1}{2} \delta^{ab}. \quad (5.25)$$

In terms of its components the fieldstrength is given by

$$F_{\mu\nu} = F_{\mu\nu}^a t^a, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c. \quad (5.26)$$

In contradistinction to QED, already the pure gauge field action is interacting. It is precisely this feature that leads to *asymptotic freedom* of QCD: the gauge coupling g weakens with increasing momentum scale and vanishes in the limit of $p^2 \rightarrow \infty$. In turn, it increases for small momenta, and this regime is not covered by perturbation theory. This has to be compared with QED, where electric coupling increases with increasing p^2 due to its interaction with fermionic matter, and finally hits a singularity (in perturbation theory), the *Landau pole*.

We have chosen the subscript as the above action is that of a *confining $SU(N)$ theory*. Only the coupling to the Higgs sector and the related spontaneous symmetry breaking lead to the weakly coupled weak interactions. The QCD action (5.23) is invariant under the combined gauge transformations of the fermions (quarks) and the gauge fields (gluons),

$$\psi \rightarrow U \psi, \quad A_\mu \rightarrow U A_\mu U^\dagger + \frac{i}{g} U (\partial_\mu U^\dagger), \quad \rightarrow \quad D_\mu \rightarrow U D_\mu U^\dagger, \quad F_{\mu\nu} \rightarrow U F_{\mu\nu} U^\dagger. \quad (5.27)$$

While the invariance of the Dirac term follows directly, that of the pure glue dynamics follows from $F_{\mu\nu}^2 \rightarrow U F_{\mu\nu}^2 U^\dagger$ and the cyclicity of the trace, $\text{tr}_f U F_{\mu\nu}^2 U^\dagger = \text{tr}_f U^\dagger U F_{\mu\nu}^2 = \text{tr}_f F_{\mu\nu}^2$. We also note that (5.27) entails that the non-Abelian fieldstrength is not gauge invariant but transforms as a tensor under gauge transformations. Only the gauge field action itself is invariant. Consequently, the chromo-electric and chromo-magnetic fields cannot be measured.

5.2. Quantisation

We expect that the gauge redundancy leaves its trace in the quantisation procedure. If the gauge field simply would be a vector field, we would treat each component as an independent field and quantise it accordingly. In a gauge theory, however, this cannot work out. We know that the theory only exhibits two physical

(transverse) polarisations and hence the gauge field comprises two redundant degrees of freedom. A first hint at the failure of the naïve canonical quantisation is given by the fact, that the perturbative treatment of a quantum field theory requires the propagator of the gauge field which is also pivotal for many applications in classical electrodynamics. Since the kinetic operator is transverse, we have to fix a gauge for inverting it, as we have done in classical electrodynamics. Standard gauges are covariant or Lorenz gauges $\partial_\mu A^\mu = 0$, the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, or axial gauges $n_\mu A^\mu = 0$.

Specifically the latter class of gauges allows for removing one of the gauge field components altogether. For example, if we choose $n_\mu = \delta_{\mu 0}$, the axial gauge fixing is given by $A^0 = 0$, the temporal or Weyl gauge. Evidently, this excludes canonical commutation relations for A_0 . Indeed, its canonical field momentum is not even present in the action.

We proceed with the quantisation of gauge fields, keeping the above intricacies in mind. The pure gauge field Lagrangian in (5.19) is given by

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (5.28)$$

with the Maxwell equations as equations of motion,

$$-\partial_\mu \frac{\partial \mathcal{L}_A}{\partial_\mu A_\nu} = \partial_\mu F^{\mu\nu} = (\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma) A_\sigma = 0. \quad (5.29)$$

In the presence of source terms, e.g. arising from the matter part in the QED action in (5.20), we are led to

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \text{QED:} \quad J^\nu = -e \bar{\psi} \gamma^\nu \psi. \quad (5.30)$$

Equation (5.30) are the inhomogeneous Maxwell equations, the homogeneous ones are encoded in the definition of the fieldstrength with the gauge field, $\partial_\mu \tilde{F}^{\mu\nu} = 0$ with the dual fieldstrength (Hodge dual)

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \quad (5.31)$$

The equation of motion (5.29) of the gauge field reflects the gauge redundancy of the gauge field A_μ , as it should: a solution of the EoMs (5.29) and (5.30) stays one under a gauge transformation with (5.10) or (5.21). For the pure gauge theory we find

$$\partial_\mu F^{\mu\nu} + e (\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma) \partial_\sigma \alpha = \partial_\mu F^{\mu\nu}, \quad (5.32)$$

where we have used the transversality of the kinetic operator of the gauge field,

$$(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma) \partial_\sigma = 0. \quad (5.33)$$

Most of our explicit computations are performed in momentum space, where the transversality relation (5.33) takes the simple form

$$(p^2 \eta^{\nu\sigma} - p^\nu p^\sigma) p_\sigma = 0. \quad (5.34)$$

where the term in the brackets is the transverse part, which we will discuss later in the section. Equation (5.29) already entails, that A^μ cannot have canonical commutation relations, the longitudinal part of the gauge field is not restricted by the equations of motion. Moreover, we have already indicated at the beginning of this Section, that the redundancy also leads to the absence of the canonical field momentum of A^0 in the action. The field momentum Π^μ is defined with

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -\frac{1}{4} \frac{\partial}{\partial(\partial_0 A_\mu)} (F_{\rho\sigma} F_{\gamma\delta} \eta^{\sigma\delta} \eta^{\rho\gamma}) = -\frac{1}{2} F_{\rho\sigma} \eta^{\sigma\delta} \eta^{\rho\gamma} \frac{\partial F_{\gamma\delta}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}, \quad (5.35)$$

and hence are simply components of the fieldstrength tensor. This leads to the vanishing of the canonical field momentum of A_0 ,

$$\Pi^0 = 0. \quad (5.36)$$

Equation (5.36) follows directly from the antisymmetry of the fieldstrength tensor, which also is the origin of the transversality of the kinetic operator, both reflecting the gauge redundancy. We now remove the gauge redundancy by fixing the gauge, which allows us to define a propagator or Green function of the gauge field. This is pivotal for the perturbative or more generally, diagrammatic, formulation of gauge theories. While gauge fixing is required, we have the free choice as physics does not depend on it. In given applications the respective choices are in most cases taken guided by computational convenience. For example, out of the three gauges mentioned in the introductory discussion,

$$\text{Covariant gauge: } \partial_\mu A^\mu = 0, \quad \text{Coloumb gauge: } \partial^i A^i = 0, \quad \text{Axial gauge: } n_\mu A^\mu = 0, \quad (5.37)$$

only the covariant gauge is Lorentz-invariant, hence the name. Due to its Lorentz-invariance, the covariant or Lorenz (by Ludwig Lorenz) gauge

$$\partial_\mu A^\mu = 0. \quad (5.38)$$

leads to significant simplifications in loop computations as it reduces the number of possible tensor structures that have to be considered. If we restrict ourself to gauge fields, that satisfy the covariant gauge (5.38), we can add the gauge condition as a simple quadratic term to the action,

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \stackrel{(5.38)}{=} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (5.39)$$

where the last identity follows for gauge fields with (5.38). The action with the gauge fixing term reads

$$S[A] = \frac{1}{2} \int d^4x A_\mu \left[\partial_\rho \partial^\rho \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu, \quad (5.40)$$

where the kinetic operator in the square brackets in (5.40) is invertible in contradistinction to that of the original action without gauge fixing. The respective propagator or Greens function in momentum space is discusses later in the context of the Feynman rules, see (5.80). However, it follows straightforwardly from (5.40) and illustrates the impact of the gauge fixing, so we already provide the Feynman propagator here,

$$D_{\mu\nu}(k) = -\frac{i}{k^2 + i\epsilon} \left[\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right], \quad (5.41)$$

the propagator is nothing but that of a scalar field, multiplied with a combination of the transverse and longitudinal projection operators. We proceed by discussing the equation of motion that follow from the gauge fixed action (5.40). It is given by

$$\partial_\mu F^{\mu\nu} = -\frac{1}{\xi} \partial^\nu (\partial_\mu A^\mu). \quad (5.42)$$

The right hand side includes a longitudinal projection and hence only contains the longitudinal part of the gauge field that vanishes on the gauge fixing condition. In turn, the left hand side includes a transverse projection and hence only contains the transversal part of the gauge field. Accordingly, both sides have to vanish separately

and a solution of the gauge fixed EoM satisfies the gauge condition (5.38). To see this more explicitly, we split the gauge field in its transverse and longitudinal parts,

$$A_\mu = (A_\perp)_\mu + (A_L)_\mu, \quad \text{with} \quad \partial_\mu (A_\perp)_\mu = 0, \quad \text{and} \quad (\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma)(A_L)_\nu = 0. \quad (5.43)$$

The gauge fixed EoM (5.42) can be written in terms of the transverse and longitudinal gauge fields,

$$(\partial_\mu \partial^\mu \eta^{\nu\sigma} - \partial^\nu \partial^\sigma)(A_\perp)_\nu = \frac{1}{\xi} \partial^\sigma \partial_\nu A_L^\nu = 0, \quad (5.44)$$

where that last identity comes from the fact that the two sides of the EoM in (5.42) are transverse (left hand side) and longitudinal (right hand side) respectively. We conclude this discussion of the gauge fixed action (5.40) and the gauge-fixed EoM with an evaluation of specific gauges. First of all, $\xi \rightarrow 0$ (Landau gauge) and $\xi \rightarrow \infty$ (unitary gauge) stick out. In the former case, the propagator (5.41) is transverse and the transversality of the theory is explicit on the level of the propagator. Obviously, in this case one only has three propagating degrees of freedom, and one of the two redundant degrees of freedom is manifestly absent. It also facilitates the inclusion of gauge symmetry in non-perturbative diagrammatic (functional) approaches and is the standard choice for diagrammatic non-perturbative (numerical) computations. In turn, for $\xi \rightarrow \infty$ the gauge fixing is removed and the propagator (5.41) exhibits a singularity. Still, one can compute in the limit of diverging ξ and observables are independent of the gauge fixing in the first place.

Finally, the propagator (5.41) is specifically simple for $\xi = 1$, where it simply is the scalar (massless) propagator, augmented with $\beta_{\mu\nu}$. This facilitates perturbative loop computations significantly and is hence the standard choice in perturbation theory. We obtain from (5.42) with $\xi = 1$,

EOM in the Feynman gauge

$$\partial_\rho \partial^\rho A^\nu = 0, \quad (5.45)$$

reminiscent of the Klein-Gordon equation (2.12), as is the propagator. Equation (5.45) suggests the definition of the quantised gauge field,

Field operator

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \left[e^{-ikx} a_\mu(\mathbf{k}) + e^{ikx} a_\mu^\dagger(\mathbf{k}) \right], \quad (5.46a)$$

with the commutation relations

Commutation relation

$$\left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}') \right] = -\eta_{\mu\nu} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \quad \text{and} \quad \left[a_\mu(\mathbf{k}), a_\nu(\mathbf{k}') \right] = 0 = \left[a_\mu^\dagger(\mathbf{k}), a_\nu^\dagger(\mathbf{k}') \right], \quad (5.46b)$$

where the $\eta_{\mu\nu}$ in the first relation in (5.46b) follows from Lorentz-symmetry. However, (5.46) is not compatible with the gauge condition (5.38) and hence with the EoM (5.42), as this requires

$$\partial_\mu A^\mu = \int \frac{d^3k}{(2\pi)^3} \frac{-i}{\sqrt{2k^0}} \left(e^{-ikx} k_\mu a^\mu(\mathbf{k}) + e^{ikx} k_\mu (a^\dagger)^\mu(\mathbf{k}) \right) \stackrel{!}{=} 0. \quad (5.47)$$

This entails, that $k_\mu a^\mu(\mathbf{k}) = 0$, which is in clear contradiction to the canonical quantisation relations (5.46b),

$$k^\mu [a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')] = -k^\nu (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \neq 0. \quad (5.48)$$

Indeed, similarly to the above one can show, that it is *not* possible to quantise the gauge field A_μ with canonical commutation relations and hence (5.46) for any (linear) gauge condition: the gauge has to be implemented on the states, and hence the analogue of (5.48) fails due to the lack of redundancy in the gauge field or rather the creation and annihilation operators a_μ, a_μ^\dagger .

We will discuss this issue later in this section, but prior to this, we construct a Fock space \mathcal{F} based on creation and annihilation operators satisfying (5.46b). Then, we will define a subspace of \mathcal{F} , where the gauge condition is satisfied. The construction of the Fock space follows that of the scalar field, and we define the vacuum state $|0\rangle$ with

$$\langle 0|0\rangle = 1. \quad (5.49)$$

One-particle states are given by

$$\sqrt{2k^0} a_\mu^\dagger(\mathbf{k}) |0\rangle, \quad (5.50)$$

and their norm follows with (5.46b) as

$$\sqrt{2k^0 2(k')^0} \langle 0| a_\nu(\mathbf{k}') a_\mu^\dagger(\mathbf{k}) |0\rangle = -\eta_{\mu\nu} (2\pi)^3 2k^0 \delta(\mathbf{k} - \mathbf{k}'). \quad (5.51)$$

n -particle states follows accordingly. Equation (5.51) entails that we have positive norm states for $\mu = \nu = i$, and negative norm states for $\mu = \nu = 0$. Consequently, \mathcal{F} is not the physical Hilbert space \mathcal{H} , as it does not allow for probability interpretation. We remark, that $\eta_{\mu\nu} \rightarrow \eta^{\mu\nu}$ does not solve the problem of negative norm states (leave aside the wrong commutators $[A^i, \Pi^i]$). But separating the positive norm subspace of \mathcal{F} will solve all problems of quantisation. This is the **Gupta-Bleuler quantisation** procedure.

Its construction starts from the Fock space above. We are interested in the subspace, where the equation of motion (5.29) is satisfied. This singled out states with

Physical states

$$\langle \text{physical states}' | \partial_\mu F^{\mu\nu} | \text{physical states} \rangle = 0, \quad (5.52)$$

that is, matrix elements of the EoM (5.29) vanish. Inserting the field operator (5.46a), the constraint (5.52) reduces to one on the annihilation and creation operators,

$$k^\mu a_\mu(\mathbf{k}) | \text{physical states} \rangle = 0. \quad (5.53)$$

Evidently, the vacuum belongs to the physical states as it is annihilated by a_μ . The above suggests to rewrite A_μ in (5.46a) in terms of transverse and longitudinal polarisations. In a first step we rewrite the annihilation and creation operators a_μ, a_μ^\dagger in terms of new creation and annihilation operators α_λ and α_λ^\dagger with $\lambda = 0, 1, 2, 3$ and

$$a_\mu = \sum_\lambda \alpha_\lambda(\mathbf{k}) \varepsilon_\mu^\lambda(k), \quad (5.54)$$

where the ε_μ^λ are unitary rotations from a_μ to a_λ and vice versa. This ensures that the novel creation and annihilation operators $\alpha_\lambda, \alpha_\lambda^\dagger$ also satisfy the canonical commutation relations. Then we arrive at

Field operator

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \sum_{\lambda=0}^3 \left[\alpha_\lambda(\mathbf{k}) \varepsilon^\lambda_\mu(k) e^{-ikx} + \alpha_\lambda^\dagger(\mathbf{k}) \varepsilon^{\lambda*}_\mu(k) e^{ikx} \right], \quad (5.55)$$

where the ε^λ_μ have the properties

Polarisation vectors

$$\varepsilon^\lambda_\mu(k) \varepsilon^{\lambda'\mu*}(k) = \eta^{\lambda\lambda'}, \quad \varepsilon^\lambda_\mu(k) \varepsilon_{\lambda\nu}(k) = \eta_{\mu\nu}. \quad (5.56)$$

This also allows us to express the "new" operators α in terms of a linear combination of the "old" operators a ,

$$\alpha_\lambda(\mathbf{k}) = a_\mu(\mathbf{k}) \varepsilon^{\lambda\mu*}(k), \quad (5.57)$$

and a similar relation holds for α^\dagger and a^\dagger . These rotations enables us to disentangle the 'physical' polarisations from the 'unphysical' ones. The former one are transverse and we define without loss of generality

$$\alpha_{1,2} : \quad k \cdot \varepsilon^i = 0, \quad \text{for } i = 1, 2. \quad (5.58)$$

The respective part of the photon field operator drops out of the equation of motion, and hence the respective creation operators $\alpha_{1,2}^\dagger$ generate states in the physical part of the Fock space. We also choose $k \cdot \varepsilon^0 = k^0 = k \cdot \varepsilon^3$, being parallel to the zero and three-direction,

$$\alpha_{0,3} : \quad k \cdot \varepsilon^0 = k \cdot \varepsilon^3, \quad \alpha_\pm = \frac{1}{\sqrt{2}} (\alpha_0 \pm \alpha_3), \quad (5.59)$$

where $\alpha_+ \propto k^\mu a_\mu$ generates the longitudinal part of the gauge field operator while α_- generates the remaining transverse part. Due to their polarisation properties, the ε 's are called *polarisation vectors*. We emphasise that the above choice is not unique, which is already clear from the selection of specific spatial and the temporal direction. Moreover, it depends on the chosen gauge. This is the reason for the quotation marks above. The constraint (5.53) can now be rewritten in terms of α_+ defined in (5.59) from $\alpha_{0,3}$. We find

$$\alpha_+ |\text{physical states}\rangle = 0, \quad (5.60)$$

We have already stated that the α 's have the same commutation relations as the a 's due to the unitary nature of the transformation, see (5.56) and (5.57). In our novel basis the canonical commutation relations (5.46b) read,

Commutation relations

$$\begin{aligned} [\alpha_i(\mathbf{k}), \alpha_j^\dagger(\mathbf{k}')] &= \delta_{ij} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), & [\alpha_+(\mathbf{k}), \alpha_-^\dagger(\mathbf{k}')] &= -(2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \\ [\alpha_\pm(\mathbf{k}), \alpha_\pm^{(\dagger)}(\mathbf{k}')] &= 0 = [\alpha_\pm(\mathbf{k}), \alpha_i^{(\dagger)}(\mathbf{k}')] . \end{aligned} \quad (5.61)$$

Note that the commutation relation of the two 'physical' polarisations $\alpha_{1,2}$ has the canonical form, while α_\pm and α_\pm^\dagger commute with each other. Note also that the relative minus sign in $\eta_{\mu\nu}$ that led to the negative norm states in the Fock space, is now carried by the latter commutation relation.

We close this investigation with the remark, that a specifically simple representation is found in the Lorentz frame with $(k^\mu) = (k^0, 0, 0, k^0)$ with

$$(\varepsilon^\lambda)_\mu = \delta^\lambda_\mu. \quad (5.62)$$

We still have to derive the physical Hilbert space \mathcal{H} . While we have defined the physical subspace $\mathcal{F}_{\text{phys}}$ of the Fock space \mathcal{F} that satisfy (5.60). This subspace still contains three polarisations and hence cannot be the physical Hilbert space. Before we can discuss this point in more detail, we first generate all states in $\mathcal{F}_{\text{phys}}$ from the vacuum. To that end we remark that with the creation operator $\alpha_{1,2}^\dagger$ we can create further states in $\mathcal{F}_{\text{phys}}$ from a given one,

$$|\psi\rangle \in \mathcal{F}_{\text{phys}} \longrightarrow \alpha_i^\dagger |\psi\rangle \in \mathcal{F}_{\text{phys}}, \quad \text{for } i = 1, 2. \quad (5.63)$$

This follows readily by showing that this state also satisfies (5.60),

$$\alpha_+ \alpha_i^\dagger |\psi\rangle = \alpha_i^\dagger \alpha_+ |\psi\rangle = 0. \quad (5.64)$$

With the same line of reasoning it follows that states generated from states in $\mathcal{F}_{\text{phys}}$ by the application of α_+^\dagger are in $\mathcal{F}_{\text{phys}}$.

$$|\psi\rangle \in \mathcal{F}_{\text{phys}} \longrightarrow \alpha_+^\dagger |\psi\rangle \in \mathcal{F}_{\text{phys}}, \quad \text{with} \quad \alpha_+ \alpha_+^\dagger |\psi\rangle = \alpha_+^\dagger \alpha_+ |\psi\rangle = 0. \quad (5.65)$$

In summary, all states that are generated from $|\psi\rangle \in \mathcal{F}_{\text{phys}}$ with the application of creation operators, that commute with α_+ , are in the physical subspace $\mathcal{F}_{\text{phys}}$. In turn, states created by applying α_-^\dagger to $|\psi\rangle \in \mathcal{F}_{\text{phys}}$ are not in $\mathcal{F}_{\text{phys}}$, as α_-^\dagger does not commute with α_+ .

$$\alpha_-^\dagger |\psi\rangle \notin \mathcal{F}_{\text{phys}}, \quad \text{with} \quad \alpha_+ \alpha_-^\dagger |\psi\rangle = \alpha_-^\dagger \alpha_+ |\psi\rangle + [\alpha_+, \alpha_-^\dagger] |\psi\rangle = [\alpha_+, \alpha_-^\dagger] |\psi\rangle \sim |\psi\rangle \neq 0. \quad (5.66)$$

This concludes our construction of states in $\mathcal{F}_{\text{phys}}$. It is simply given by the span of states that are created by applying $\alpha_{1,2,+}^\dagger$ to the vacuum.

Physical subspace

$$\mathcal{F}_{\text{phys}} = \text{span} \left[\left(\alpha_+^\dagger \right)^{n_+} \left(\alpha_1^\dagger \right)^{n_1} \left(\alpha_2^\dagger \right)^{n_2} |0\rangle \right]. \quad (5.67)$$

and all states in $\mathcal{F}_{\text{phys}}$ have a semi-positive norm

$$\langle \psi | \psi \rangle \geq 0, \quad (5.68)$$

as we have eliminated the longitudinal polarisation. However, $\mathcal{F}_{\text{phys}}$ also contains zero norm states which are related to the fact that α_+^\dagger generates states in $\mathcal{F}_{\text{phys}}$. Indeed we find that any state, that is generated by the application of α_+^\dagger to a state in the physical subspace, $|\psi\rangle \in \mathcal{F}_{\text{phys}}$, has vanishing norm,

$$\| \alpha_+^\dagger |\psi\rangle \|^2 = \langle \psi | \alpha_+ \alpha_+^\dagger |\psi\rangle = \langle \psi | \alpha_+^\dagger \alpha_+ |\psi\rangle = 0. \quad (5.69)$$

In turn, any state which is only generated by applying $\alpha_{1,2}^\dagger$ to the vacuum, has a non-vanishing norm,

$$\left\| \left(\alpha_1^\dagger \right)^{n_1} \left(\alpha_2^\dagger \right)^{n_2} |0\rangle \right\| > 0, \quad (5.70)$$

due to the canonical commutation relation of $\alpha_{1,2}$. Obviously, the zero norm states comprise the remaining redundancy of our construction: In the physical subspace we cannot distinguish two states $|\psi_1\rangle, |\psi_2\rangle$, if their difference is a zero norm state, $\| |\psi_1\rangle - |\psi_2\rangle \| = 0$. Then, all matrix element of an operator $\mathcal{O}(\alpha_i^{(\dagger)}, \alpha_+^{(\dagger)})$ vanish, $\langle \psi | \mathcal{O}(|\psi_1\rangle - |\psi_2\rangle)$. Hence we can identify these two states as physically equivalent. This leads us to the definition of the physical Hilbert space as the space of equivalence classes of the physical subspace $\mathcal{F}_{\text{phys}}$ defined in (5.67),

a) $\mathcal{L}_I = e \bar{\psi} \not{A} \psi :$

$= i e \gamma^\mu$

b) $\mathcal{L}_I = D_\mu \phi (D^\mu \phi)^* - \partial_\mu \phi \partial^\mu \phi^* :$

$= -i e (p_\mu + p'_\mu),$

$= 2i e^2 \eta^{\mu\nu}.$
(5.78)

In a final step we now generalise the setting from the Feynman gauge with the gauge fixing parameter $\xi = 1$ to general gauge fixing parameters. The whole construction of the physical Hilbert space entails that we can add a longitudinal part to the field A_μ , without changing the physics,

$$A_\mu \rightarrow A_\mu + \alpha \partial_\mu \frac{1}{\partial^\rho \partial^\rho} \partial^\nu A_\nu, \quad (5.79)$$

which goes hand in hand with only changing the prefactor of the gauge fixing term in the gauge-fixed action (5.40) in the Landau gauge by $1/2 \rightarrow 1/2(1 + \alpha)^2$. This is simply a change of the gauge fixing parameter $1/\xi$ and respective general propagator is given by (5.41),

$$= -\frac{i}{k^2 + i\epsilon} \left(\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right). \quad (5.80)$$

This finalises the discussion of the quantisation of Abelian gauge theories. With the Feynman rules above we now can compute processes in QED and the Abelian Higgs model. We close this Chapter with a few remarks on the gauge invariance of observables. A first example is given by scattering amplitudes such as the given by the scattering of an electron-positron pair into some final state, e.g. $e^+ e^- \rightarrow \mu^+ \mu^-$, which is discussed in detail in Section 6.2.

$$\simeq \langle 0 | T A_\mu A_\nu | 0 \rangle (k) \bar{v}(-p-k) \gamma^\mu u(p), \quad (5.81)$$

where we have taken into account that the momentum arguments in $u(p)$ and $\delta v(p')$ are counted as incoming, see (4.139) for the Yukawa theory and (6.4) in Chapter 6 for QED. Hence $p' = -(p+k)$.

Seemingly, this process depends linearly on the gauge fixing parameter via (5.80). Evidently, the on-shell condition for the in- and out-states has to eliminate this dependence and hence the parts of the propagator proportional to $k_\mu k_\nu$. We concentrate on this part and derive

$$\begin{aligned} (1 - \xi) k_\mu k_\nu \bar{v}(-p-k) \gamma^\mu u(p) &= (1 - \xi) k_\nu \bar{v}(-p-k) \not{k} u(p) = (1 - \xi) k_\nu \bar{v}(-p-k) (\not{k} + \not{p} - \not{p}) u(p) \\ &\stackrel{(4.93)}{=} (1 - \xi) k_\nu \bar{v}(-p-k) (\not{k} + \not{p} - m) u(p) \stackrel{(4.93)}{=} 0. \end{aligned} \quad (5.82)$$

Equation (4.93) entails $(\not{p} + m)u(p) = 0$ and $(\not{p} - m)v(p) = 0$. The first relation is used directly in (5.82). The latter relation implies

$$v^\dagger(p)(\not{p}^\dagger + m)\gamma^0 = v^\dagger(p)\gamma^0(\gamma^0 \not{p}^\dagger \gamma^0 + m) = \bar{v}(p)(\not{p} + m) = 0, \quad (5.83)$$

which leads to $\bar{v}(-p-k)(\not{k} + \not{p}) = \bar{v}(-p-k)m$ in the last step in (5.82).

We conclude that we can drop the $k_\mu k_\nu$ -term and the computation reduces to that in the Feynman gauge. Particularly, the exchange photon is proportional to $\eta_{\mu\nu}$ as if there are four polarisations. Note that the ξ -independence only holds on-shell, i.e. general Feynman diagrams are ξ -dependent as they do not involve the on-shell projection.

The second example concerns gauge invariant observables in terms of matrix elements of gauge invariant operators, such as the electric and magnetic fields, the E and B -fields. We have

$$E^i = -F^{0i} = -(\partial^0 A^i - \partial^i A^0), \quad B^i = \epsilon^{ijk} F_{jk}. \quad (5.84)$$

Using (5.46a), we are led to the electric fieldstrength

$$\begin{aligned} \mathbf{E} &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} ik^0 \left[\left(\mathbf{a} - \frac{\mathbf{k}}{k^0} a_0 \right) e^{-ikx} - \left(\mathbf{a}^\dagger - \frac{\mathbf{k}}{k^0} a_0^\dagger \right) e^{ikx} \right] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} ik^0 \left[(\boldsymbol{\epsilon}_1 \alpha_1 + \boldsymbol{\epsilon}_2 \alpha_2) e^{-ikx} - (\boldsymbol{\epsilon}_1^\dagger \alpha_1^\dagger + \boldsymbol{\epsilon}_2^\dagger \alpha_2^\dagger) e^{ikx} \right. \\ &\quad \left. - \left(\frac{\mathbf{k}}{k^0} \alpha_+ e^{-ikx} - \frac{\mathbf{k}}{k^0} \alpha_+^\dagger e^{ikx} \right) \right], \end{aligned} \quad (5.85)$$

with the physical polarisations $\epsilon_{1,2}$. Applied on a state in the physical Hilbert space (5.71), the third line proportional to $\alpha_+, \alpha_+^\dagger$ drops out. Analogously, we find for the magnetic field

$$B^i(x) = \epsilon^{ijl} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} ik^j \left[(\epsilon_1^l \alpha_1 + \epsilon_2^l \alpha_2) e^{-ikx} - (\epsilon_1^l \alpha_1^\dagger + \epsilon_2^l \alpha_2^\dagger) e^{ikx} \right]. \quad (5.86)$$

As for the electric field, applying B to a state in the physical Hilbert space (5.71), the terms to $\alpha_+, \alpha_+^\dagger$ drops out. In summary, only $\alpha_{1,2}$ and α_+ appear in \mathbf{E} and \mathbf{B} , and sandwiched between physical states $|\psi\rangle \in \mathcal{H}$, the $\alpha_+, \alpha_+^\dagger$ parts drops out.

The Hamiltonian reads with $\Pi^i = E^i$ and $1/4 F_{\mu\nu} F^{\mu\nu} = 1/2(\mathbf{E}^2 - \mathbf{B}^2)$,

$$H = \int d^3x \left[\boldsymbol{\Pi} (\partial_0 \mathbf{A} - \nabla A_0) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \stackrel{\nabla \mathbf{E}=0}{=} \int d^3x \left[\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) + \nabla (\mathbf{E} A_0) \right] = \frac{1}{2} \int d^3x (\mathbf{E}^2 + \mathbf{B}^2), \quad (5.87)$$

where we have used

$$\nabla \mathbf{E} = \left(-\partial^0 \partial^i A^i + (\partial^i)^2 A^0 \right) \stackrel{\partial_\mu A^\mu=0}{=} \left[-(\partial^0)^2 + (\partial^i)^2 \right] A^0 = 0. \quad (5.88)$$

Now we insert the operators of the E- and the B-field, (5.85) and (5.86), and arrive at

$$P^0 = H \simeq \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{k_0}{2k^0} k^0 \sum_{i=1}^2 (\alpha_i \alpha_i^\dagger + \alpha_i^\dagger \alpha_i) \simeq \int \frac{d^3k}{(2\pi)^3} \frac{k_0}{2k^0} k^0 \sum_{i=1}^2 \alpha_i^\dagger(\mathbf{k}) \alpha_i(\mathbf{k}), \quad (5.89)$$

where we have dropped the α_+ -terms in the first line, and the vacuum terms in the second line. Similarly, we get for \mathbf{P}

$$\mathbf{P} = \int d^3x \mathbf{E} \times \mathbf{B} \simeq \int \frac{d^3k}{(2\pi)^3} \mathbf{k} \sum_{i=1}^2 \alpha_i^\dagger(\mathbf{k}) \alpha_i(\mathbf{k}), \quad (5.90)$$

where we have dropped the vacuum terms.

6. QED

In the lecture course so far we have discussed the quantisation of scalar fields, [Chapter 2](#), fermionic theories, [Chapter 4](#) and gauge fields, [Chapter 5](#). [Chapter 3](#) was dedicated to the computation of the cross section of scattering processes within perturbation theory.

In combination this allows us to discuss a plethora of quantum phenomena in condensed matter, statistical and particle physics. A first physics application is given by scattering processes in Quantum Electro Dynamics (QED), which also serves as a blue print for many scattering processes in the Standard Model. Specifically, we shall compute the tree-level contribution of the annihilation of an e^+e^- pair into a $\mu^+\mu^-$ pair in [Section 6.2](#).

6.1. Action and Feynman rules

QED describes the electromagnetic interaction of the Leptons and quarks via the exchange of photons γ , described by the U(1) gauge boson A_μ of the electromagnetic force. We collect the respective fermionic matter fields where the masses are taken from the Review of Particle Physics 2022 [1].

	1 st Gen.	2 nd Gen.	3 rd Gen.	Charge
Lepton	e^\pm	μ^\pm	τ^\pm	± 1
	ψ_e	ψ_μ	ψ_τ	
Mass [MeV]	0.511	105.658	1776,86(12)	
Mass [MeV]	1.5-4	1150-1350	170×10^3	
Quark	u	c	t	$\pm \frac{2}{3}$
Quark	d	s	b	$\mp \frac{1}{3}$
Mass [MeV]	4-8	80-130	$(4.1-4.4) \times 10^3$	

Table 6.1.: Masses and electric charges of quarks and leptons.

The action is a sum of the Dirac actions of e , μ , τ and the gauge field action of the photon, see (5.20). We collect

$$S_{\text{QED}}[A, \psi_e, \psi_\mu, \psi_\tau] = S_A[A] + S_{\text{gf}}[A] + \sum_{i=e,\mu,\tau} S_D[A, \psi_i], \quad (6.1a)$$

with the Dirac actions

$$S_D[A, \psi_i] = \int d^4x \bar{\psi}_i (i \not{D} - m_{e,\mu,\tau}) \psi_i, \quad \text{with } D_\mu = \partial_\mu - ieA_\mu, \quad (6.1b)$$

and the gauge field action

$$S_A[A] = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (6.1c)$$

The gauge fixing term $S_{\text{gf}}[A]$ in the covariant gauge is

$$S_{\text{gf}}[A] = -\frac{1}{2\xi} \int d^4x (\partial_\mu A^\mu)^2, \quad (6.1d)$$

with the gauge fixing parameter ξ . We have left out the Dirac action of the quarks, which can be simply added by a sum of Dirac actions with $\psi = u, d, s, c, b, t$. The action (6.1) is invariant under the combined gauge transformations

$$\psi_i(x) \rightarrow e^{ie\alpha(x)} \psi_i(x), \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x). \quad (6.2)$$

Next, we collect the Feynman rules of the fermions, (4.145), and the gauge field (5.77) and (5.78). We also remark, that any other coupling of leptons and photon introduces dimension-full couplings to the theory, e.g. the spin-coupling

$$\frac{e}{\Lambda} \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi, \quad (6.3)$$

where Λ carries momentum dimension one. Such a term makes the theory non-renormalisable.

Feynman rules for QED

Propagators

Leptons:
$$\eta \xrightarrow{p} \eta' = i \left(\frac{\not{p} + m_\psi}{p^2 - m_\psi^2 + i\epsilon} \right)_{\eta\eta'}$$

Photon:
$$\mu \text{---} \text{---} \text{---} \nu \xrightarrow{k} = -\frac{i}{k^2 + i\epsilon} \left(\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right).$$

Vertex:
$$\begin{array}{c} \eta' \\ \nearrow \\ \text{---} \text{---} \text{---} \mu \\ \nwarrow \\ \eta \end{array} = i e (\gamma_\mu)_{\eta\eta'}$$

incoming lepton: $\xrightarrow{p} \bullet = u(p)$ outgoing lepton: $\bullet \xleftarrow{p} = \bar{u}(p)$

incoming anti-lepton: $\bullet \xleftarrow{p} = \bar{v}(p)$ outgoing anti-lepton: $\bullet \xrightarrow{p} = v(p)$

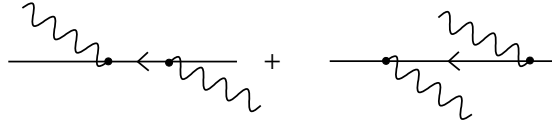
incoming photon: $\mu \text{---} \text{---} \text{---} \bullet \xrightarrow{k} = \epsilon_\mu(k)$ outgoing photon: $\bullet \text{---} \text{---} \text{---} \mu \xrightarrow{k} = \epsilon_\mu^*(k).$ (6.4)

We also remind the reader on the rules iii) (minus sign for fermion loops) and iv) (total momentum conservation) in (4.145). They are not repeated in (6.4), as they apply to any theory.

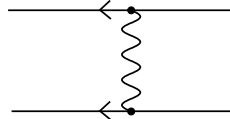
6.2. Elementary Processes

With the Feynman rules for QED collected in (6.4) at hand, we can compute the cross sections of elementary processes. We encourage the reader to compute some or all of the cross sections of the processes listed below,

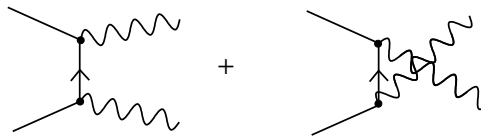
i) Compton scattering: $e^- \gamma \rightarrow e^- \gamma$



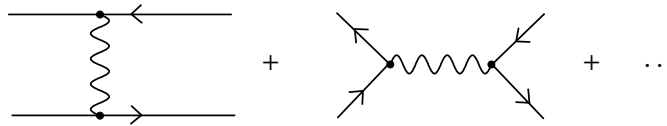
ii) Elastic $e^- e^-$ -scattering:



iii) Pair-annihilation/creation: $e^+ e^- \rightarrow \gamma \gamma$



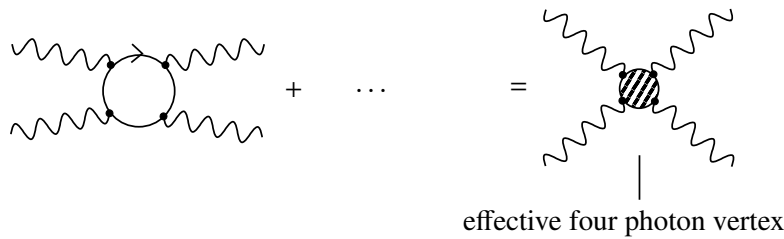
iv) Bhabha-scattering: $e^+ e^- \rightarrow e^+ e^-$



tree level processes

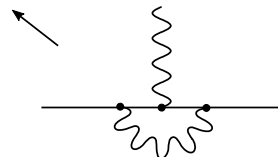
(6.5)

v) light-by-light scattering: (non-linear electrodynamics)



vi) Landé factor (gyromagnetic ratio):

$$i\mathcal{D} - m_e \rightarrow i\mathcal{D} - m_e + \frac{\Delta g}{2} \frac{e}{4m_e} \sigma_{\mu\nu} F^{\mu\nu}, \quad \Delta g = \frac{\alpha}{\pi}, \quad \alpha = \frac{e^2}{4\pi}$$



loop effects

(6.6)

In this lecture we compute the simplest one in detail, that of $e^+ e^- \rightarrow \mu^+ \mu^-$ at tree level in the highly-relativistic limit.

Example 6-17: Electron-positron annihilation into a muon–anti-muon pair ($e^+ e^- \rightarrow \mu^+ \mu^-$).

We have chosen this example, because there is only a single Feynman diagram for this process at tree level,

$$(6.7)$$

As we look at the highly relativistic case of 2-2 scattering we can use (3.166),

$$d\sigma = \frac{1}{2s} |M_{fi}|^2 \int d\Pi_2, \quad \text{with} \quad \int d\Pi_2 = \frac{1}{2} \frac{s/4}{(2\pi)^2 4p_3^0 p_4^0} d\Omega, \quad s = (p_1 + p_2)^2. \quad (6.8)$$

We find

$$|M|^2 = \frac{1}{2} \sum_r \frac{1}{2} \sum_{r'} \sum_{s,s'} |M(r, r', s, s')|^2, \quad (6.9)$$

where we computed the spin averages by summing over r, r' and summed over all possible splits s, s' . The scattering amplitude is read off from the Feynman rules,

$$iM = \bar{u}_{\mu_s}(p_3) (i e \gamma_\rho) v_{\mu_{s'}}(p_4) \left[\frac{\eta^{\rho\sigma}}{s} \right] \bar{v}_{e_{r'}}(p_2) (i e \gamma_\sigma) u_{e_r}(p_1), \quad (6.10)$$

where the term in square bracket corresponds to the photon propagator in the diagram (6.7), taken from the Feynman rules (6.4). We have shown in (5.82), that the term proportional to $(1 - \xi)$ in the propagator drops out and effectively the propagator is that in the Feynman gauge, independent of the gauge taken. The other terms correspond to the $e + e^-$ and $\mu^+ \mu^-$ currents, connected to the in- and out-states in the diagram (6.7). It follows from (6.10), that

$$|M|^2 = \frac{e^4}{4s^2} (T_\mu)_{\alpha\beta} (T_e)^{\alpha\beta}, \quad (6.11)$$

with

$$(T_\mu)_{\alpha\beta} = \sum_{s,s'} \bar{u}_{\mu_s}(p_3) \gamma_\alpha v_{\mu_{s'}}(p_4) \left[\bar{u}_{\mu_s}(p_3) \gamma_\beta v_{\mu_{s'}}(p_4) \right]^*$$

$$(T_e)^{\alpha\beta} = \sum_{r,r'} \bar{v}_{e_{r'}}(p_2) \gamma^\alpha u_{e_r}(p_1) \left[\bar{v}_{e_{r'}}(p_2) \gamma^\beta u_{e_r}(p_1) \right]^*. \quad (6.12)$$

The tensors $T_{e/\mu}^{\alpha\beta}$ can be computed with the completeness relations (4.104). For the computation of T_μ we need the relations for the muons,

$$\sum_s u_{\mu_s}(p_3) \bar{u}_{\mu_s}(p_3) = (\not{p}_3 + m_\mu), \quad \sum_s v_{\mu_{s'}}(p_4) \bar{v}_{\mu_{s'}}(p_4) = (\not{p}_4 - m_\mu), \quad (6.13)$$

and the respective one of the electrons for T_e . Inserting these relations in $T_{e/\mu}^{\alpha\beta}$ in (6.12), leads us to

$$\sum_{s,s'} \bar{u}_{\mu_s}(p_3) \gamma_\alpha \left[v_{\mu_{s'}}(p_4) \bar{v}_{\mu_{s'}}(p_4) \right] \gamma_\beta^* u_{\mu_s}(p_3) = \text{tr} (\not{p}_3 + m_\mu) \gamma_\alpha (\not{p}_4 - m_\mu) \gamma_\beta, \quad (6.14)$$

where we have used that

$$\left[\bar{u}_s(p) \gamma_\alpha v_{s'}(q) \right]^* = v_{s'}^\dagger(q) \gamma^0 \gamma^\alpha \gamma^0 \gamma_\alpha^\dagger \gamma^0 \gamma^0 \bar{u}_s^\dagger(p) = \bar{v}_{s'}(q) \gamma_\alpha u_s(p). \quad (6.15)$$

with $\gamma^0 \gamma^0 = \mathbb{1}$ and $\gamma^0 \gamma^\dagger \gamma^0 = \gamma^0$. With the above results we arrive at

$$(T_\mu)_{\alpha\beta} = \text{tr} \left(\not{p}_3 + m_\mu \right) \gamma_\alpha \left(\not{p}_4 - m_\mu \right) \gamma_\beta \stackrel{\text{tr} \gamma^{2n+1}=0}{=} \text{tr} \not{p}_3 \gamma_\alpha \not{p}_4 \gamma_\beta + \text{tr} \gamma_\alpha \gamma_\beta m_\mu^2, \quad (6.16)$$

which can be computed with the traces of two and four γ -matrices, that follow straightforwardly by repeated application of the Clifford algebra,

$$\text{tr} \gamma^\rho \gamma^\sigma = \frac{1}{2} \text{tr} \{ \gamma^\rho, \gamma^\sigma \} = \frac{1}{2} \text{tr} (2\eta^{\rho\sigma}) = 4\eta^{\rho\sigma}$$

$$\text{tr} \gamma^\rho \gamma^\sigma \gamma^\alpha \gamma^\beta = 2\eta^{\rho\sigma} \text{tr} \gamma^\alpha \gamma^\beta - \text{tr} \gamma^\sigma \gamma^\rho \gamma^\alpha \gamma^\beta = 8\eta^{\rho\sigma} \eta^{\alpha\beta} - \text{tr} \gamma^\sigma \gamma^\rho \gamma^\alpha \gamma^\beta = 4 \left(\eta^{\rho\sigma} \eta^{\alpha\beta} - \eta^{\rho\alpha} \eta^{\beta\sigma} + \eta^{\rho\beta} \eta^{\alpha\sigma} \right). \quad (6.17)$$

Furthermore, as we consider the highly relativistic limit, m_μ , m_e -contributions are sub-leading, and we obtain our final results for T_μ with

$$(T_\mu)_{\alpha\beta} = 4 \left(p_{3\alpha} p_{4\beta} + p_{3\beta} p_{4\alpha} - \eta_{\alpha\beta} p_3 p_4 \right) - 4\eta_{\alpha\beta} m_\mu^2 \stackrel{s \gg m_\mu^2}{\simeq} 4 \left[p_{3\alpha} p_{4\beta} + p_{3\beta} p_{4\alpha} - \eta_{\alpha\beta} p_3 p_4 \right]. \quad (6.18)$$

Similarly, we get

$$(T_e)^{\alpha\beta} \simeq 4 \left[p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha - \eta^{\alpha\beta} p_1 p_2 \right]. \quad (6.19)$$

Inserting these results in (6.11), we arrive at

$$|M|^2 = \frac{e^4}{4s^2} 2 \times 16 \left[(p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right] = \frac{8e^4}{s^2} \left[(p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right]. \quad (6.20)$$

In summary, and after inserting (6.20) in (6.8), we find

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2}{p_3^0 p_4^0} \frac{1}{s} \left[(p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right], \quad \text{with} \quad \alpha = \frac{e^2}{4\pi}, \quad (6.21)$$

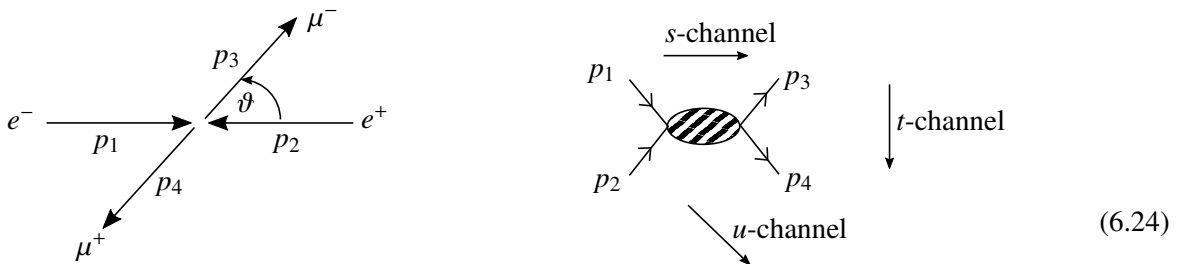
Here, α is the *fine structure constant*, and the results depends on the scattering angle ϑ with

$$\cos \vartheta = \frac{\mathbf{p}_1 \mathbf{p}_3}{|\mathbf{p}_1| |\mathbf{p}_3|}. \quad (6.22)$$

Equation (6.27) can be simplified further by using the *Mandelstam variables*

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \quad (6.23)$$

These are the momenta squared of the three different momentum channels in a $2 \rightarrow 2$ scattering.



Again using the highly relativistic limit, we obtain,

$$p_1 p_3 = p_1^0 p_3^0 - \mathbf{p}_1 \mathbf{p}_3 \approx \frac{1}{4}s - \frac{1}{4}s \cos \vartheta = \frac{1}{4}s(1 - \cos \vartheta) = p_2 p_4, \quad p_1 p_4 = \frac{1}{4}s(1 + \cos \vartheta) = p_2 p_3. \quad (6.25)$$

Using these limits, the sum of products of the different momenta in (6.21) reduces to

$$(p_1 p_4)(p_2 p_3) + (p_1 p_3)(p_2 p_4) = \frac{1}{16}s^2(2 + 2\cos^2 \vartheta) = \frac{1}{8}s^2(1 + \cos^2 \vartheta). \quad (6.26)$$

The final result for $|M|^2$ is

$$|M|^2 = e^4(1 + \cos^2 \vartheta) = 16\pi^2\alpha^2(1 + \cos^2 \vartheta), \quad \alpha = \frac{e^2}{4\pi}. \quad (6.27)$$

The matrix element squared in (6.27) has an angular dependence, that is absent in the respective result for scalar 2-2 scattering in (3.47) with $|M|^2 = \lambda^2$. Inserting (6.27) in (6.8) and using $4p_3^0 p_4^0 \approx s$ yields the differential cross section

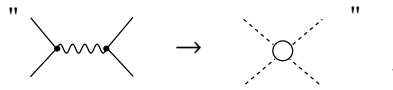
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2 \vartheta), \quad (6.28)$$

in comparison to the differential cross section of scalar 2-2 scattering, $d\sigma/d\Omega = 1/4(\lambda/(4\pi))^2 1/s$, provided in (3.166).

We remark, that in the high energy limit also $(p_1 - p_2)_\mu \bar{v}(p_2) \gamma^\mu u(p_1) \approx \sim^{m_e} 0$. Only the physical polarisations ϵ_1 and ϵ_2 play a role, ϵ_3 drops out, see (5.56). This argument also applies to $\bar{u}(p_3) \gamma^\nu v(p_4)$. In summary we have

$$\bar{u}(p_3) \gamma^\nu v(p_4) (p_{3/4})_\nu \approx 0 \quad \bar{v}(p_2) \gamma^\mu u(p_1) (p_{1/2})_\mu \approx 0.$$

So if $p_{3,4}$ are orthogonal to the beam axis, defined by $p_{1/2}$, the related polarisation ϵ_1 or ϵ_2 also 'drops out of the game'. In this case, $\vartheta = \pi/2$, only one polarisation contributes to the scattering, for $\vartheta = 0$, both. Lastly, note, that in the highly relativistic case and for $\vartheta = \pi/2$,



This closes our discussion of elementary processes in QED. The evaluation done for the tree-level contribution of $e^+e^- \rightarrow \mu^+\mu^-$ scattering carries over straightforwardly to other tree-level processes discussed in the beginning of this Section, i.e. the processes i) - iv) depicted in (6.5). The last two processes discussed there and depicted in (6.6) are loop processes and the correlation functions involved are the one-loop contribution of the four-photon vertex, and the one-loop correction to the electron-photon vertex. The former correction has no counterpart in the classical action and turns out to be finite. These statements also holds true for the contribution to the electron-photon vertex proportional to $\sigma_{\mu\nu} F^{\mu\nu}$ that is relevant for the Landé factor: it also has no counterpart in the classical action and turns out to be finite. However, the respective diagram also includes a correction with the tensor structure of the classical vertex, γ_μ Naïvely, this correction diverges, and has to be *regularised*. For example we could cut off the spatial momentum integral with $\mathbf{p}^2 \leq \Lambda^2$ with a large momentum cutoff Λ . However, then we are left with a divergent Λ -dependence, in this case proportional to $\ln \Lambda/\mu$, where μ is some reference scale. This contribution can be absorbed in a redefinition of the classical vertex, which is called *renormalisation*, and its consistent setup and application is the subject of Chapter 7.

7. Renormalisation

In the previous chapters we have encountered many occasions, where the naïve expressions we encountered, involved singular momentum integrals. The first prominent example was the 'vacuum contribution' or vacuum loop in the Hamiltonian, we have also encountered in perturbation theory. Moreover, in the discussion of the latter we also came across many loop corrections to correlation functions, whose momentum integrals were not well-defined. This cumulated in the loop examples for the elementary processes discussed in [Section 6.2](#), and in particular the loop correction of the photon-electron vertex. Moreover, in [Sections 3.5](#) and [3.5.2](#) we have also encountered factors Z , that originated in potentially divergent loop corrections of the external propagators, and are related to redefinitions of the fields.

In the present chapter we will discuss the mathematically consistent setup of quantum field theories within perturbation theory, in which these divergences of momentum integrals are first *regularised*, the simplest example being a (spatial) momentum cutoff regularisation: only loop momenta with $p^2 \leq \Lambda^2$ are allowed in the loops. This leaves us with finite, but Λ -dependent expressions for the loops. If divergences only occur for correlation functions that have a classical counter part such as the correction to the electron-photon, or the two-point function of the electron and photon, these divergences can be consistently removed by a redefinition of the 'classical' couplings, masses and fields. If these properties hold true for all orders in perturbation theory, the theory is called perturbatively renormalisable.

In the present chapter we shall show, that both, the ϕ^4 -theory as well as QED are indeed perturbatively renormalisable. Again, we start with the ϕ^4 -theory in [Section 7.1](#) in four space-time dimensions as the simplest model theory. In [Section 7.3](#) we extend our analysis to QED.

7.1. Renormalisation in the ϕ^4 -theory

The perturbative setup of the ϕ^4 -theory has been provided in [Section 3.3](#) with the action [\(3.126\)](#) and the ensuing Feynman rules [\(3.127\)](#). In view of potential redefinitions of the field, the mass and the coupling we write the action as

$$S[\phi] = -\frac{1}{2} \int d^4x \phi_0 (\partial^2 + m_0^2) \phi_0 - \frac{\lambda_0}{4!} \int d^4x \phi_0^4, \quad (7.1)$$

with *bare* fields ϕ_0 and parameters/couplings m_0^2 and λ_0 .

7.1.1. Renormalisation conditions and Feynman rules

Now we anticipate, that we want to absorb the Λ -dependence of corrections of the mass, the coupling and the kinetic term in the classical parameters of the theory. To that end we introduce factors for all the quantities, ϕ_0, m_0, λ_0 in the bare action [\(7.1\)](#). We write

$$\phi_0 = Z_\phi^{1/2} \phi, \quad m_0^2 = Z_m m^2, \quad \lambda_0 = Z_\lambda \lambda, \quad (7.2)$$

with *renormalised* or physical fields ϕ , parameters m^2, λ and *multiplicative* renormalisation factors Z_ϕ, Z_m, Z_λ . Inserting the relations [\(7.2\)](#) in [\(7.1\)](#), leads us to

$$S[\phi] = -\frac{1}{2} \int d^4x Z_\phi \phi (\partial^2 + Z_m m^2) \phi - Z_\phi^2 Z_\lambda \frac{\lambda}{4!} \int d^4x \phi^4, \quad (7.3)$$

Evidently, the Feynman rules in terms of the renormalised fields involve the prefactors Z_ϕ , Z_m and Z_λ . For example, the Feynman propagator of the renormalised field is now given by

$$\mathcal{D}_F(p) = \frac{1}{Z_\phi} \frac{i}{p^2 - Z_m m^2 + i\epsilon}, \quad (7.4)$$

while the Feynman rule for the vertex gives

$$-iZ_\phi^2 Z_\lambda \lambda. \quad (7.5)$$

In perturbation theory the Z 's are expanded in powers of the coupling, and the classical part is of power λ^0 and is simply unity. We write

$$Z = 1 + \delta Z, \quad \delta Z = \delta Z_1 \lambda + \delta Z_2 \lambda^2 + \dots, \quad (7.6)$$

Let us now related the renormalised quantities to physics. This is best done on-shell and we recall the LSZ-formalism with (3.179) for the bare fields ϕ_0 ,

$$\langle T \phi_0 \phi_0 \rangle(p) \Big|_{\text{pole}} = \frac{iZ}{p^2 - m_{\text{pole}}^2} + \text{finite} = Z_\phi \langle T \phi \phi \rangle(p) \Big|_{\text{pole}}. \quad (7.7)$$

Fixing $Z_\phi = Z$ and identifying the renormalised mass m^2 with the pole mass m_{pole} leads us to

$$\langle T \phi \phi \rangle(p) \Big|_{\text{pole}} = \frac{i}{p^2 - m^2} + \text{finite}. \quad (7.8)$$

Evidently, this links m^2 to a physical quantity, the pole mass squared, m_{pole}^2 with $m^2 = m_{\text{pole}}^2$. Note also, that Z is a physical quantity, that enters the spectral sum rule (3.184). It carries the spectral weight of the particle pole and is smaller than one, (3.185). In turn, $1 - Z$ is the spectral weight of all the scattering states. Note that this relation implies that the amplitude of the pole of the renormalised propagator is unity, and the sum rule (3.184) does not hold for the renormalised propagator. Indeed, it satisfies the sum rule

$$1 = Z + \int_{m_1^2}^{\infty} \frac{dM^2}{2\pi} Z \rho_{\text{ren}}(M^2), \quad (7.9)$$

where ρ_{ren} is the spectral function of the renormalised fields, and m_1 is the onset of the scattering continuum. Equations (7.7) and (7.8) can be cast into conditions for the renormalised propagator $\langle T \phi \phi \rangle(p)$, the renormalisation conditions.

Renormalisation conditions for the inverse propagator

$$\left[\langle T \phi \phi \rangle(p) \right]_{p^2=m^2}^{-1} = 0 \quad (7.10a)$$

$$i \frac{\partial}{\partial p^2} \left[\langle T \phi \phi \rangle(p) \right]_{p^2=m^2}^{-1} = 1. \quad (7.10b)$$

This fixes the constants Z_ϕ and Z_m at the momentum scale $p^2 = m_{\text{pole}}^2$. We have chosen the on-shell condition as this makes the relation to the physical mass most obvious. However, the procedure clearly also goes through if choosing any other scale. In general, we fix $\langle T \phi \phi \rangle$ at some scale $p^2 = \mu^2$, where μ is called renormalisation scale. We remark, that the inverse propagator is one-particle irreducible: it has a representation in terms of diagrams, that are still connected after cutting one line. Its inverse, the propagator, is only connected.

Let us now elucidate the content of the renormalisation conditions for the inverse propagator, (7.10a) and (7.10b) within an explicit parametrisation. The full, momentum dependent two-point function is written as

$$G_\phi(p) = \frac{i D_\phi(p^2)}{p^2 - m_{\text{pole}}^2 + i\epsilon}, \quad (7.10c)$$

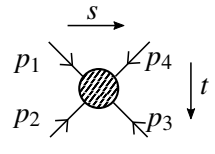
with the *dressing function* $D_\phi(p^2)$. Equation (7.10c) reduces to the classical *finite* renormalised propagator for $D_\phi(p^2) \rightarrow 1$. The propagator has a pole at $p^2 = m_{\text{pole}}^2$ (we assume $D(m_{\text{pole}}^2) \neq 0$), and hence m_{pole}^2 is the physical pole mass. The momentum dependence of $D_\phi(p^2)$ carries the non-trivial quantum corrections of the diagrams. Inserting the inverse G_ϕ^{-1} in the renormalisation conditions (7.10a) and (7.10b) with $\mu^2 = m^2$ leads us to

$$m^2 = m_{\text{pole}}^2, \quad \text{and} \quad \left[\frac{1}{D_\phi(p^2)} + (p^2 - m_{\text{pole}}^2) \frac{\partial D_\phi^{-1}(p^2)}{\partial p^2} \right]_{p^2=m^2} = 1. \quad (7.10d)$$

where we have left the RG-scale μ^2 general in the RG-condition (7.10b). As discussed before, the on-shell RG-condition (7.10a) fixes the renormalised mass parameter to be the physical pole mass. The second condition, (7.10b), fixes the global amplitude of the propagator. In contradistinction to the first condition is simply fixes a parametrisation freedom. For $\mu^2 = m_{\text{pole}}^2$ is reduces to the simple condition $D(m_{\text{pole}}^2) = 1$. Evidently, the spectral sum rule (3.184) does not hold any more and we are left with the modified one, (7.9).

It is left to fix the one remaining free parameter, the coupling renormalisation Z_λ . This is usually done by a renormalisation condition for the one-particle irreducible part of the four-point function, in order to avoid redundancies. In the present case of a ϕ^4 -theory the one-particle irreducible part of the four-point function is simply the amputated four-point function. Note that this is different for theories with non-vanishing three-point functions, and we shall discuss this in the context of QED.

A common choice is the symmetric point with $s = t = u = m^2$ with the Mandelstam variables (6.23),



$$\left. \begin{array}{l} \text{!} \\ = -i\lambda. \\ \text{!} \\ s=t=u=m^2 \end{array} \right| \quad (7.10e)$$

If we write this in terms of the two- and four-point function, we obtain the third renormalisation condition

Renormalisation condition for the vertex

$$\prod_i \left[\langle T \phi \phi \rangle(p_i) \right]^{-1} \cdot \langle T \phi(p_1) \cdots \phi(p_4) \rangle \Big|_{s=t=u=m^2} = -i\lambda, \quad (7.10f)$$

where $\lambda = \lambda_{\text{phys}}|_{\text{symmetric point}}$ and we have dropped the overall δ -function that carries the momentum conservation on the vertex. As in the case of the renormalisation conditions for the propagator we discuss that for the vertex within a parametrisation of the amputated vertex. We write

$$\prod_i \left[\langle T \phi \phi \rangle(p_i) \right]^{-1} \cdot \langle T \phi(p_1) \cdots \phi(p_4) \rangle = -i\lambda V_\phi(p_1, p_2, p_3) (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) \quad (7.11)$$

with all momenta incoming and $p_4 = -(p_1 + p_2 + p_3)$. Inserting (7.11) into (7.10f) leads us to the RG-condition for the vertex dressing,

$$V_\phi(p_1, p_2, p_3) \Big|_{s=t=u=m^2} = 1. \quad (7.12)$$

Equation (7.12) entails that the full (on-shell) vertex strength is given by λ . In summary, (7.10d) and (7.12) relate the renormalised parameters to physical ones: the physical pole mass and the full scattering strength at a given momentum configuration.

Moreover, on the technical side, the renormalisation conditions (7.10) also fix the map between the bare quantities ϕ_0, m_0, λ_0 to the renormalised (finite) quantities ϕ, m, λ . The finiteness of correlation functions of the renormalised fields ϕ follows from the finiteness of (7.10). Hence, the Z 's have to cancel the loop divergences and carry (minus) the singularities themselves. Note, that in renormalisable theories, it is sufficient to introduce the Z 's, and similar renormalisation factors for parameters (couplings, masses, fields) in the classical action, for getting a manifestly finite theory.

In turn, if the perturbative loop expansion leads to further divergences, for example a hypothetical one for the $\sigma_{\mu\nu}F^{\mu\nu}$ or for the ϕ^6 correlation function in a ϕ^4 -theory, these terms have to be added to the classical action if we want to cancel the emergent singularities as described above. However, these terms in the classical action may generate further singularities and in the two examples they indeed do. Evidently, if we have to add infinitely many terms to the classical action in order to get finite results, all these terms have their own renormalisation conditions similar to (7.10). They have to be fixed by linking them to physical observables, and we loose the predictive power of the theory. Such theories are called (perturbatively) non-renormalisable and cannot be used as fundamental theories as they lack predictive power.

Note also that the freedom of (re)-normalising fields and couplings encodes, that correlation functions are *not* by themselves physical observables, even though they are linked to them. For example, we could have renormalised the theory at some other momentum scale $p^2 = \mu^2$ with the renormalisation conditions, with

$$\lambda = \lambda_{\text{phys}}|_{p^2=\mu^2}, \quad m^2 = m_{\text{phys}}^2|_{p^2=\mu^2}. \quad (7.13)$$

Importantly, physics is invariant under changing the renormalisation scales μ . This is expressed in the renormalisation group equation.

Renormalisation group equation

$$\mu \frac{d}{d\mu}(\text{phys. observables}) = 0. \quad (7.14)$$

Note, that the renormalisation conditions encode the reparametrisation invariance of the theory and the insensitivity of physics to the specific renormalisation scheme used. The scale μ is called *renormalisation group (RG) scale*. We remark, that the generator of the RG is the logarithmic scale derivative $\mu \frac{d}{d\mu}$, and the RG is a one-parameter Abelian semi group (see QFT II). In the founding paper of renormalisation group theory, [2], the reparametrisations discussed also include conformal transformations, and the respective group is non-Abelian.

Let us now reformulate the Feynman rules for the renormalised theory, (7.4) and (7.5) in terms of classical propagators and the δZ 's, that carry the singularities.

Feynman rules for renormalised fields

Propagator:

$$\left[\phi \text{---} \phi \right]^{-1} = Z_\phi \frac{p^2 - Z_m m^2}{i} = \left[\frac{i}{p^2 - m^2 + i\epsilon} \right]^{-1} - \text{---} \otimes \text{---} ,$$

$$\text{where } \text{---} \otimes \text{---} = (-i) \left[\underbrace{(1 - Z_\phi) p^2}_{< 1} - (1 - Z_\phi Z_m) m^2 \right] .$$

Vertex:

$$\text{---} \bullet \text{---} = -i Z_\lambda Z_\phi^2 \lambda = -i \lambda + \text{---} \otimes \text{---} ,$$

$$\text{where } \text{---} \otimes \text{---} = i \lambda (1 - Z_\phi^2 Z_\lambda) . \quad (7.15)$$

The δZ -dependent quantities, $\text{---} \otimes \text{---}$, $\text{---} \otimes \text{---}$ are called counter terms, as the renormalisation factors Z_ϕ, Z_m, Z_λ cancel or counter the singularities of the loop contributions, that are proportional to p^2, m^2 and λ respectively.

7.1.2. Generating funtionals and one-particle irreducible vertices

We have mentioned in the context of the renormalisation condition for the vertex, (7.10f), that these conditions are usually applied to one-particle irreducible (1PI) correlation functions. In this chapter we briefly elucidate, how to generate these correlation functions as well as the connected one. A more detailed discussion will be done in QFT II. To begin with, the generating functional of full correlation functions, $Z[J]$, is given by

$$Z[J] = \left\langle \Omega \left| T \exp \left\{ i \int d^4 x J(x) \phi(x) \right\} \right| \Omega \right\rangle . \quad (7.16)$$

which is simply the S-matrix, where a source term has been inserted. Taking n derivatives with respect to the current J provides the n -point functions,

$$\langle T \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{Z[0]} \frac{i^n \delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} . \quad (7.17)$$

where the normalisation with $Z[0]$ normalises the zero-point function to unity, see (3.78) in Chapter 3. As discussed there, the correlation functions in (7.17) also contain disconnected diagrams. These diagrams are removed if taking derivatives of the logarithm of Z , the *Schwinger functional* $W[J]$,

$$\langle T \phi(x_1) \cdots \phi(x_n) \rangle_c = \frac{i^n \delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} , \quad \text{with } W[J] = \log Z[J] , \quad (7.18)$$

where the subscript c indicates the connected part of correlation functions. A simple example is the Feynman propagator $\mathcal{D}_F(x_1 - x_2)$, which is the connected two-point function. From we arrive at

$$\mathcal{D}_F(x_1 - x_2) = -\frac{\delta^2 W[J]}{\delta J(x_1) \cdots \delta J(x_2)} = \langle T \phi(x_1) \phi(x_2) \rangle - \bar{\phi}(x_1) \bar{\phi}(x_2) = \langle T \phi(x_1) \phi(x_2) \rangle_c , \quad (7.19)$$

with $\bar{\phi}(x) = \langle \phi(x) \rangle$. The connected two-point function only contains the part of the two-point function, that is not already contained in the one-point function (squared). The latter is the disconnected part of the two-point function. This subtraction property carries over to general n -point function.

Hence, going from the full correlation function to the connected correlation functions removes a redundancy, as the information of lower correlation functions is subtracted. Clearly, in a renormalisation programme it is advantageous (but not necessary) to implement renormalisation conditions on the correlation functions with the least redundancies. A final step in this direction is done by going from connected correlation functions to 1PI correlation functions. This is done by deriving the generating functional of 1PI correlation functions, the effective action $\Gamma[\bar{\phi}]$ from the Schwinger functional. This is done by a Legendre transformation,

$$i\Gamma[\bar{\phi}] = \sup_J \left[i \int d^4x J(x)\bar{\phi}(x) - W[J] \right], \quad (7.20)$$

where for our purposes here the supremum in (7.20) is a maximum, leading to the determination of J as

$$\bar{\phi} = -i \left. \frac{\delta W[J]}{\delta J(x)} \right|_{J_{\text{sup}[\bar{\phi}]}} , \quad \left. \frac{\delta^2 \Gamma[\bar{\phi}]}{\delta \bar{\phi}(x_1) \delta \bar{\phi}(x_2)} \right|_{\bar{\phi}=0} = \mathcal{D}_F^{-1}(x_1 - x_2), \quad (7.21)$$

where in general $\bar{\phi}$ has to be evaluated on the equations of motion. In the present ϕ^4 -theory we assume $\bar{\phi} = \langle \phi \rangle = 0$. The inverse propagator is indeed 1PI. It is also straightforward to show that the four-point function, derived from $\Gamma[\bar{\phi}]$ with four derivatives w.r.t. $\bar{\phi}$, is 1PI. A general proof will be given in QFT II.

7.1.3. One-loop renormalisation in the ϕ^4 -theory

For illustration, we work out the renormalisation programme of the ϕ^4 -theory at one loop. First, we consider the mass correction, see (3.207). At one loop we have graphically

$$\begin{aligned} \text{---} \text{---} \text{---} &= \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---} + \mathcal{O}(\lambda^2) \\ &= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} \left[-i\Pi(p) \right] \frac{i}{p^2 - m^2} + \dots, \end{aligned}$$

with the finite renormalised self-energy

$$\begin{aligned} -i\Pi(p) &= \left[\frac{1}{2} \text{---} \text{---} \text{---} + \text{---} \otimes \text{---} \right] \\ &= \left[\frac{1}{2} \text{---} \text{---} \text{---} + i(1 - Z_\phi)p^2 + i(1 - Z_\phi Z_m)m^2 \right]. \end{aligned}$$

finite

The loop diagram follows with the Feynman rules as

$$\text{---} \text{---} \text{---} = -i\lambda \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2}. \quad (7.22)$$

In summary, the renormalised self-energy $\Pi(p)$ reads

$$-i\Pi(p) = -\frac{i\lambda}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2} - i(1 - Z_\phi)p^2 + i(1 - Z_\phi Z_m)m^2. \quad (7.23)$$

Note, that $\text{---} \text{---} \text{---}$ has no dependence on the external momentum p . Hence, the renormalisation condition (7.10b) leads to a vanishing correction term $-i(1 - Z_\phi)p^2 = 0$ with $Z_\phi|_{1\text{-loop}} = 1$. Then, the constant term reduces to $i(1 - Z_\phi Z_m)m^2 \rightarrow i(1 - Z_m)m^2$, and (7.10a) directly fixes Z_m .

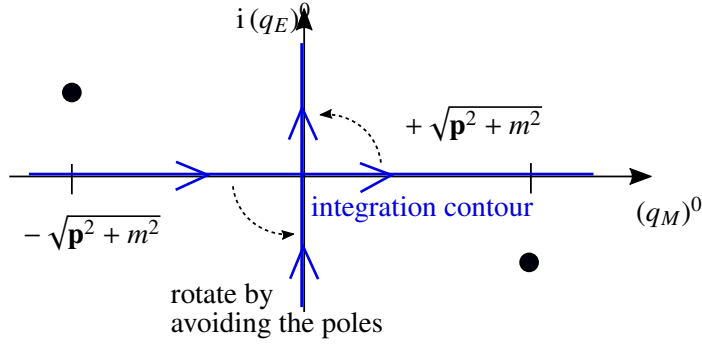


Figure 7.1.: Recall the $i\epsilon$ of time-ordering: $\frac{1}{q^2 - m^2 + i\epsilon}$. The rotated Euclidean contour runs $i(q_E)^0$ from $-\infty$ to $+\infty$, or $(q_M)^0$ from $-\infty$ to $+\infty$. As this rotation does not swipe the poles, the integration stays the same.

The renormalisation condition (7.10a) implies the vanishing of the on-shell inverse propagator. Graphically this is given by

$$\left[\text{tadpole diagram} \right]^{-1} \Big|_{p^2=m^2} = 0. \quad (7.24)$$

As the tadpole is momentum-independent, the renormalisation condition (7.10a) or (7.24) implies that the renormalised self-energy vanishes on one-loop,

$$\Pi(p) \Big|_{1\text{-loop}} \equiv 0. \quad (7.25)$$

It follows, that

$$1 - Z_m = \frac{1}{2} \frac{\lambda}{m^2} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon}. \quad (7.26)$$

This concludes the derivation of the one-loop renormalisation of the mass, including the manifest finiteness of the result. Note that while (7.26) is only a formal relation without the loop integral being finite, the latter is easily achieved by introducing a spatial momentum cutoff with $\mathbf{p}^2 \leq \Lambda^2$. Then, $Z_m \propto \lambda \Lambda^2 / m^2$, however, the total result for the mass, namely m^2 , is manifestly Λ -independent.

While the result is trivial, the tadpole is a simple toy example to explain two computationally important tricks that are commonly used within perturbative loop computations. To begin with, we notice that the spatial cutoff regularisation breaks Lorentz invariance. For the present computation this is not important as it is momentum-independent. However, in general such a cutoff will introduce a cutoff-dependent separation of frequency and spatial momentum dependences, which then has to be removed via the definition of appropriate counter terms, that have to break Lorentz invariance themselves in order to restore it on the level of the final results. This is not very appealing. Seemingly it is tempting to introduce a four-momentum cutoff with $p^2 \leq \Lambda^2$. However, then the integrand of the tadpole, (7.22) or (7.26), still diverges on-shell ($q^2 = m^2$). This intricacy can be resolved with the Wick rotation, see also (8.10). Within the Wick rotation we rotate the loop frequency in Minkowski space to Euclidean space such, that we do not sweep over the poles in the Feynman propagator in the integrand, for a depiction see Figure 7.1. Then, the result for the integral stays the same according to the residue theorem. The Wick rotation can be implemented in the loop integral via the substitution,

$$(q_M)^0 = i(q_E)^0, \quad (7.27)$$

which implies

$$(q_M)_\mu (q_M)^\mu = -(q_E)_\mu (q_E)^\mu = -(q_E)_\mu (q_E)_\mu, \quad \text{and} \quad \int_{\mathbb{R}^4} d^4 q_M = i \int_{\mathbb{R}^4} d^4 q_E, \quad (7.28)$$

With the relations (7.27) and (7.28), the tadpole turns into

$$\int \frac{d^4 q_M}{(2\pi)^4} \frac{i}{q_M^2 - m^2} = \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{q_E^2 + m^2}. \quad (7.29)$$

Note that with the Wick rotation the non-compact Lorentz group turns into the compact Euclidean symmetry group $O(4)$. The Euclidean momentum integral is still divergent, but we can now use a Euclidean four-momentum cutoff with $q^2 \leq \Lambda^2$, where we have dropped the subscript E and $q^2 = q_0^2 + \mathbf{q}^2$. This regularisation preserves the Euclidean $O(4)$ symmetry, and hence implicitly it also preserves Lorentz symmetry. It is implemented via

$$\int_{\mathbb{R}^4} \frac{d^4 q}{(2\pi)^4} \rightarrow \int_{q^2 \leq \Lambda^2} \frac{d^4 q}{(2\pi)^4}. \quad (7.30)$$

Using (7.30) in the Euclidean tadpole (7.29) leads us to

$$\int_{q^2 \leq \Lambda^2} \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2} = \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\Lambda dq \frac{q^3}{q^2 + m^2} = \frac{1}{8\pi^2} \int_0^\Lambda dq \frac{q^3}{q^2 + m^2} = \frac{1}{16\pi^2} \left[\Lambda^2 + m^2 \ln \frac{m^2}{\Lambda^2 + m^2} \right], \quad (7.31)$$

where we have used that $\int d\Omega_4 = 2\pi^2$. Equation (7.31) is readily used in (7.26), leading to

$$Z_m = 1 - \frac{1}{2} \frac{\lambda}{16\pi^2} \left[\frac{\Lambda^2}{m^2} + \ln \frac{1}{1 + \frac{\Lambda^2}{m^2}} \right], \quad (7.32)$$

the desired explicit finite result. As discussed before, using the finite Z_m and the regularised finite tadpole (7.31) leads to the trivial result (7.25). Importantly, all expressions are manifestly finite in all steps involved. Hence, the result is mathematically sound.

The mathematical foundation of the momentum cutoff regularisation is intriguingly simple to understand. It is closely related Bogoliubov-Parasiuk-Hepp-Zimmermann renormalisation scheme, which has been used for many proofs or renormalisability. Indeed, in modern renormalisation theory (functional renormalisation) used for non-perturbative numerical computations, a generalised combination of the two is used.

However, for the present purpose of analytic perturbative computations another regularisation scheme, dimensional regularisation, is more tailor-made, and is indeed used for the overwhelming majority of the respective computations. It is based on the analytic properties of loop integrals, that can be understood as analytic functions of a continuous dimension d . Similarly to the introduction of the momentum cutoff regularisation we introduce dimensional regularisation at the example of the Euclidean tadpole (7.29). We rewrite the four-dimensional integral as

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2} \rightarrow \left[(\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d q}{(2\pi)^d} \right] \frac{1}{q^2 + m^2} = \frac{\Omega_d}{(2\pi)^d} (\bar{\mu}^2)^{\frac{4-d}{2}} \int_0^\infty dq q^{d-1} \frac{1}{q^2 + m^2}, \quad (7.33)$$

with $d = 4 - 2\epsilon$ and $\epsilon \rightarrow 0$. Equation (7.33) depends on the d -dimensional angular volume Ω_d and a momentum scale $\bar{\mu}$. The latter scale had to be introduced in order to keep the momentum dimension two of the loop integral:

the term in square brackets in has dimension 4 due to the scaling factor in front of the d -dimensional integral. Note that one should understand the right hand side in (7.33) as the definition of the d -dimensional integral for $d \in \mathbb{R}$. For $d < 2$ the integral in the last line is finite, and we can compute (7.33) analytically. The result defines an analytic function, that can be continued to $d \geq 2$.

We first compute the d -dimensional angular volume Ω_d . For this purpose we consider

$$\sqrt{\pi}^d = \left(\int dx e^{-x^2} \right)^d = \int d^d x e^{-x_i^2} = \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} = \frac{1}{2} \int d\Omega_d \Gamma\left(\frac{d}{2}\right), \quad (7.34)$$

where we have used an integral representation of the Γ -function: $\Gamma[t] = \int_0^\infty ds s^{t-1} \exp\{-s\}$ with $s = x^2$ and $ds = dx^2 = 2dx x$. The radial momentum integral in (7.33) is readily computed also for higher powers of the propagator. We collect both results,

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}, \quad \int_0^\infty dq q^{d-1} \frac{1}{(q^2 + m^2)^n} = \frac{1}{2} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-\frac{d}{2}}, \quad (7.35)$$

which can be combined in one of the important master integrals in dimensional regularisation,

$$\int_0^\infty \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-\frac{d}{2}}. \quad (7.36)$$

With the general relation (7.36) we proceed with the computation of the tadpole diagram. It is given by (7.36) with $n = 1$ and $d = 4 - 2\epsilon$ with $\epsilon \rightarrow 0$, multiplied by $\bar{\mu}^{4-d}$,

$$\left(\bar{\mu}^2\right)^{\frac{4-d}{2}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{\bar{\mu}^2}{m^2}\right)^{2-\frac{d}{2}} m^2 = \frac{m^2}{16\pi^2} \left[-\frac{1}{\epsilon} + \gamma - 1 + \ln 4\pi + \ln \frac{m^2}{\bar{\mu}^2}\right], \quad (7.37)$$

where we have used that

$$\Gamma(-1 + \epsilon) \stackrel{x\Gamma(x)=\Gamma(x+1)}{=} \frac{1}{-1 + \epsilon} \Gamma(\epsilon) = -\frac{1}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon), \quad (7.38)$$

and the Euler-Mascheroni constant $\gamma = 0.577\dots$. Similarly to the derivation of Z_m within the momentum cutoff regularisation, (7.32), we also can extract Z_m within dimensional regularisation,

$$Z_m = 1 - \frac{1}{2} \frac{\lambda}{16\pi^2} \left(-\frac{1}{\epsilon} + \gamma - 1 + \ln \frac{m^2}{4\pi\bar{\mu}^2}\right). \quad (7.39)$$

The equivalence of (7.32) and (7.39) is best seen with $\ln 1/(1 + \Lambda^2/m^2) = \ln m^2/\Lambda^2 + \ln 1/(1 + m^2/\Lambda) \rightarrow \ln m^2/\Lambda^2$, as the latter logarithm vanishes in the limit $\Lambda \rightarrow \infty$. This leads to the same relation for the logarithmic mass derivative of Z_m , derived from both (7.32) and (7.39),

$$m \frac{dZ_m}{dm} = -\frac{\lambda}{16\pi^2}. \quad (7.40)$$

Note also, that the factor $\lambda/(16\pi^2)$ in both (7.32) and (7.39) is the universal coupling coefficient in the perturbative expansion of the ϕ^4 -theory. Similar coefficients are present in other theories such as QED. There we encounter the fine structure constant $\alpha = e^2/(4\pi)$ or rather $\alpha/(4\pi)$. Finally, for both regularisation schemes we find the renormalised one-loop propagator

$$\text{---} \circ \text{---} = \frac{i}{p^2 - m^2 + i\epsilon} + \mathcal{O}(\lambda^2). \quad (7.41)$$

This concludes the one loop renormalisation programme for the two point function in the ϕ^4 -theory: the wave function renormalisation is trivial, $Z_\phi = 1$, and the mass renormalisation is given by (7.32) and (7.39). Evidently, the renormalised propagator (7.41) satisfied the renormalisation conditions (7.10a) and (7.10b).

This leaves us with the third renormalisation condition (7.10f) which fixes the coupling renormalisation Z_λ . Note that the counter term (and the vertex) also depends on the wave function renormalisation. With the trivial one-loop result for the wave function renormalisation, the vertex counter term reduces to $i\lambda(1 - Z_\phi^2 Z_\lambda) \rightarrow i\lambda(1 - Z_\lambda)$, similarly to the reduction of the mass counter term. With this reduction we get

$$\begin{aligned} \text{---} \times \text{---} &= \text{---} \times \text{---} + \frac{1}{2} \text{---} \circ \text{---} + \frac{1}{2} \text{---} \circ \text{---} + \frac{1}{2} \text{---} \circ \text{---} + \mathcal{O}(\lambda^3) \\ &= \text{---} \times \text{---} + \left[\frac{1}{2} \text{---} \circ \text{---} + \dots + \text{---} \circ \text{---} \right] + \mathcal{O}(\lambda^3) \\ Z_\phi|_{1\text{-loop}} = 1 \rightarrow &= \text{---} \times \text{---} + \underbrace{\left[\frac{1}{2} \text{---} \circ \text{---} + \dots + i\lambda(1 - Z_\lambda) \right]}_{=0 \quad (\mu=0)} + \mathcal{O}(\lambda^3) \end{aligned} \quad (7.42)$$

where we now have chosen $\mu = 0$ for the sake of simplicity. This implies $t = s = u = 0$, and the renormalisation condition follows from (7.10f) and (7.42) as

$$1 - Z_\lambda = \frac{3\lambda}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{(q^2 - m^2 + i\epsilon)^2}. \quad (7.43)$$

We can solve the integral on the right hand side of (7.43) with cutoff regularisation, dimensional regularisation or other regularisation schemes as discussed in the case of the mass renormalisation Z_m . Dimensional regularisation is by far the simplest choice, in particular if we would go to $\mu \neq 0$. Moreover, we can simply read off the result from the master integral (7.36) with $n = 2$ and $2 - \frac{d}{2} = \epsilon$. This yields

$$-\frac{3}{2} \lambda \bar{\mu}^{\frac{4-d}{2}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^2} = -\frac{3}{2} \lambda \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\epsilon)}{\Gamma(2)} \left(\frac{m^2}{\bar{\mu}^2} \right)^{-\epsilon} = -\frac{3\lambda}{2} \frac{1}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\bar{\mu}^2} \right), \quad (7.44)$$

where we used the expansion $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$ as in (7.38). In summary we obtain at the renormalisation group scale $\mu = 0$,

$$Z_\lambda = 1 + \frac{3}{2} \frac{\lambda}{16\pi^2} \left(\frac{1}{\epsilon} - \gamma - \ln \frac{m^2}{4\pi\bar{\mu}^2} \right). \quad (7.45)$$

If we want to make the μ -dependence explicit, we have to choose a non-vanishing renormalisation point $p^2 = \mu^2 \neq 0$, and the only change in (7.45) is a respective shift in the mass, $\ln m^2/\bar{\mu}^2 \rightarrow \ln(m^2 + \mu^2)/\bar{\mu}^2$.

This concludes the one-loop renormalisation programme in the ϕ^4 -theory. The results for the renormalisation factors Z_ϕ, Z_m, Z_λ are provided with $Z_\phi = 1$, (7.39) and (7.45).

We remark, that the renormalised correlation functions $\langle \phi(p_1) \phi(p_2) \rangle_{1\text{-loop}}, \langle \phi(p_1) \cdots \phi(p_4) \rangle_{1\text{-loop}}$ are *finite*, but depend on the renormalisation scale μ . Furthermore, for (one-loop) renormalisability we also have to show, that all other correlation functions are finite. For this purpose we consider the six-point function at $p_i = 0$,

$$\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle_{1\text{-loop}} \sim \lambda^3 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2)^3}. \quad (7.46)$$

where we have performed a Wick rotation. The integrand decays with $1/(q^2)^3$ or large loop momenta and hence the integral is finite. This decay is also present for external momenta $p_i^2 \neq 0$ and hence the six point function is finite for all external momenta. Higher correlation functions are also finite as their loop integrands decay with even larger power. Indeed, the integrand of a $2n$ -point function decays with $(1/p^2)^{2n}$ and hence the one-loop contributions for n -point functions are manifestly finite for $n > 4$.

We close the discussion of one-loop renormalisation of the ϕ^4 -theory with the remark, that a singularity in $\langle \phi_0(p_1) \cdots \phi_0(p_6) \rangle_{1\text{-loop}}$ would be disastrous, because there is no counter term for it. Hence, perturbative renormalisability (in ϕ^4 -theory) implies, that all correlation functions *to all orders* in perturbation theory are finite, by adjusting Z_ϕ, Z_m, Z_λ .

7.1.4. The renormalisation group and the running coupling

In the last Chapter we have computed the two- and four-point function, that, together with higher correlation functions, give us access to scattering events in quantum field theories. The renormalisation condition (7.10) also entail, that changing the renormalisation scale μ provides some information about the momentum dependence of physics. This is most apparent from the renormalisation condition for the vertex, (7.10f), where we have set the vertex strength to the interaction strength of the physical scattering process at the momentum scale $p^2 = \mu^2$. This allows us to extract the physical momentum dependence of the interaction strength (*running coupling*) from the renormalisation-scale dependence of the renormalised coupling. While the two are often used as synonyms, the latter cannot be physical as it depends on the renormalisation scale. We have already discussed this below (7.14): observables cannot depend on the renormalisation procedure, and in particular they cannot depend on the renormalisation scale μ . We recast (7.14) in terms of the generating functionals for renormalised correlation functions,

$$\mu \frac{dW[J]}{d\mu} = 0 = \mu \frac{d\Gamma[\bar{\phi}]}{d\mu}. \quad (7.47)$$

We emphasise that (7.47) does not imply that the correlation functions are μ -independent. Indeed, they are as the renormalised fields depend on μ , which is discussed below.

Note also that changing the scale μ is a scale transformation, the logarithmic derivative w.r.t. μ is the generator of (renormalisation) scale transformations, that form a (semi) group, hence the name renormalisation group. Equation (7.47) readily allows for a more general interpretation, the theory or rather its physics predictions should be invariant under general reparametrisations of the theory, and these reparametrisations can be labeled by the scale parameter μ , called the renormalisation group scale. This general point of view incorporates generic renormalisation group setups and is a very versatile framework.

The renormalisation group equations (7.47) or (7.14) should be augmented with a similar one concerning the cutoff dependence. Evidently, the generating functionals do not depend on the cutoff scale Λ . Moreover, the renormalised correlation function are Λ -independent,

$$\Lambda \frac{d}{d\Lambda} \langle T \phi(x_1) \cdots \phi(x_n) \rangle = 0. \quad (7.48)$$

With (7.47) and (7.48) we are now in the position to find concise differential equations that cover the μ -dependence of the renormalised fields, masses and couplings. We start with the observation, that the bare quantities know nothing of the renormalisation point μ ,

$$\mu \frac{d\phi_0}{d\mu} = 0, \quad \mu \frac{dm_0}{d\mu} = 0, \quad \mu \frac{d\lambda_0}{d\mu} \lambda_0 = 0. \quad (7.49)$$

but the renormalisation factors Z_ϕ, Z_m, Z_λ depend on it. Inserting the relation (7.2) between renormalised and bare quantities in (7.49), leads us to

$$\gamma_\phi = \mu \frac{d\phi}{d\mu} \frac{1}{\phi}, \quad \gamma_m = \mu \frac{dm}{d\mu} \frac{1}{m}, \quad \beta_\lambda = \mu \frac{d\lambda}{d\mu} \frac{1}{\lambda}. \quad (7.50)$$

where the anomalous dimensions γ_ϕ, γ_m and the β -function β_λ can be computed from the μ -dependence of the renormalisation factors,

$$2\gamma_\phi = -\mu \frac{dZ_\phi}{d\mu} \frac{1}{Z_\phi}, \quad \gamma_m = -\mu \frac{dZ_m}{d\mu} \frac{1}{Z_m}, \quad \beta_\lambda = -\mu \frac{dZ_\lambda}{d\mu} \frac{1}{Z_\lambda}, \quad (7.51)$$

Equation (7.51) are referred to as either *beta-functions* or *anomalous dimensions*. Specifically, Equations (7.50) and (7.51) allow us to compute the μ -dependence of the renormalised coupling $\lambda(\mu)$ upon integration of the β -function β_λ . Before we proceed with this computation at one loop, we remark, that the cutoff-independence of the renormalised quantities,

$$\Lambda \frac{d}{d\Lambda} \phi = \Lambda \frac{d}{d\Lambda} m = \Lambda \frac{d}{d\Lambda} \lambda = 0, \quad (7.52)$$

implies, that Λ and μ scaling of the renormalisation factors are directly related for asymptotically large scales: they are dimensionless and hence can only depend on ratios of scales. For asymptotically large scales we are left with μ and Λ , all mass and coupling scales can only lead to a subleading behaviour. Accordingly, we have $Z_i = \gamma(\mu/\Lambda)$ for $i = \phi, m, \lambda$.

We close this Chapter with the computation of the running coupling and a discussion of the general structure of our findings. As already indicated above, we first compute the renormalised coupling by integrating its β -function, and then we related its μ -running for asymptotically large scale to the momentum running of the running coupling. The renormalisation group equation for the running coupling is given in (7.50) with (7.51). There, Z_λ was computed for $\mu = 0$, and the general result is obtained by shifting m^2 with μ^2 , see also the remark below (7.50). Hence, the one-loop β -function in the asymptotic scaling regime (or $m \rightarrow 0$) can be read off from (7.45) as $\beta_\lambda = \bar{\mu} \partial_{\bar{\mu}} \log Z_\lambda$. Note also, that at $\mu = 0$ this is also equal to $-m \frac{d \ln Z_\lambda}{dm}$. In any case we arrive at the one-loop β -function,

$$\beta(\mu) = -\mu \frac{d \ln Z_\lambda}{d\mu} = \frac{3}{16\pi^2} \lambda + O(\lambda^2), \quad (7.53)$$

where $O(\lambda^2)$ comprises all the higher loop orders, the n -loop order being proportional to λ^n . Inserting β_λ from (7.53) in the RG-equation for the coupling in (7.51) leads us to

$$\mu \frac{d\lambda}{d\mu} = \frac{3}{16\pi^2} \lambda^2 + O(\lambda^3). \quad (7.54)$$

Equation (7.54) is readily integrated and its solution is the renormalised coupling at one loop as a function of the renormalisation scale,

$$\lambda(\mu) = \frac{\bar{\lambda}}{1 + \frac{3\bar{\lambda}}{16\pi^2} \ln \frac{\mu_{\text{ref}}}{\mu}}, \quad \text{with} \quad \bar{\lambda} = \lambda(\mu_{\text{ref}}). \quad (7.55)$$

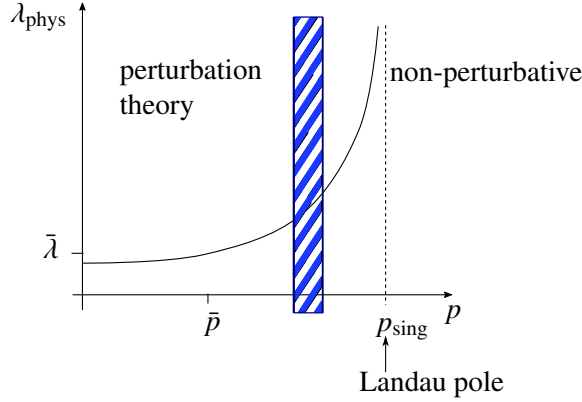


Figure 7.2.: Sketch of the running coupling.

Equation (7.55) is proven correct by insertion into the RG-equation (7.54). Here, μ_{ref} is some reference scale. At $\mu = \mu_{\text{ref}}$, the right hand side reduces to $\bar{\lambda} = \lambda(\mu_{\text{ref}})$, and the coupling increases with the RG-scale μ , dictated by the sign of the β -function.

Finally, we relate the renormalised coupling $\lambda(\mu)$ in (7.55) to the running coupling $\lambda_{\text{phys}}(p)$. With our renormalisation condition, we have fixed the renormalised coupling $\lambda(\mu)$ to the physical running coupling $\lambda_{\text{phys}}(p^2 = \mu^2)$. This allows us, in the absence of further scales, and in particular for asymptotically large momenta, to simply identify $\mu^2 = p^2$ in (7.55), that $\lambda_{\text{phys}}(p) = \lambda(\mu^2 = p^2)$. With (7.55) we arrive at

$$\lambda_{\text{phys}}(p) = \frac{\bar{\lambda}}{1 + \frac{b_0}{2} \frac{\bar{\lambda}}{16\pi^2} \ln \frac{\bar{p}^2}{p^2}}, \quad \text{with} \quad b_0 = 3, \quad (7.56)$$

with the coefficient b_0 of the one-loop β -function. We emphasise that (7.56) does not depend on μ , nor does it depend on Λ . The only input in (7.56) is the value $\bar{\lambda}$ of the coupling at the momentum scale \bar{p} .

The normalisation and the notation of the coefficients b_i of the $i + 1$ -loop coefficients of the β -function is a commonly used one, but by no means unique. Another choice is using $\lambda/(4\pi)$ instead of $\lambda/(16\pi^2)$. The former choice reflects the choice of the fine structure constant $\alpha = e^2/(4\pi)$ as the expansion parameter in QED, or $\alpha_s = g_s^2/(4\pi)$ in QCD with the strong coupling g_s .

Equation (7.56) is linked to the triviality of ϕ^4 -theory: If the theory is a fundamental theory, it has to be well-defined for all momenta, and hence we conclude $\lambda_{\text{phys}}(p) < \infty$ for all p . However, for any finite value $\bar{\lambda}$ for λ_{phys} at some momentum scale \bar{p}^2 , it follows from (7.56), that the running coupling diverges at some finite momentum scale,

$$p_{\text{sing}}^2 = \bar{p}^2 e^{\frac{32\pi^2}{3\bar{\lambda}}}, \quad (7.57)$$

see also **Figure 7.2**. This singularity is called the Landau pole. Lowering the value of $\bar{\lambda}$ shifts the Landau pole towards larger scales. Only for $\bar{\lambda} = 0$ its position is at infinity, and hence it follows that

$$\lambda_{\text{phys}} \equiv 0. \quad (7.58)$$

In turn, if the β -function would be negative,

$$\beta_\lambda = b_0 \frac{\lambda}{16\pi^2}, \quad \text{with} \quad b_0 < 0, \quad (7.59)$$

the running coupling (7.56) would decrease with increasing momenta, finally approaching zero. This behaviour is called *asymptotic freedom* and is observed in QCD, where the one-loop β -function of QCD reads

$$\mu \frac{d\alpha_s}{d\mu} = b_0 \frac{\alpha_s}{4\pi} + O(\alpha_s^2), \quad \text{with} \quad \alpha_s = \frac{g_s^2}{4\pi} \quad \text{and} \quad b_0 = -\left(\frac{22}{3}N_c - \frac{4}{3}N_f\right), \quad (7.60)$$

and the respective running coupling,

$$\alpha_{s,\text{phys}}(p) = \frac{\bar{\alpha}_s}{1 + \frac{b_0}{2} \frac{\bar{\alpha}_s}{4\pi} \ln \frac{\bar{p}^2}{p^2}}, \quad \text{with} \quad \bar{\alpha}_s = \alpha_{s,\text{phys}}(\bar{p}). \quad (7.61)$$

with $\alpha_{s,\text{phys}}(p \rightarrow \infty) = 0$. It is this feature with admits QCD as a fundamental theory, in contradistinction to the ϕ^4 -theory. With $\alpha_{s,\text{phys}}(p \rightarrow \infty) = 0$, the expansion point of perturbation theory, the free theory is achieved, and one can hope for the convergence of the asymptotic series. The one-loop result in (7.61) has been shown by David Gross and Frank Wilczek [3], and David Politzer [4]. It was awarded with the Nobel prize in 2004. Note however, that the perturbative QCD coupling runs into a Landau pole at small momenta, and there QCD can only be treated (numerically) with non-perturbative methods.

We close with the remark, that the persistence of the Landau pole (7.57) beyond one-loop in the full theory can only be proven non-perturbatively, as perturbation theory certainly fails for sufficiently large couplings, see Figure 7.2. The insufficiency of (7.56) for a proof of triviality can already be discussed with the next order of perturbation theory. This leads us to

$$\mu \frac{d}{d\mu} \frac{\lambda}{16\pi^2} = b_0 \left(\frac{\lambda}{16\pi^2} \right)^2 + b_1 \left(\frac{\lambda}{16\pi^2} \right)^3, \quad \text{with} \quad b_0 = 3, \quad b_1 = -\frac{17}{3}. \quad (7.62)$$

with the one-loop and two-loop coefficients b_0 and b_1 respectively. If (7.62) would capture the full running of the coupling at least qualitatively, the coupling would rise with momenta p until it saturates at a so-called UV fixed point with the value $\lambda_*^{(\text{UV})}$,

$$\lambda_*^{(\text{UV})} = -\frac{b_0}{b_1} = \frac{9}{17} 16\pi^2. \quad (7.63)$$

Equation (7.63) looks intriguing, but the fixed point value λ_* is large, and even that of $\lambda_*/(16\pi^2)$ is of order one. Indeed, the alternating sign is not surprising and a common feature of Taylor expansions, a trivial example being the geometric series $1/(1-x) - 1 = x - x^2 + O(x^3)$. For the ϕ^4 -theory, the next orders in perturbation theory and fully non-perturbative computation with either the lattice or non-perturbative diagrammatic methods do not support the fixed point behaviour but the steady increase of the coupling, and the ϕ^4 -theory most probably suffer from a physical Landau pole.

However, if a ultraviolet fixed point such as (7.63) is present in the full theory, this theory is called *asymptotically safe*. While such a theory does not run into a perturbative regime for asymptotic large momenta, its coupling does not diverge by stays finite. One may hope, that then a (perturbative) expansion about the interacting theory or fixed point in powers of $\lambda_*^{(\text{UV})} - \lambda$ works. This scenario encompasses asymptotic freedom as a special case. Asymptotic safety may be realised in quantum gravity with a non-trivial fixed point for the Newton constant. It also is realised in model theories.

7.1.5. A glimpse at two loops and loop computations

In the last section we discussed the potential impact of the two-loop β -function on the physics of the ϕ^4 -theory, or more general, the quantum field theory at hand. At one-loop the RG-equation of the coupling was equivalent to that of the vertex as the wave function renormalisation was trivial, $Z_\phi = 1$. In general this is not correct and the vertex incorporates the renormalisation of the coupling *and* the field: the vertex in the Feynman rules of the renormalised theory (7.15) has the counter term $(1 - Z_\phi^2 Z_\lambda) \lambda$. Accordingly, the amputated correlation function (7.10e) with the renormalisation condition (7.10f) has a singular part $Z_{\phi^4} = Z_\phi^2 Z_\lambda$. Consequently, the RG-equation of the coupling reads

$$\beta_\lambda = -\mu \left[\frac{d}{d\mu} \frac{Z_{\phi^4}}{Z_\phi^2} \right] \frac{Z_\phi^2}{Z_{\phi^4}} = -\mu \frac{dZ_{\phi^4}}{d\mu} \frac{1}{Z_{\phi^4}} + 2\mu \frac{dZ_\phi}{d\mu} \frac{1}{Z_\phi} = \gamma_{\phi^4} - 4\gamma_\phi, \quad \gamma_{\phi^4} = -\mu \frac{dZ_{\phi^4}}{d\mu} \frac{1}{Z_{\phi^4}}, \quad (7.64)$$

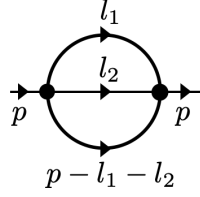


Figure 7.3.: Sunset diagram, contributing to the momentum-dependence of the (inverse) propagator at two-loop.

where γ_{ϕ^4} is the anomalous dimension of the vertex. Equation (7.64) entails, that only for $\gamma_{\phi} = 0$ the β -function β_{λ} agrees with the anomalous dimension of the vertex, while in general the latter also encodes the running of the fields attached. Hence, we have to compute both, the RG-running of the vertex and the anomalous dimension of the field for the RG-running of the coupling. Only at one-loop in the ϕ^4 -theory this reduces to the computation of the vertex running. Here we only sketch the computation of the two-loop anomalous dimension. The inverse propagator only has one one-loop contribution, the sunset diagram, see Figure 7.3. The self-energy is defined with

$$\langle \phi\phi \rangle(p) = \frac{i}{p^2 - m^2 - \Pi(p) + i\epsilon}, \quad (7.65)$$

and is the quantum part of the 1PI two-point function. We have already discussed the mass correction, and at two-loop the self-energy is given by

$$\Pi_0(p) \simeq -\frac{\lambda^2}{6} \int \frac{d^4 l_1}{(4\pi)^4} \frac{d^4 l_2}{(4\pi)^4} \frac{1}{l_1^2 - m^2 + i\epsilon} \frac{1}{l_2^2 - m^2 + i\epsilon} \frac{1}{l_3^2 - m^2 + i\epsilon}, \quad \text{with } l_3 = p - l_1 - l_2, \quad (7.66)$$

with the loop momenta l_1 and l_2 . The subscript $_0$ indicates that (7.66) is not renormalised yet, and we dropped constant parts related to two-loop tadpole corrections. As in the one-loop case the respective renormalisation can be done implicitly. The mass on-shell renormalisation condition (7.10a), enforce that the mass counterterm has to cancel the loop contribution for $p^2 = m^2$. Accordingly, we are led to

$$\Pi_0(p) \rightarrow \Pi_0(p) - \Pi_0(p^2 = m^2), \quad (7.67)$$

which eliminates the quadratic divergence and guarantees (7.10a).

It is left to enforce (7.10b). To that end we compute the momentum-dependence of $\Pi_0(p)$ and take a p^2 -derivative. The latter step annihilates the subtraction $\Pi_0(p^2 = m^2)$ and we ignore this term from now on. First we use the Feynman trick to reduce the integral to one with a uniform denominator,

$$\prod_{i=1}^n \frac{1}{A_i} = (n-1)! \left[\prod_{i=1}^n \int_0^1 d\alpha_i \right] \frac{\delta(1 - \sum_{i=1}^n \alpha_i)}{(\sum_{i=1}^n \alpha_i A_i)^n}. \quad (7.68)$$

In the present case with $n = 3$ we obtain for the integrand of (7.66),

$$\Pi_0(p) \simeq -\frac{\lambda^2}{6} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int \frac{d^4 l_1}{(4\pi)^4} \frac{d^4 l_2}{(4\pi)^4} \frac{2\theta(1 - \alpha_1 - \alpha_2)}{(\alpha_1 l_1^2 + \alpha_2 l_2^2 + (1 - \alpha_1 - \alpha_2) l_3^2 - m^2 + i\epsilon)^3}. \quad (7.69)$$

The iterative solution of the loop integral requires a further master integral, that can be derived readily from the master integral (7.36), which yields in Euclidean space,

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + 2pq + m^2)^n} = \frac{\Gamma(n - \frac{d}{2})}{(4\pi)^{d/2} \Gamma(n)} \frac{1}{(m^2 - p^2)^{n-d/2}}. \quad (7.70)$$

The left hand side of (7.70) can be reduced to the form of the master integral (7.36) by using $q^2 + 2pq + m^2 = (q + p)^2 + (m^2 - p^2)$ and the right hand side of (7.70) follows readily. Equation (7.70) can now be used twice on (7.69), keeping track of the α_i -dependent prefactors.

Here we follow a slightly different route and perform the 'diagonalisation' in loop momentum, that was key for the derivation of (7.70) from (7.36), directly on the integrand in (7.69). We arrive at

$$\alpha_1 l_1^2 + \alpha_2 l_2^2 + (1 - \alpha_1 - \alpha_2) l_3^2 = b_1 q_1^2 + b_2 q_2^2 + b_3 p^2, \quad (7.71)$$

with the new momentum variables

$$q_1 = l_1 + \frac{\alpha_3}{\alpha_1 + \alpha_3} (l_2 - p), \quad q_2 = l_2 - \frac{\alpha_1 \alpha_3}{\alpha_1 \alpha_3 + \alpha_2 (\alpha_1 + \alpha_3)} p, \quad (7.72)$$

with $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and the coefficients

$$b_1 = \alpha_1 + \alpha_3, \quad b_2 = \alpha_2 + \frac{\alpha_1 \alpha_3}{\alpha_1 + \alpha_3}, \quad b_3 = \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}. \quad (7.73)$$

We proceed by inserting (7.71) into the self-energy and rewriting the l_1, l_2 -integrals as q_1, q_2 -integrals, where we use that the respective Jacobian is unity as the transformation matrix is triangular, see (7.72). Taking also the p^2 -derivative as in the renormalisation condition (7.10b) leads us to

$$\frac{\partial \Pi_0(p)}{\partial p^2} = \frac{\lambda^2}{6} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int \frac{d^4 q_1}{(4\pi)^4} \frac{d^4 q_2}{(4\pi)^4} \frac{2b_3 \theta(1 - \alpha_1 - \alpha_2)}{(b_1 q_1^2 + b_2 q_2^2 + b_3 p^2 - m^2 + i\epsilon)^4}. \quad (7.74)$$

The loop integrals in (7.74) can be rotated into Euclidean space and are solved with the master integral (7.36). After a tedious but straightforward computation we are led to

$$\frac{\partial \Pi_0(p)}{\partial p^2} = -\frac{1}{12} \left[\frac{\lambda}{16\pi^2} \right]^2 \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \theta(1 - \alpha_1 - \alpha_2) \frac{b_3}{(b_1 b_2)^2} \left(\frac{1}{\epsilon} + 2 \ln \frac{4\pi \bar{\mu}^2}{m^2} - 2\gamma + \ln \frac{b_1 b_2}{\left(1 - b_3 \frac{p^2}{m^2}\right)^2} \right). \quad (7.75)$$

A further simplification is obtained by going to the renormalisation point with $p^2 = m^2$. Then, the last term in parenthesis in (7.75) reduces to $\ln b_1 b_2 / (1 - b_3)^2$. In a final step we solve the two α_i integrals with $i = 1, 2$ and arrive at our final result for the p^2 -derivative of the self-energy,

$$\left. \frac{\partial \Pi_0(p)}{\partial p^2} \right|_{p^2=m^2} = -\frac{1}{24} \left[\frac{\lambda}{16\pi^2} \right]^2 \left(\frac{1}{\epsilon} + 2 \ln \frac{4\pi \bar{\mu}^2}{m^2} - 2\gamma - \frac{3}{2} \right), \quad (7.76)$$

which also confirms the previous statement about the expansion parameter being $\lambda/(16\pi^2)$. Now we insert (7.76) in the renormalisation condition (7.10b) and determine the wave function renormalisation Z_ϕ and the anomalous dimension: The sum of the p^2 -derivative of the self-energy and the counter term contribution $1 - Z_\phi$ have to sum up to zero. We conclude with (7.23),

$$Z_\phi = 1 - \frac{1}{24} \left[\frac{\lambda}{16\pi^2} \right]^2 \left(\frac{1}{\epsilon} + 2 \ln \frac{4\pi\bar{\mu}^2}{m^2} - 2\gamma - \frac{3}{2} \right). \quad (7.77)$$

In a final step we compute the (asymptotic) anomalous dimension γ_ϕ from its $\bar{\mu}$ -dependence,

$$\gamma_\phi = \frac{1}{2} \bar{\mu} \frac{dZ_\phi}{d\bar{\mu}} = \frac{1}{24} \left[\frac{\lambda}{16\pi^2} \right]^2. \quad (7.78)$$

Note that often the factor 1/2 in the definition of the anomalous dimension in (7.78) is dropped.

7.2. Renormalisability

In this Chapter we provide a brief discussion of the consistency of the renormalisation programme at all loop orders in both, the ϕ^4 -theory and in QED. In particular, this requires that no further divergence is produced. The mathematical proofs of perturbative renormalisability are rather involved and are not subject of the present lecture course. Instead we concentrate on the general counting of singularities. This is facilitated in the presence of a mass, as otherwise we also would have to discuss infrared singularities.

7.2.1. Renormalisability of the ϕ^4 theory in d dimensions

Ultraviolet divergences in loop integrals in the ϕ^4 -theory are present if the sum D of the momentum dimension of the integrand and the loop integrations is positive. The number D is called the *superficial degree of divergence*, and only counts the overall divergence. A full proof of renormalisability also has to take into account potential sub-divergences triggered by a sub-diagram in the diagram considered. Then it also has to be shown, that if present, these subdivergences can be absorbed into the renormalisation constants $Z_\phi, Z_\psi, Z_\lambda$. It is this aspects, which requires much work.

If we have $D < 0$, the diagram is finite while for $D \geq 0$ the diagram is ultraviolet divergent. The respective operator with a positive (total) momentum dimension is called (UV) *relevant*, that with a vanishing momentum dimension are called *marginal* with *logarithmic* divergences, and that with a negative momentum dimension are called *irrelevant*. We emphasise that presently we count canonical momentum dimensions, but the total momentum dimension also takes into account the anomalous scaling.

The latter integrations have the momentum dimension d and in general we have L of them. The P propagators have the momentum dimension -2 and we consider general diagrams with N external lines. With this information we can link the number of loops L to that of the vertices V and the propagators P : each propagators comes with a momentum integration and each Vertex comes with a momentum conserving δ -function. However, there is also the momentum conservation for the external momenta, and we arrive at

$$L = P - V + 1. \quad (7.79)$$

Moreover, there are 4 lines emanating from each vertex and we find

$$4V = N + 2P, \quad (7.80)$$

as the propagators are connected to two lines from the vertex. We note that (7.79) is independent of the number n of lines emanating from a vertex, and (7.80) is readily generalised with $4 \rightarrow n$.

Finally, the degree of divergence D in the ϕ^4 -theory in d space-time dimensions is obtained as

$$D = dL - 2P = d + \left[4 \frac{d-2}{2} - d \right] V - \frac{d-2}{2} N. \quad (7.81)$$

where the second relation follows by inserting (7.79) and (7.80) into $dL - 2P$. The factor $d_P = 2$ in front of the number of propagators P is the momentum dimension of its dispersion or kinetic operator \mathcal{P} , in the scalar field theory $\mathcal{P} = \partial_\mu \partial^\mu$. This leads to the counting factor $-d_P$ for the propagator. For a fermion we find $d_P = 1$ from its dispersion $\mathcal{P} = \gamma_\mu \partial^\mu$. In general we have

$$d_P = -[S_F], \quad (7.82)$$

where S_F is the propagator of the scalar field and $[O]$ is the momentum or mass dimension of the operator or parameter O . The $+d$ in front of the number of loops L is the dimension of the momentum integral.

We can easily generalise (7.81) to a theory with a ϕ^n -interaction, for example the ϕ^3 -theory with $n = 3$. With $4 \rightarrow n$ we deduce from (7.81)

$$D = dL - 2P = d + \left[n \frac{d-2}{2} - d \right] V - \frac{d-2}{2} N. \quad (7.83)$$

For the interpretation of (7.81) and (7.83) we note that the dimension $[\phi]$ of the field ϕ , that of the dispersion, d_P in (7.82), and the dimension $[\lambda]$ of the coupling are linked with

$$[\phi] = \frac{d - d_P}{2}, \quad [\lambda] = d - n[\phi] = d - n \frac{d - d_P}{2}. \quad (7.84)$$

With (7.84) we rewrite (7.83) in terms of the dimension of the coupling and the field,

$$D = d - [\lambda]V - [\phi]N. \quad (7.85)$$

Coming back to the ϕ^4 -theory in $d = 4$ space-time dimensions, the relation (7.81) is given by

$$D = d - N. \quad (7.86)$$

The term with the number of vertices V in (7.83) has dropped out in $d = 4$ dimensions as the prefactor vanishes. This is related to the vanishing momentum dimension of the coupling λ in $d = 4$ dimensions. In short, the superficial degree of divergence D in the ϕ^4 -theory decays with the number of external legs. Apart from $D = 4$ for vacuum diagrams, we have the quadratic divergence of the two-point function (Z_m), which also contains a logarithmic divergence for its p^2 -derivative (Z_ϕ). The latter is easily seen as the p^2 -derivative of a propagator has a momentum dimension -4 . Finally, we have the logarithmic divergence of the 4-point function (Z_λ).

In $d > 4$ the expression in the square bracket is larger than zero: $d - 4 > 0$. In this case the coupling has a negative momentum dimension $[\lambda] = 4 - d$. In this case we encounter more and higher superficial divergences at each loop order. Such a theory is perturbatively *non-renormalisable*. In turn, for $d < 4$ only the mass requires renormalisation. One also can show that there is only a finite number of divergent diagrams. Such a theory is called *super-renormalisable*. Finally, in $d = 4$ dimensions the theory has a dimensionless coupling and is in its *critical* dimension.

7.2.2. Renormalisability of QED in d dimensions

A similar analysis as done in Section 7.2.1 for the ϕ^4 -theory can be done for QED: For the sake of simplicity we restrict ourselves to the photon and the electron. Then the diagrams contain P_γ photon and P_e electron propagators and N_γ, N_e external legs. The relations for the loop order, (7.79), the counting of vertices, (7.80), and the superficial degree of divergence in d dimensions are given by

$$\begin{aligned} L &= P_e + P_\gamma - V + 1 \\ V &= 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e), \\ D &= dL - P_e - 2P_\gamma = d + \frac{d-4}{2}V - \frac{d-2}{2}N_\gamma - \frac{d-1}{2}N_e. \end{aligned} \quad (7.87)$$

From the prefactor of the vertex number V in (7.87) we conclude that the critical dimension of QED is $d = 4$ as for the ϕ^4 -theory. For $d = 4$, the superficial degree of divergence in (7.87) reduces to

$$D = 4 - N_\gamma - \frac{3}{2}N_e. \quad (7.88)$$

Moreover, the correlation functions with a superficial degree of divergence are the electron two-point function with $N_\gamma = 0, N_e = 2$ and $D = 1$, the photon two-point function with $N_\gamma = 2, N_e = 0$ and $D = 2$, and finally the electron-photon vertex with $N_\gamma = 1, N_e = 2$ and $D = 0$. Potentially, this leaves us with mass renormalisation conditions for electron ($D=1$) and photon ($D=2$), wave function renormalisations for electron and photon ($D=0$ in both cases), and the vertex renormalisation ($D=0$) via the coupling renormalisation. We shall see in the next Chapter, Section 7.3, that the mass renormalisation condition for the photon drops out.

Furthermore, we conclude that in $d > 4$, QED is not perturbatively renormalisable, while in $d < 4$ the theory is super-renormalisable. We emphasise once more, that the difficult part in proofs of renormalisability is not the check of a positive superficial degree of divergence $D > 0$ is only present for a finite number of correlation functions. The difficult part is to show that the divergences are local and can be absorbed in the renormalisation constants in the classical action. Moreover, this iterative (in orders of perturbation theory) procedure renders all sub-diagrams in a given diagram finite.

7.3. QED

In this Chapter we put to work the general renormalisation programme for QED, and we restrict ourselves to the electron, leaving out the muons and the τ -lepton. In Section 7.2.2 we have already shown that QED is in its critical dimension in $d = 4$ dimensions with a dimensionless coupling. We introduce the renormalisation constants for the electron and photon fields, the electron mass and the coupling via the QED action (6.1a), written in terms of the bare fields and coupling parameters,

$$A_{0\mu}, \quad \psi_0, \bar{\psi}_0, \quad m_0, \quad e_0, \quad \xi_0, \quad (7.89)$$

and the bare gauge fixing parameter ξ_0 . The respective bare action is given by

$$S_{\text{QED}}[A_0, \psi_0, \bar{\psi}_0] = \int d^4x \bar{\psi}_0 (i\mathcal{D} - m_0) \psi_0 - \frac{1}{4} \int d^4x F_{\mu\nu}(A_0) F^{\mu\nu}(A_0) - \frac{1}{2\xi_0} \int d^4x (\partial_\mu A_0^\mu)^2, \quad (7.90)$$

with $D_\mu = \partial_\mu - ie_0 A_{0\mu}$ and $\psi_0 = \psi_{0e}$. The action (without the gauge fixing term) is gauge invariant under

$$A_{0\mu} \rightarrow A_{0\mu} + \frac{1}{e_0} \partial_\mu \alpha \quad \psi_0 \rightarrow e^{-i\alpha} \psi_0, \quad (7.91)$$

of the bare fields $A_{0\mu}$ and ψ_0 , see also (6.2). We introduce the respective finite renormalised fields and coupling parameters analogously to that in the ϕ^4 -theory in (7.2) with

$$A_{0\mu} = Z_A^{1/2} A_\mu, \quad \psi_0 = Z_\psi^{1/2} \psi, \quad m_0 = Z_m m, \quad e_0 = Z_e e, \quad \xi_0 = Z_\xi \xi. \quad (7.92)$$

With the relations (7.92), the QED action takes the form

$$S_{\text{QED}}[A, \psi, \bar{\psi}] = \int d^4x Z_\psi \bar{\psi} (i\mathcal{D} - Z_m m) \psi - \frac{Z_A}{4} \int d^4x F_{\mu\nu}(A) F^{\mu\nu}(A) - \frac{Z_A}{Z_\xi} \frac{1}{\xi} \int d^4x (\partial_\mu A^\mu)^2, \quad (7.93a)$$

with

$$D_\mu = \partial_\mu - Z_e Z_A^{1/2} i e A_\mu. \quad (7.93b)$$

Note that in (7.93a) we have used $S_{\text{QED}}[A_\mu, \psi]$ with the renormalised fields with a slight abuse of notation. Equation (7.93b) poses a problem insofar that the covariant derivative in terms of the finite renormalised gauge

field and the finite renormalised coupling contains the product of the cutoff-dependent renormalisation constants Z_e and $Z_A^{1/2}Z_e$. Moreover, *physical gauge invariance* is obtained in terms of the renormalised quantities and hence *physical gauge invariance* should apply to renormalised quantities, with the covariant derivative

$$D_\mu = \partial_\mu - i e A_\mu . \quad (7.94)$$

Both, (7.93b) and the gauge symmetry in the renormalised quantities with (7.94) enforce

$$\mu \frac{d}{d\mu} (Z_A^{1/2} Z_e) = 0 , \quad \leftrightarrow \quad \mu \frac{d}{d\mu} e A_\mu = 0 . \quad (7.95)$$

Equation (7.95) and similar relations for correlation functions are called *Ward-Takahashi identities*. Note that the above simply argument is implicitly based on the simple additive structure of the gauge invariant F^2 term and the gauge fixing term. To that end we evaluate the difference of the expectation value of a gauge transformation of the gauge-fixed action and that without a gauge transformation,

$$\langle S_{\text{QED}}[A^\alpha, \psi^\alpha, \bar{\psi}^\alpha] - S_{\text{QED}}[A, \psi, \bar{\psi}] \rangle = -\frac{Z_A}{Z_\xi} \frac{1}{\xi} \int d^4x \langle \partial_\rho A^\rho \rangle \partial_\rho \partial^\rho \alpha . \quad (7.96)$$

The right hand side vanishes on the gauge fixing condition and in any case it simply is the gauge transformation of the gauge fixing term of the mean field $\langle A_\mu \rangle$: It does not involve any higher order correlation functions. We conclude, that (7.95) holds for general linear gauge fixings. Moreover, (7.96) also entails that the gauge fixing term receives no quantum corrections, leading to

$$Z_\xi = Z_A . \quad (7.97)$$

We derived the absence of quantum corrections for the longitudinal two-point functions expressed by (7.97) from gauge invariance of the action. Indeed, any correction to the longitudinal part of the two point function violates gauge invariance, but specifically this holds true for a mass term. Hence, we already conclude, that a gauge invariant regularisation scheme such as dimensional regularisation has to preserve transversality of the quantum corrections to the inverse propagator. In [Section 7.3.1](#) this is confirmed within an explicit one-loop computation. A welcome benefit is, that the mass renormalisation of the photon drops out and we are only left with renormalisation conditions for the electron mass, the electron and photon wave functions and the vertex. Note that the present rather heuristic argument for (7.95) will be derived from the effective action in a systematic way in QFT II.

In non-Abelian gauge theories the non-linear nature of the gauge transformations complicates matter, and (7.95) does not hold any more, as (7.96) fails due to the non-linearity of the gauge transformation in field space. The underlying symmetry of the gauge fixed theory is the Becchi-Rouet-Stora-Tyutin (BRST) symmetry, and is encoded in terms of Slavnov-Taylor identities, for more details see [Chapter 13](#) in [Section 13.4](#). They can be written concisely in terms of the effective action Γ introduced in (7.20) in [Section 7.1.2](#), the BRST master equations, for example the Zinn-Justin equation (13.151).

We proceed with the Feynman rules in terms of renormalised quantities, that can be read off from (7.93).

Feynman rules of QED for renormalised fields

Propagators:

$$\left[\text{---} \xrightarrow{p} \text{---} \right]^{-1} = \frac{1}{i} Z_\psi (\not{p} - Z_m m) = \left[i \frac{\not{p} + m^2}{p^2 - m} \right]^{-1} - \text{---} \otimes \text{---} ,$$

where:

$$\text{---} \otimes \text{---} = -i(1 - Z_\psi) \not{p} + i(1 - Z_\psi Z_m) m .$$

$$\begin{aligned} \left[\text{---} \xrightarrow{k} \text{---} \right]_{\mu\nu}^{-1} &= i Z_A \left[k^2 \eta_{\mu\nu} - k_\mu k_\nu \left(1 - \frac{1}{Z_A \xi} \right) \right] \\ &= \left[-\frac{i}{k^2} \left(\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \right]^{-1} - \text{---} \otimes \text{---} , \end{aligned}$$

where:

$$\text{---} \otimes \text{---} = i(1 - Z_A)(k^2 \eta_{\mu\nu} - k_\mu k_\nu) .$$

↑
only transversal modes
get renormalised

Vertex:

$$\text{---} \text{---} \text{---} \gamma = i Z_\psi Z_A^{1/2} Z_e e \gamma_\mu = i e \gamma_\mu + \text{---} \otimes \text{---} ,$$

where:

$$\text{---} \otimes \text{---} = -i e \gamma_\mu (1 - Z_\psi Z_A^{1/2} Z_e) . \tag{7.98}$$

This concludes the discussion of renormalised perturbation theory in QED.

7.3.1. One-loop renormalisation in QED and the running fine-structure constant

QED features a running coupling, typically expressed in terms of the fine-structure constant $\alpha(p)$. On one-loop its momentum dependence can be inferred from the μ -dependence of the renormalised

$$\alpha_{\text{ren}}(\mu) = \frac{e^2(\mu)}{4\pi} . \tag{7.99}$$

The RG-running of $e(\mu)$ can be computed from its β -function, and with (7.92) we relate the renormalisation group running of the fields and renormalised coupling e and electron mass m to that of the respective renormalisation constants,

$$\gamma_A = -\frac{1}{2}\mu \frac{dZ_A}{d\mu} \frac{1}{Z_A}, \quad \gamma_\psi = -\frac{1}{2}\mu \frac{dZ_\psi}{d\mu} \frac{1}{Z_\psi}, \quad \beta_e = -\mu \frac{dZ_e}{d\mu} \frac{1}{Z_e}, \quad (7.100)$$

The respective renormalisation conditions are given by

Renormalisation conditions in QED

$$-i\frac{1}{3} \frac{\partial}{\partial p^2} \left[\langle T A_\mu A_\nu \rangle^{-1}(p) \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \right] \Big|_{p^2=\mu^2} = 1, \quad (7.101a)$$

$$i\frac{1}{4} \frac{1}{p^2} \text{tr} \not{p} \left[\langle T \psi \bar{\psi} \rangle(p) \right]^{-1} \Big|_{p^2=\mu^2} = 1, \quad (7.101b)$$

$$-i\frac{1}{4} \text{tr} \left[\langle T \psi \bar{\psi} \rangle(p) \right]^{-1} \Big|_{p^2=\mu^2} = m, \quad (7.101c)$$

$$-i\frac{1}{4d} \left[\langle T \bar{\psi} \gamma^\mu A_\mu \psi \rangle_{\text{1PI}}(p_1, p_2, p_3) \right] \Big|_{p_i^2=\mu^2} = e, \quad (7.101d)$$

where we have chosen a general renormalisation point μ and $p_3 = -(p_1 + p_2)$ in (7.101d). The vertex is evaluated at the symmetric point with $p_i^2 = \mu^2$ with $i = 1, 2, 3$ with all momenta counted incoming and $p_i \cdot p_j = (3\delta_{ij} - 1)p^2/2$. We emphasise that for $\mu^2 \neq m_{\text{pole}}^2$, the mass m in (7.101d) is not the pole mass, but has to be tuned such that the electron propagator diverges for $p^2 = m_{\text{pole}}^2$.

Before we come to applications, we write the renormalisation conditions (7.101) in an explicit form. To that end we first parametrise the full electron propagator $G_A(p)$ and photon propagators $G_\psi(p)$

$$G_{A\mu\nu}(p) = \langle T A_\mu A_\nu \rangle(p), \quad G_{\psi\eta\bar{\eta}}(p) = \langle T \psi_\eta \bar{\psi}_{\bar{\eta}} \rangle(p), \quad (7.102a)$$

where $G_A(p^2)$, and $G_\psi(p^2)$ are the two-point correlation functions without the δ -function $(2\pi)^4 \delta(p+q)$ which carries momentum conservation,

$$\langle T A_\mu(q) A_\nu(p) \rangle = \langle T A_\mu A_\nu \rangle(p) (2\pi)^4 \delta(p+q). \quad (7.102b)$$

We parametrise the propagators now in terms of dressing functions that carry their non-trivial momentum dependence,

$$G_A(p) = -i \frac{1}{p^2} \left[D_A(p^2) \Pi_\perp + \xi \Pi_L \right] \quad G_\psi(p) = i \frac{1}{A_\psi(p^2)} \frac{\not{p} + M_\psi(p^2)}{p^2 + M_\psi^2(p^2)}, \quad (7.102c)$$

with the longitudinal and transverse projection operators

$$\Pi_{\perp\mu\nu}(p) = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \quad \Pi_{L\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}, \quad (7.103)$$

with

$$\Pi_{\perp/L}^2 = \Pi_{\perp/L}, \quad \Pi_\perp + \Pi_L = \mathbb{1} \quad \Pi_\perp \cdot \Pi_L = 0. \quad (7.104)$$

For $D_A(p^2), A_\psi(p^2) \rightarrow 1$ and $M_\psi(p^2) \rightarrow m$ the full propagators reduce to the classical *finite* renormalised ones: $G_A(p) \rightarrow D_F(p)$ in (5.41) and $G_\psi(p) \rightarrow S_F(p)$ in (4.129). In turn, the momentum-dependence of the dressing functions $D_A(p^2), A_\psi(p^2), M_\psi(p^2)$ in (7.102c) originates in the loops. The renormalisation conditions (7.101) use the inverse propagators, which are readily obtained from

$$G_A^{-1}(p) = i \left[\frac{p^2}{D_A(p^2)} \Pi_\perp + \frac{1}{\xi} p^2 \Pi_L \right], \quad G_\psi^{-1}(p) = -i A_\psi(p^2) [\not{p} - M_\psi(p^2)]. \quad (7.105)$$

Let us now reformulate the renormalisation conditions (7.101a) to (7.101c) for the inverse propagators as conditions for the respective dressings.

We start with the photon two-point function: its parametrisation in (7.105) already contains non-trivial information: the longitudinal part is trivial, it is simply the classical gauge fixing term, which receives no quantum corrections. The latter only contribute to the transverse part and are comprised in $1/D(p^2)$. The function $D(p^2)$ is also called the dressing function of the propagator, as it 'dresses' the classical one. The transversality of the quantum corrections is a consequence of gauge invariance as carried by the Ward identity,

$$p_\mu \left[G_A^{-1}{}_{\mu\nu}(p) - i \frac{1}{\xi} p^2 p_\nu \right] = 0. \quad (7.106)$$

If we contract $G_A^{-1}(p)$ with the transverse projection operator, we obtain

$$G_A^{-1}{}_{\mu\nu}(p) \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) = 3i \frac{p^2}{D(p^2)}, \quad (7.107)$$

and the renormalisation condition (7.101a) takes the form

$$\left[\frac{1}{D_A(p^2)} + p^2 \frac{\partial D_A^{-1}(p^2)}{\partial p^2} \right]_{p^2=\mu^2} = 1. \quad (7.108)$$

A preferred choice is $\mu = 0$ and (7.108) reduces to $D_A(0) = 1$, fixing the amplitude of the propagator at its pole to one. With (7.108), the physics content of the renormalisation condition is apparent, it fixes the amplitude of the propagator at $p^2 = \mu^2$. Hence, apart from removing the singularities, Z_A or rather its finite part adjusts the global amplitude of the photon propagator.

The inverse fermion propagator $G_\psi^{-1}(p)$ in (7.105) depends on the momentum-dependent functions $A_\psi(p^2)$ and $M_\psi(p^2)$. While the former contains the full momentum dependence of the Dirac part of the propagator, the latter is the mass function, and the pole position is determined by

$$m_{\text{pole}}^2 = M_\psi^2(m_{\text{pole}}^2). \quad (7.109)$$

Note that such a split is unique for the Dirac fermion due to the Dirac tensor structure of the kinetic term and the scalar tensor structure of the mass term.

Inserting the parametrisation (7.105) for G_ψ^{-1} into the second renormalisation condition, (7.101b), we readily obtain

$$A_\psi(\mu^2) = 1, \quad (7.110)$$

analogously to the renormalisation condition for the wave function of the photon, (7.108). The wave function renormalisation Z_ψ adjusts the global amplitude of the fermion propagator, apart from removing the singularities in the diagrams that are proportional to the Dirac structure. Note that we could have also chosen a division by $1/p^2$ instead of a derivative for the photon renormalisation condition (7.108). However, for $\mu = 0$, the two conditions agree anyway. Moreover, for massive bosons, using the derivative is far more convenient.

The fermion propagator also requires a renormalisation condition for the mass, (7.101c). With (7.105) we obtain

$$M_\psi(\mu^2) = m^2, \quad (7.111)$$

and for on-shell renormalisation with $\mu^2 = m_{\text{pole}}^2$ we are led to (7.109). The counter term $i(1 - Z_\psi Z_m)m$ in $M(p^2)$ is used both for removing potential singularities from the diagrams as well as adjusting (7.101c). As Z_ψ is already fixed by (7.110), Z_m is fixed by (7.111).

In summary we have shown, that the renormalisation conditions (7.101a) and (7.101b) simply fix the overall amplitude of the photon and electron propagator, and as such do not determine physical parameters such as the couplings or the masses. In contradistinction, (7.101c) adjusts the physical electron mass.

The renormalisation condition for the vertex (or vertices in general) is slightly more complicated, as the electron-positron-photon three-point function not only has a contribution that is proportional to the classical tensor structure γ_μ , but also further ones, i.e. that of the Landé factor, see process vi) in (6.6). This makes the projection on the classical tensor structure non-trivial. For example, if contracting the three-point function with γ_μ , this process drops out, but a (potential) part of the vertex being proportional to $\gamma^\rho \sigma^{\mu\nu}$, remains. In any case, the vertex has a part that is proportional to the classical tensor structure and we write

$$\langle T \bar{\psi}_\eta A^\mu \psi_\eta \rangle_{\text{1PI}}(p_1, p_2) \simeq -ie\gamma_{\eta\bar{\eta}}^\mu V(p_1, p_2) + \dots \quad (7.112)$$

where \dots stands for the other tensor structures, and p_1 and p_2 are the incoming momenta of fermion ψ and gauge field A_μ . Due to momentum conservation at the vertex we have $p_3 = -(p_1 + p_2)$. The scalar function $V(p_1, p_2)$ is the dressing function of the classical tensor structure analogously to the dressings $D(p)$, $A(p)$, $M(p)$ of the propagators. As for the propagators, we have dropped the δ -function $(2\pi)^4 \delta(p_1 + p_2 + p_3)$, that carries total momentum conservation at the vertex. For $V \equiv 1$, this term reduces to the classical *finite* renormalised vertex. We project on this term with

$$\langle T \bar{\psi} \gamma_\mu A^\mu \psi \rangle_{\text{1PI}}(p_1, p_2, p_3) \simeq ieV(p_1, p_2, p_3) \text{tr} \gamma_\mu \gamma^\mu = ieV(p_1, p_2, p_3) 4d. \quad (7.113)$$

Inserting (7.113) into (7.101d) leads us to

$$V(p_1, p_2, p_3)|_{p_i^2 = \mu^2} = 1, \quad (7.114)$$

with $p_i^2 = \mu^2$ for $i = 1, 2, 3$, all momenta counted incoming and $p_i \cdot p_j = (3\delta_{ij} - 1)p^2/2$. The vertex dressing V includes potential regularised singularities from the diagrams as well as the counter term $-ie\gamma_\mu(1 - Z_\psi Z_A^{1/2} Z_e)$. In dimensional regularisation the singularity is a term $\text{const.}/\epsilon$ which can be removed by an appropriate choice of the counter term or rather Z_e , as the other two constants have already been fixed with (7.108) and (7.110). Z_e can be also used to reduce $V(p_1, p_2, p_3)$ can be reduced to unity at the renormalisation point, leading us to (7.114).

In summary, the renormalisation constants Z_A, Z_ψ, Z_m, Z_e are used for removing the potential singularities from the diagrams as well as adjusting to finite parts of the correlation functions such, that they satisfy (7.108), (7.110), (7.111) and (7.114), which link the finite renormalised couplings and masses e, m to the physical ones. As in the ϕ^4 -theory the coupling renormalisation, Z_e , is linked to that of the electron-photon vertex, $Z_{\bar{\psi}A\psi}$. The latter is given by the product of the wave function renormalisations of electron and photon and that of the coupling, see (7.93),

$$Z_{\bar{\psi}A\psi} = Z_e Z_\psi Z_A^{1/2}, \quad \longrightarrow \quad Z_e = \frac{Z_{\bar{\psi}A\psi}}{Z_\psi Z_A^{1/2}}. \quad (7.115)$$

With (7.115) we compute the one-loop β -function of the fine-structure constant, β_α , by computing the μ -running of the propagators and the vertex. From its definition in (7.100) we conclude

$$\beta_e = -\mu \frac{d}{d\mu} \ln \left[\frac{Z_{\bar{\psi}A\psi}}{Z_\psi Z_A^{1/2}} \right] = \gamma_{\bar{\psi}A\psi} - 2\gamma_\psi - \gamma_A, \quad \text{with} \quad \gamma_{\bar{\psi}A\psi} = -\mu \frac{dZ_{\bar{\psi}A\psi}}{d\mu} \frac{1}{Z_{\bar{\psi}A\psi}}. \quad (7.116)$$

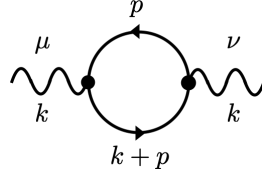


Figure 7.4.: One-loop vacuum polarisation $\Pi_{\mu\nu}^{(\text{diag})}(k)$ of the photon in QED.

Equation (7.116) is the analogue of (7.64) in the scalar theory, and follows directly from the relation of the renormalised quantities and renormalisation constants to the bare quantities.

A far simpler way is provided by using the relation (7.95): the μ -running of Z_e is inverse to that of $Z_A^{1/2}$, and the latter only requires the computation of the μ -running of the photon propagator. This leads us to

$$\beta_e = \frac{1}{2} \mu \frac{dZ_A}{d\mu} \frac{1}{Z_A} = -\gamma_A, \quad (7.117)$$

which also implies

$$\gamma_{\bar{\psi}A\psi} = 2\gamma_\psi, \quad (7.118)$$

the anomalous dimension of the electron-photon vertex and the electron are related. We emphasise that this is also a non-trivial constraint for the singularity structure of the diagrams: Z_A , Z_ψ , and in particular its singular parts, are adjusted by (7.108) and (7.110), and (7.117) and (7.118) link the singularities of the vertex to that of the fermion propagator.

It is left to compute the renormalisation constant of the photon. The only one-loop diagram is the vacuum polarisation Figure 7.4, and the full renormalised vacuum polarisation $\Pi_{\mu\nu}$ is obtained by the sum of the regularised diagram $\Pi_{\mu\nu}^{(\text{diag})}$ in Figure 7.4 and the counter term in (7.98),

$$i\Pi_{\mu\nu}(k) = i\Pi_{\mu\nu}^{(\text{diag})}(k) - i(1 - Z_A) (k^2 \eta_{\mu\nu} - k_\mu k_\nu). \quad (7.119)$$

We already inferred from gauge invariance and the Ward identities, that the vacuum polarisation is transversal, see (7.106) and also (7.105). Hence, it can be written in terms of the transversal projection operator, multiplied by the scalar part $\Pi(k)$ of the vacuum polarisation,

$$i\Pi_{\mu\nu}(k) = i \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k), \quad \text{with} \quad \Pi(k) = \frac{1}{d-1} \Pi_\mu^\mu(k). \quad (7.120)$$

The vacuum polarisation is now computed with the Feynman rules (7.98) as

$$i\Pi_{\mu\nu}^{(\text{diag})}(k) = -e^2 (\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d p}{(2\pi)^d} \text{tr} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \gamma_\mu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\epsilon} \gamma_\nu. \quad (7.121)$$

For the computation of $\Pi(k)$ in (7.120) we contract the Lorentz indices in (7.121) and compute evaluate the Dirac trace in $\Pi^{(\text{diag})}(k)$,

$$\text{tr} (\not{p} + m) \gamma_\mu (\not{p} + \not{k} + m) \gamma_\nu. \quad (7.122)$$

The contraction and the tracing reduces the loop integral to a standard one of the type already discussed in the scalar theory. In $d = 4 - 2\epsilon$ dimensions we have

$$\eta_\mu^\mu = d, \quad \gamma_\mu \gamma^\mu = d \mathbb{1}, \quad \text{tr} \mathbb{1} = 4, \quad \gamma^\mu \not{p} \gamma_\mu = 2\not{p} - \gamma^\mu \gamma_\mu \not{p} = (2-d)\not{p}. \quad (7.123)$$

With the relations in (7.123), the Dirac trace in (7.122),

$$\text{tr}((\not{p} + m)\gamma_\mu(\not{p} + \not{k} + m)\gamma_\nu) = \text{tr}[(2-d)\not{p} + dm](\not{p} + \not{k} + m) = 4[2-d)p(p+k) + dm^2]. \quad (7.124)$$

With (7.124), the scalar part $\Pi^{(\text{diag})}$ of the vacuum polarisation diagram takes the form

$$i\Pi^{(\text{diag})}(k) = \frac{4}{d-1}e^2(\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d p}{(2\pi)^d} \frac{(d-2)p(p+k) - dm^2}{[p^2 - m^2 + i\epsilon][(p+k)^2 - m^2 + i\epsilon]}. \quad (7.125)$$

Equation (7.125) has the standard form of a scalar integral and can be computed from the respective master integrals such as (7.36) and (7.70). For their application we have to convert (7.125) into a form with a single denominator, using the Feynman trick (7.68) for $i = 1$ with

$$\frac{1}{AB} = \int_0^1 d\alpha \frac{1}{[\alpha A + (1-\alpha)B]^2} \quad (7.126)$$

With (7.126), $\Pi(k)$ takes the form

$$i\Pi^{(\text{diag})}(k) = \frac{4e^2}{d-1} \int_0^1 d\alpha (\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d p}{(2\pi)^d} \frac{(d-2)p(p+k) - dm^2}{[(1-\alpha)(p^2 - m^2 + i\epsilon) + \alpha((p+k)^2 - m^2 + i\epsilon)]^2}. \quad (7.127)$$

Equation (7.127) is of the form of the master integral (7.70) after rescaling the loop momentum p . Instead of this rescaling we use a shift $p \rightarrow p - \alpha k$, that converts (7.127) into the form of the master integral (7.36). We emphasise that the use of either procedures is a matter of taste and it is suggestive to take that which is the least error prone. With $p \rightarrow p - \alpha k$ we get for the integrand in (7.127)

$$\frac{(d-2)p(p+k) - dm^2}{[(1-\alpha)(p^2 - m^2) + \alpha((p+k)^2 - m^2) + i\epsilon]^2} \xrightarrow{p \rightarrow p - \alpha k} \frac{(d-2)(p^2 + (1-2\alpha)kp - \alpha(1-\alpha)k^2) - dm^2}{[p^2 - \Delta + i\epsilon]^2}, \quad (7.128)$$

where the 'mass squared' parameter Δ in the denominator is given by,

$$\Delta = \alpha(1-\alpha)k^2 - m^2. \quad (7.129)$$

Inserting (7.128) into (7.127) leads us to the final expression that has the form (7.36) after a Wick rotation. We use that the denominator in (7.128) does not depend on the angles in the loop integral and hence the part proportional to kp vanishes. This leads us to

$$i\Pi^{(\text{diag})}(k) = \frac{4e^2}{d-1} \int_0^1 d\alpha (\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d p}{(2\pi)^d} \frac{(d-2)(p^2 - \alpha(1-\alpha)k^2) - dm^2}{[p^2 - \Delta + i\epsilon]^2}, \quad (7.130)$$

and after the Wick rotation (7.27) and Figure 7.1 we get

$$\Pi^{(\text{diag})}(k) = \frac{4e^2}{d-1} \int_0^1 d\alpha (\bar{\mu}^2)^{\frac{4-d}{2}} \int \frac{d^d p}{(2\pi)^d} \frac{(d-2)(p^2 - \alpha(1-\alpha)k^2) - dm^2}{[p^2 + \Delta]^2}, \quad (7.131)$$

with the Euclidean mass squared parameter

$$\Delta = \alpha(1 - \alpha)k^2 + m^2. \quad (7.132)$$

Equation (7.131) is readily computed with (7.36) for $d = 4 - 2\epsilon$ and we arrive at

$$\Pi^{(\text{diag})}(k) = -\frac{4e^2}{d-1} \frac{1}{(4\pi)^{d/2}} \int_0^1 d\alpha \left[\frac{\bar{\mu}^2}{\Delta} \right]^\epsilon \left\{ (d-2) \left[\Gamma(-1 + \epsilon)\Delta - \Gamma(\epsilon) (\Delta + \alpha(1 - \alpha)k^2) \right] + d\Gamma(\epsilon)m^2 \right\}. \quad (7.133)$$

An expansion in $\epsilon \rightarrow 0$ leads us to the final result

$$\Pi^{(\text{diag})}(k) = -\frac{1}{3\pi} \frac{e^2}{4\pi} k^2 \left[-\frac{1}{\epsilon} + \gamma - \ln(4\pi) + 6 \int_0^1 \alpha(1 - \alpha) \ln \frac{\Delta}{\bar{\mu}^2} \right]. \quad (7.134)$$

The remaining α -integral can also be performed, leading to

$$6 \int_0^1 \alpha(1 - \alpha) \ln \frac{\Delta}{\bar{\mu}^2} = \left[-\frac{5}{3} \frac{4m^2}{k^2} + \frac{2(k^2 - 2m^2) \sqrt{k^2 + 4m^2} \operatorname{arctanh} \left(\sqrt{\frac{k^2}{k^2 + 4m^2}} \right) + \ln \frac{m^2}{\bar{\mu}^2} \right], \quad (7.135)$$

which can be rewritten in terms of logarithms with $\operatorname{artanh}(x) = 2[\ln(1 + x) - \ln(1 - x)]$.

Equation (7.134) is the regularised results for the vacuum polarisation diagram. We insert it into the full vacuum polarisation (7.119) and determine Z_A with the renormalisation condition (7.101d). We choose $\mu = 0$ for convenience and compute the k^2 -derivative of $\Pi^{(\text{diag})}$,

$$\frac{\partial \Pi^{(\text{diag})}(k)}{\partial k^2} = -\frac{1}{3\pi} \frac{e^2}{4\pi} \left[-\frac{1}{\epsilon} + \gamma + \ln \frac{m^2}{4\pi \bar{\mu}^2} \right], \quad (7.136)$$

where we have used that the k^2 -expansion of (7.135) about $k^2 = 0$ is given by $\ln m^2/\bar{\mu}^2 + O(k^2)$. Collecting all results leads us to

$$Z_A = 1 + \frac{1}{3\pi} \frac{e^2}{4\pi} \left[-\frac{1}{\epsilon} + \gamma + \ln \frac{m^2}{4\pi \bar{\mu}^2} \right]. \quad (7.137)$$

We use (7.137) in the β -function of QED, (7.117), and arrive at the one-loop result,

$$\beta_e = \frac{1}{12\pi^2} e^2, \quad (7.138)$$

where we have used the asymptotic scaling argument. Equation (7.138) is just the matter part of the QCD β -function. The β -function is $\alpha_s = g_s^2/(4\pi)$ is given in (7.60) and the one of g_s has 1/2 of the coefficient given there. It is reduced to that of QED with $N_c = 0$ (no gauge boson self interaction), $N_f = 1$ and an additional normalisation factor 2. The relative factor 1/2 comes from the group trace, $\operatorname{tr}_f t^a t^b = 1/2 \delta^{ab}$. This agreement is expected as the matter part of the QCD β -function originates from the same polarisation diagram, only with quarks running through the loop. The respective renormalised coupling $e(\mu)$ is obtained by integrating β_e in (7.138) from $\mu = 0$ to a given μ . We obtain the asymptotic running coupling e_{phys} with

$$e_{\text{phys}}(k) = \frac{\bar{e}}{1 + \frac{\bar{e}^2}{24\pi^2} \ln \frac{\bar{k}^2}{k^2}}, \quad \text{with} \quad \bar{e} = e_{\text{phys}}(\bar{k}), \quad (7.139)$$

with

$$k \frac{de_{\text{phys}}(k)}{dk} = \beta_e \bar{e}, \quad (7.140)$$

as required. From (7.139) we also deduce the one-loop fine structure $\alpha(k) = e_{\text{phys}}^2(k)/(4\pi)$,

$$\alpha(k) = \frac{\bar{\alpha}}{1 + \frac{\bar{e}^2}{12\pi^2} \ln \frac{\bar{k}^2}{k^2}}, \quad \text{with} \quad \bar{\alpha} = \alpha(\bar{k}). \quad (7.141)$$

Its value at vanishing momentum is measure very accurately, [?],

$$\alpha(k=0) = 1/137.035999084(21), \quad \alpha(m_Z) \approx 1/127, \quad (7.142)$$

and the second value is the approximate one at the electroweak or Z -boson scale $m_Z \approx 91.2$ GeV. Note that $k \rightarrow 0$ is not in the validity regime of (7.139) which was obtained for asymptotically large momenta, relative to any mass scale. While the Standard Model below the electroweak scale is described by a combination of QED, QCD and the weak interaction due to spontaneous electroweak symmetry breaking via the Higgs, above the electroweak scale we have to discuss the running of the $U(1)$ - hypercharge. This is deferred to QFT II. Here we simply collect all the different one-loop terms in the 1-loop coefficient b_0 and write

$$e_{\text{phys}}(k) = \frac{\bar{e}}{1 + \frac{b_0}{2} \frac{\bar{\alpha}}{4\pi} \ln \frac{k^2}{\bar{k}^2}}, \quad \text{with} \quad \bar{\alpha} = \frac{\bar{e}^2}{4\pi}. \quad (7.143)$$

with $b_0 = 4/3$ for QED with an electron, following from (7.138). The position of the Landau pole is then given by

$$p_{\text{sing}}^2 = \bar{p}^2 e^{\frac{8\pi}{b_0 \bar{\alpha}}}, \quad (7.144)$$

analogously to (7.57). A rough estimate in the Standard model (3 families of leptons and quarks as well as the charged Higgs) provides us with a position in the Landau pole very far above the Planck scale $m_{\text{pl}} \approx 1.22 \times 10^{19}$ GeV, a computation within the Standard Model is done in QFT II.

8. Anomalies*

In this Chapter we reconsider the invariance of a massless fermionic theory under axial U(1) transformations as introduced in (4.87) and discussed below. The respective conservation law was given in (4.89) with the Noether charge (4.90). We consider a fermionic theory, coupled to a background SU(N) or U(1) gauge field with $A_\mu = A_\mu^a t^a$. The action is given by the fermionic part of (5.23),

$$S[A, \psi, \bar{\psi}] = \int d^4x \bar{\psi} \left(i\mathcal{D} - m_\psi \right) \psi, \quad \text{with} \quad D_\mu = \partial_\mu - igA_\mu, \quad (8.1)$$

and the U(1) case in (6.1a) is achieved for the U(1) photon field and $g = e$. The properties under gauge transformations are discussed in the last Chapter, Section 5.1.

First we discuss the classical current conservation for both, the vector and the axial current,

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi. \quad (8.2)$$

with the help of the equations of motion (EoM). The EoM can be written as

$$\not{\partial}_\mu \psi(x) = (igA - im_\psi)\bar{\psi}, \quad \text{and} \quad \partial_\mu \bar{\psi}(x) \gamma_\mu = \bar{\psi} (-igA + im_\psi). \quad (8.3)$$

The respective steps will also be of use for discussion current conservation and anomalies in the quantised fermionic theory.

Now we restrict ourselves to the massless theory with $m_\psi = 0$. For the vector current we find

$$\partial_\mu [\bar{\psi}(x) \gamma^\mu \psi(x)] = [\partial_\mu \bar{\psi}(x)] \gamma^\mu \psi(x) + ig \bar{\psi}(x) A \psi(x) + \bar{\psi}(x) \not{\partial} \psi(x), \quad (8.4)$$

where the first two terms on the right hand side of (8.4) are the EoM for $\bar{\psi}$ contracted from the right with ψ , and the last one is the EoM for ψ , contracted from the left with $\bar{\psi}$. For later use it is more convenient to cast (8.4) in form of a Dirac and gauge group trace tr , where the group trace is in the fundamental representation.

$$\partial_\mu j^\mu(x) = -\text{tr} \left[\not{\partial} \psi(x) \bar{\psi}(x) - \psi(x) \bar{\psi}(x) \overleftarrow{\not{\partial}} \right], \quad (8.5)$$

where $\bar{\psi} \overleftarrow{\partial}_\mu = -\partial_\mu \bar{\psi}$, and $\psi(x) \bar{\psi}(x)$ is a tensor in Dirac and gauge group space with

$$[\psi(x) \bar{\psi}(x)]_{\xi \bar{\xi}}^{AB} = \psi_\xi^A(x) \bar{\psi}_{\bar{\xi}}^B(x), \quad (8.6)$$

with the spinor indices $\xi, \bar{\xi} = 1, 2, 3, 4$ and the gauge group indices $A, B = 1, \dots, N$ in the fundamental representation. The expectation value of (8.6) is nothing but the fermion propagator (4.129), or rather its components (4.130). Evidently, (8.5) vanished on the equations of motion.

Similarly it follows for the axial current,

$$\partial_\mu \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x) = [\partial_\mu \bar{\psi}(x)] \gamma^\mu \gamma_5 \psi(x) + ig \bar{\psi}(x) A \gamma_5 \psi(x) - \bar{\psi}(x) \gamma_5 \not{\partial} \psi(x), \quad (8.7)$$

or in the trace form,

$$\partial_\mu j_5^\mu(x) = \text{tr} \left[\gamma_5 \left(\not{\partial} \psi(x) \bar{\psi}(x) + \psi(x) \bar{\psi}(x) \overleftarrow{\not{\partial}} \right) \right], \quad (8.8)$$

which also vanishes on the equations of motions. Note that there is a crucial difference between (8.5) and (8.8): for the vector current there is a relative minus sign between the EoM contribution of ψ and $\bar{\psi}$. Accordingly, potential mass contributions to the EoM cancel in (8.5) and the vector current is also conserved for the massive theory. In turn, mass contributions add up in (8.8) and current conservation is lost. This difference is triggered by the γ_5 in the axial current trace.

We have already mentioned there that the classical conservation law is broken on the quantum level. Indeed it is the fermionic quantum effects that trigger an anomaly in the conservation law (8.7), leading to

$$\partial_\mu \langle j_5^\mu \rangle = \mathcal{A}. \quad (8.9)$$

The term \mathcal{A} is the axial anomaly, which vanishes in the classical conservation law (8.7). Equation (8.9) and its cousins for chiral transformations have far reaching physics consequences. For instance, (8.9) is pivotal for the decay of a neutral pion into two photons, $\pi^0 \rightarrow 2\gamma$, as well as for anomalous chiral symmetry breaking. In its integrated form it constitutes the baby version of the Atiyah-Singer index theorem, relating the *analytic* index of the Dirac operator to a *topological* invariant.

In the present lecture we concentrate on the latter topological aspects of anomalies, and we aim at the derivation of a simple version of the Atiyah-Singer index theorem. Applied to the present fermionic theory with the Dirac operator \mathcal{D} , this theorem connects the topological index of the Dirac operator to the analytic one. It holds true on compact manifolds, and Minkowski space-time is not. Accordingly, the current derivations should be done in Euclidean space-time, which we obtain from Minkowski space time by the *Wick rotation*, already mentioned before as an efficient tool for numerical computation of Feynman diagrams: we rotate a positive Minkowski frequency p_0 towards positive imaginary values, thus avoiding the poles in the Feynman propagator, see Figure 3.5. In particular, this does not change the results of momentum loops. Such a rotation with the angle $\pi/2$ leads us to $p_0 \rightarrow ip_0^E$, and we obtain

$$-(p_0^2 - \vec{p}^2 - m_\phi^2) \rightarrow (p_0^E)^2 + \vec{p}^2 + m_\phi^2, \quad -i(\gamma_E^0 p_0 - \boldsymbol{\gamma} \mathbf{p}) - m_\psi \rightarrow i(\gamma^0 p_0 + \boldsymbol{\gamma} \mathbf{p}) + m_\psi. \quad (8.10)$$

with $\gamma_E^0 = i\gamma^0$. Note that the Wick rotation also implies $x^0 \rightarrow ix_0^E$ and $\bar{\psi} \rightarrow i\bar{\psi}_E$. The Euclidean Clifford algebra and the Euclidean γ_5 are given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \quad \gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \gamma_5^2 = \mathbb{1}, \quad (8.11)$$

with Hermitian γ^μ . The respective Euclidean metric is $\eta^E = -\mathbb{1}$, which can be readily converted to $\eta^E = \mathbb{1}$, which leaving us with a standard flat Euclidean space \mathbb{R}^4 . In a final step we compactify \mathbb{R}^4 by identifying all points at infinity, this leaves us with the four-dimensional sphere S^4 .

Importantly one sees directly, that the Euclidean theory has no infrared problem for non-vanishing masses. Furthermore, the topological considerations below including the respective topological invariants are defined for compact Euclidean spaces. However, for the sake of convenience and the direct connection to the lecture course we simply stick to the Minkowski conventions. However, it is implicitly understood that space-time integrals are taken in the compact Euclidean space S^4 . We first consider the integrated form of (8.9),

$$\int d^4x \langle \partial_\mu j_5^\mu \rangle = \int d^4x \partial_\mu \langle j_5^\mu \rangle = - \int d^4x \partial_\mu \text{tr} [\gamma_\mu \gamma_5 \langle \psi(x) \bar{\psi}(x) \rangle] = \int d^4x \mathcal{A}, \quad (8.12)$$

The expectation value in (8.12) is simply that of (8.6). These are the components of the fermion propagator at equal space-time points, traced over Dirac indices and gauge group indices (in the fundamental representation). Naively the integral over the divergence of the current vanishes (if the integrand is globally well-defined and no boundary terms are present) as it is an integral of a total derivative. The formal expression in (8.12) is potentially subject to both ultraviolet and infrared divergences, and we shall regularise both of them. The right hand side is the integrated anomaly, and the latter, \mathcal{A} may and will arise within the fermionic quantisation.

We proceed by introducing a small mass m_ψ for the fermions, which regularises the infrared divergence: $\bar{\psi} i\mathcal{D}\psi \rightarrow \bar{\psi}(i\mathcal{D} - m_\psi)\psi$. Then, the axial current is not conserved, see (4.91). Its integrated form reads,

$$\int d^4x \left(\partial_\mu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle - \mathcal{A} \right) = 2im_\psi \int d^4x \langle \bar{\psi} \gamma_5 \psi \rangle = -2im_\psi \int d^4x \text{tr} [\gamma_5 \langle \psi \bar{\psi} \rangle]. \quad (8.13)$$

Now we use that the propagator of a field is nothing but the inverse or Greens function of the kinetic operator, and for the fermionic propagator we find

$$\langle \psi(x) \bar{\psi}(y) \rangle = \frac{i}{i\mathcal{D} - m_\psi}(x, y), \quad \text{with} \quad (i\mathcal{D}_x - m_\psi) \langle \psi(x) \bar{\psi}(y) \rangle = i\delta(x - y), \quad (8.14)$$

where on the right hand side the identity matrix in spinor and gauge group indices is implied (as with the mass term). Note that the on-shell condition used for the derivation of the fermionic field operators now depends on the background gauge field and the momentum space integral in (4.105) has to be substituted by an integral/sum over eigenvalues of the Dirac operator \mathcal{D} . Luckily, we do not need this explicitly and only use generic properties of the field operators. For $A_\mu = 0$ the reduces to the free propagator, that is the Greens function of $i\mathcal{D} - m$.

With these preparations, the right hand side of (8.13) can be written as an operator trace,

$$2im_\psi \int d^4x \langle \bar{\psi} \gamma_5 \psi \rangle = 2m_\psi \text{Tr} \left[\gamma_5 \frac{1}{i\mathcal{D} - m_\psi} e^{\varepsilon \mathcal{D}^2} \right], \quad (8.15)$$

where the operator trace Tr now also includes the integration over space-time, $\text{Tr} = \int d^4x \text{tr}$. We also have introduced a regularisation of the operator trace with the damping factor $\exp\{\varepsilon \mathcal{D}^2\}$ with $\varepsilon \rightarrow 0$, as the integral in (8.15) is potentially cubically divergent (best seen in momentum space for $A_\mu = 0$). Note that ε has momentum dimension -2.

Now we use that a trace can be represented in terms of a complete sum of all basis vectors in any orthonormal basis of the Hilbert space. Evidently, the most convenient choice are the eigenstates $|\varphi_n\rangle$ of the Dirac operator,

$$i\mathcal{D}|\varphi_n\rangle = \lambda_n |\varphi_n\rangle, \quad \langle \varphi_n | \varphi_m \rangle = \delta_{nm}, \quad \text{with} \quad \mathbb{1} = \sum_n |\varphi_n\rangle \langle \varphi_n|, \quad \text{and} \quad \varphi_n(x) = \langle x | \varphi_n \rangle, \quad (8.16)$$

with $n \in \mathbb{N}$ and the eigenfunctions $\varphi(x)$, that are the projection of the eigenstate $|\varphi_n\rangle$ on the position state. This leads us to

$$2m_\psi \text{Tr} \left[\gamma_5 \frac{e^{\varepsilon \mathcal{D}^2}}{i\mathcal{D} - m} \right] = 2m_\psi \sum_n \langle \varphi_n | \gamma_5 \frac{e^{\varepsilon \mathcal{D}^2}}{i\mathcal{D} - m_\psi} | \varphi_n \rangle = 2m_\psi \sum_n \frac{e^{-\varepsilon \lambda_n^2}}{\lambda_n - m_\psi} \langle \varphi_n | \gamma_5 | \varphi_n \rangle, \quad (8.17)$$

where we assume that $m_\psi \neq \lambda_n$ for all n . It is left to compute the overlap of $\gamma_5 |\varphi_n\rangle$ with $|\varphi_n\rangle$: first we infer from the anti-commutation relation $\{\mathcal{D}, \gamma_5\} = 0$, that the state $\gamma_5 |\varphi_n\rangle$ is also an eigenstate of the Dirac operator with the eigenvalue $-\lambda_n$. Accordingly, the overlap vanishes for non-vanishing eigenvalues. This is proven straightforwardly by

$$\mathcal{D} \gamma_5 |\varphi_n\rangle = -\gamma_5 \mathcal{D} |\varphi_n\rangle = -\lambda_n \gamma_5 |\varphi_n\rangle \quad \longrightarrow \quad \langle \varphi_n | \gamma_5 | \varphi_n \rangle = 0, \quad \text{for} \quad \lambda_n \neq 0. \quad (8.18)$$

In consequence, the sum over the full Hilbert space in (8.17) reduces to the sum over the subspace spanned by the zero modes $|\varphi_{n_0}\rangle$ with vanishing eigenvalues of the Dirac operators, $\mathcal{D} |\varphi_{n_0}\rangle = 0$ and hence $\lambda_{n_0} = 0$. These states are labelled by n_0 . With these preparations we arrive at

$$2m_\psi \text{Tr} \left[\gamma_5 \frac{e^{\varepsilon \mathcal{D}^2}}{i\mathcal{D} - m_\psi} \right] = -2 \sum_{n_0} \langle \varphi_{n_0} | \gamma_5 | \varphi_{n_0} \rangle, \quad (8.19)$$

where we have used that $e^{\varepsilon \mathcal{D}^2} |\varphi_{n_0}\rangle = |\varphi_{n_0}\rangle$. As the Dirac operator commutes with γ_5 on the zero mode subspace, we can diagonalise both operators simultaneously on this space. Thus, we can define φ_{n_0} such, that they have can positive and negative chirality,

$$\gamma_5 |\varphi_{n_0}\rangle = \pm |\varphi_{n_0}\rangle, \quad \longrightarrow \quad \langle \varphi_{n_0} | \gamma_5 | \varphi_{n_0} \rangle = \pm 1, \quad (8.20)$$

and $|\varphi_{n_0}\rangle$ live in the positive and negative chirality subspaces, and can be labelled by their chirality, $|\varphi_{n_0^\pm}\rangle$ with $(1 \pm \gamma_5)|\varphi_{n_0^\pm}\rangle = |\varphi_{n_0^\pm}\rangle$ and $(1 \mp \gamma_5)|\varphi_{n_0^\pm}\rangle = 0$. In summary we are led to a remarkable identity,

$$2m_\psi \text{Tr} \left[\gamma_5 \frac{1}{i\mathcal{D} - m_\psi} \right] = 2(N_- - N_+) \quad (8.21)$$

where N_+ , N_- are the number of zero modes with positive and negative chirality respectively. This leaves us with the exciting result the the right hand side of (8.13) is non-zero and independent of the mass. Indeed it is the analytic index of \mathcal{D} , while the left hand side is the space-time integral of a total derivative and bound to vanish. As long as the fermion is massive, we certainly have

$$\int d^4x \partial_\mu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = 0, \quad (8.22)$$

as correlations and fields in a massive theory decay exponentially with the spacial distance r , in the present case with $e^{-m_\psi r}$. This leaves us with the equation

$$\int d^4x \mathcal{A} = 2(N_+ - N_-), \quad (8.23)$$

which relates the presence of an anomaly in the axial conservation law for background fields A_μ with a non-vanishing index of the Dirac operator. Note that seemingly this is very puzzling as the axial transformation is a unitary rotation that naïvely should be an invariance on fermionic the Hilbert space. This leaves us with the question which of the implicit assumptions of the above seemingly trivial statement is wrong. At its root it is the property (8.18), which fails on the zero mode space: an axial rotation rotates states with eigenvalues λ_n into states with eigenvalues $-\lambda_n$. However, on the zero mode subspace with $N_+ \neq N_-$ the states cannot be paired and the rotation is 'incomplete'. Hence the phase rotation is not unitary on the Hilbert space, but only on the subspace spanned by states with $\lambda_n \neq 0$.

This observation is the starting point of the derivation of the anomaly within the path integral (Fujikawa trick), and we shall pick up this very elegant and powerful derivation in the second part of the lecture course. With the present perturbative techniques we may resort to perturbation theory and compute the expectation value of (8.8),

$$\langle \partial_\mu \tilde{J}_5^\mu \rangle = \text{tr} \left[\gamma_5 \left(\mathcal{D} \langle \psi(x) \bar{\psi}(x) \rangle + \langle \psi(x) \bar{\psi}(x) \rangle \overleftarrow{\mathcal{D}} \right) \right], \quad (8.24)$$

in terms of perturbation theory: the first non-trivial term is proportional to two gauge fields, and is computed from the triangle diagram with two fermion-gauge field vertices that are proportional to γ_μ and one axial vector vertex, proportional to $\gamma_5 \gamma_\mu$. Again, this loop seemingly vanishes for symmetry reasons. However, as it is linearly divergent, it has to be regularised as we did in (8.15). Then it leads to a finite result. This can be read as the fact, that the regularisation breaks the axial symmetry, but at its root it is the non-unitarity of the rotation. The computation of the triangle diagram is a standard one contained in QFT textbooks, in particular for its importance for the $\pi^0 \rightarrow 2\gamma$ decay and spontaneous chiral symmetry breaking. We refer the interested reader to the literature, while we resort to a more hands-on approach here. To begin with, we regularise the expression in (8.24) as we have done in (8.15) with $\psi \rightarrow \exp\{\varepsilon/2\mathcal{D}^2\}\psi$ and $\bar{\psi} \rightarrow \bar{\psi} \exp\{\varepsilon/2\overleftarrow{\mathcal{D}}^2\}$. This is damping the

contributions of the large eigenvalues of the Dirac operator and renders (8.25) finite. Also using (8.14) leads us to

$$\langle \partial_\mu j_5^\mu \rangle = \text{tr} \left[\langle x | \gamma_5 \left(\not{D} \frac{1}{\not{D}} + \frac{1}{\not{D}} \not{D} \right) e^{\varepsilon \not{D}^2} | x \rangle \right] = 2 \text{tr} \left[\langle x | \gamma_5 e^{\varepsilon \not{D}^2} | x \rangle \right]. \quad (8.25)$$

Note that the same computation for the conservation law of the vector current leads us to

$$\langle \partial_\mu j_5^\mu \rangle = \text{tr} \left[\langle x | \left(\not{D} \frac{1}{\not{D}} - \frac{1}{\not{D}} \not{D} \right) e^{\varepsilon \not{D}^2} | x \rangle \right] = 0. \quad (8.26)$$

Naïvely, (8.25) also vanishes identically as the trace of γ_5 does, $\text{tr} \gamma_5 = 0$. However, (8.25) is defined with states in the Hilbert space, and the use of the position states $|x\rangle$ are a bit deceiving. We know already that the trace of γ_5 on the Hilbert space is not vanishing, if the number of right- and left-handed zero modes of the Dirac operator do not agree, see (8.21). This relation is more explicit if using the eigenstates of the Dirac operators, (8.16). Inserting the completeness relation in (8.16) twice, leads us to

$$\langle \partial_\mu j_5^\mu \rangle = 2 \sum_n \varphi_n^\dagger(x) \gamma_5 e^{\varepsilon \not{D}^2} \varphi_n(x), \quad (8.27)$$

with is the (intermediate) expression for the anomaly within the Fujikawa trick, where it arises as the Jacobian of the transformation of the fields $\psi, \bar{\psi}$ to the fields $\psi e^{i\alpha\gamma_5}, \bar{\psi} e^{i\alpha\gamma_5}$. Here it has been obtained in a hands-on approach that lacks the rigorosity of the path integral derivation, but only makes use of the results obtained so far in the QFT lecture course.

The final step concerns the explicit computation of (8.25). For that purpose we switch to momentum states with

$$\begin{aligned} 2 \text{tr} \left[\langle x | \gamma_5 e^{\varepsilon \not{D}^2} | x \rangle \right] &= 2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \text{tr} \langle x | q \rangle \langle q | \gamma_5 e^{\varepsilon \not{D}^2} | p \rangle \langle p | x \rangle \\ &= 2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \gamma_5 e^{\varepsilon [(ip_\mu + \partial_\mu - igA_\mu)^2 - \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu}]}. \end{aligned} \quad (8.28)$$

where we have used where we have used that $\langle q | p \rangle = (2\pi)^4 \delta(p - q)$ and

$$\not{D}^2 = \left(\frac{1}{2} \{\gamma_\mu, \gamma_\nu\} + \frac{1}{2} [\gamma_\mu, \gamma_\nu] \right) D_\mu D_\nu = D^2 - \frac{g}{2} \sigma_{\mu\nu} F^{\mu\nu}, \quad \text{with} \quad \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]. \quad (8.29)$$

The derivatives in the exponent in the second line of (8.28) hit the field A_μ and the fieldstrength $F^{\mu\nu}$ in the series expansion of the exponential. Now we use that the first non-trivial Dirac trace is $\text{tr} \gamma_5 \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4}$ stemming from $\varepsilon^2/2 (\sigma_{\mu\nu} F^{\mu\nu})^2$, the traces with less γ -matrices vanish. The factor ε^2 cancels the $1/\varepsilon^2$ factor from the momentum integral. Other terms with non-vanishing traces vanish in the limit $\varepsilon \rightarrow 0$ as they are either proportional to $\varepsilon^n (\sigma_{\mu\nu} F^{\mu\nu})^n$ with $n > 2$, or proportional to additional powers of derivatives of the gauge field, $\varepsilon \partial_\nu A_\mu$ or derivatives of the field strength. This leaves us with

$$\text{tr} \left[\langle x | \gamma_5 e^{\varepsilon \not{D}^2} | x \rangle \right] = \frac{1}{8} \int \frac{d^4 p}{(2\pi)^4} \text{tr} \gamma_5 \sigma_{\mu\nu} \sigma_{\rho\sigma} F^{\mu\nu} F^{\rho\sigma} e^{-\varepsilon p^2} + \mathcal{O}(\varepsilon) \rightarrow -\frac{i}{2} \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{tr}_f F^{\mu\nu} F^{\rho\sigma}, \quad (8.30)$$

with the gauge group trace tr_f in the fundamental representation and $\varepsilon_{0123} = 1$. In (8.30) we have used

$$\text{Minkowski: } \text{tr}_D \gamma_5 \sigma_{01} \sigma_{23} = -i, \quad (8.31)$$

with the trace tr_D in spinor space. Then, the occurrence of $\varepsilon_{\mu\nu\rho\sigma}$ follows by the antisymmetry of $\sigma_{\mu\nu}$. We have also provided the Euclidean analogue, leading to

$$\text{Euclidean: } \text{tr}_D \gamma_5 \sigma_{01} \sigma_{23} = -1, \quad \longrightarrow \quad \text{tr} \left[\langle x | \gamma_5 e^{\varepsilon \not{D}^2} | x \rangle \right]_{\varepsilon=0} = -\frac{1}{2} \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{tr}_f F^{\mu\nu} F^{\rho\sigma}, \quad (8.32)$$

In summary we arrive at the important relation

$$\sum_n \varphi_n^+(x) \gamma_5 e^{\varepsilon D^2} \varphi_n(x) = -\frac{i}{2} \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{tr}_f F^{\mu\nu} F^{\rho\sigma}, \quad (8.33)$$

where the limit $\varepsilon \rightarrow 0$ is understood. In (8.33), we have also used the representation of the left hand side of (8.30) in terms of the eigenfunctions of the Dirac operator, (8.27). The final axial current conservation law for massive Dirac fermions reads

$$\partial_\mu j_5^\mu - 2m \langle \bar{\psi} \gamma_5 \psi \rangle = -i \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{tr}_f F^{\mu\nu} F^{\rho\sigma}, \quad (8.34)$$

with the chiral limit

$$\partial_\mu j_5^\mu = -i \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. \quad (8.35)$$

The right hand side of can also be expressed in terms of the Hodge dual (5.31) as $-ig^2/(8\pi^2) \text{tr}_f F_{\mu\nu} \tilde{F}^{\mu\nu}$. The Euclidean version of (8.34) and (8.35) imply the (baby) Atiyah-Singer index theorem upon integration over a given compact Euclidean manifold (without boundary),

Atiyah-Singer index theorem

$$-\frac{1}{32\pi^2} \int d^4x \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(\mathcal{A}) F^{\rho\sigma}(\mathcal{A}) = N_+ - N_-, \quad \text{with} \quad \mathcal{A}_\mu = gA_\mu, \quad (8.36)$$

where we have absorbed the coupling in the gauge field. The Atiyah-Singer index theorem connects the topological Pontryagin number of the manifold (left hand side, topological index of the Dirac operator) to the analytical index of the Dirac operator (right hand side).

Part II.

Advanced quantum field theory

9. Functional integral approach

The lecture course so far has been based on the operator approach to quantum field theory. We have introduced this approach within the many-body limit of operators in quantum mechanics, see [Section 2.3](#), [Figure 2.1](#). It is based on the operator algebra of creation operators a^\dagger and annihilation operators a of a given quantum field as well as the Hilbert space, that can be generated from a vacuum state $|\Omega\rangle$ by applying sums of products of the creation operators on this state. This allowed us to describe scalar, fermionic and vector/gauge field quantum field theories, and in particular their scattering events. Certainly, the astute reader has realised that we have been leaving out or cutting short specific topics, starting from the conceptual discussion of the fate of symmetries on the quantum level including quantum symmetry identities, with the exception of the brief discussion of Ward identities in QED in [Section 7.3](#). A further important topic is renormalisation theory, which is best done within a functional path integral approach; again first indications for this have been included in [Section 7.1.2](#) on Generating Functionals.

Moreover, we have already learned, that (numerical) computations in perturbation theory are best done after a Wick rotation from Minkowski to Euclidean space time, or more generally from real-time QFT to imaginary-time QFT. The Wick rotation leads us from a unitary time evolution to an exponential damping as we encounter in statistical theories. This is a very fruitful analogy which is also behind the standard non-perturbative computational approach to QFT with lattice field theories. In the latter, the statistical theory after the Wick rotation is sampled numerically over all states.

Both, the conceptual advantages for symmetry considerations in particular in quantum gauge theory and renormalisation theory as well as the numerical applicability in particular in non-perturbative settings are naturally realised within the path integral approach to QFTs.

Before we derive the path integral for quantum mechanical systems in the next Section, [Section 9.1](#), we briefly elucidate the idea as well as some of the statements above at the example of the S-matrix element in a quantum theory between an *initial* state $|i\rangle$ at a time t_0 and a *final* state $|f\rangle$ at a time t of the time evolution operator [\(3.34\)](#) in [Section 3.1](#). We recall the matrix element for the sake of convenience,

$$T_{fi} = \langle f | U(t, t_0) | i \rangle = \langle f | T e^{-i \int_{t_0}^t dt' H_{\text{int}}(t')} | i \rangle. \quad (9.1)$$

[Equation \(9.1\)](#) has been obtained in [Section 3.1](#) by considering products of infinitesimal time evolution operators from times $t_0 + n\Delta t$ to times $t_0 + (n+1)\Delta t$, see [\(3.30\)](#). This can be rewritten in terms of products of infinitesimal matrix elements which will be done in detail in [Section 9.1](#). Here we simply consider an insertion of a complete set of states $|\lambda\rangle$ at the time $t_0 < t' < t$. Here λ can be a discrete or continuous index of the basis, and we have

$$T_{fi} = \sum_{\lambda} \langle f | U(t, t') |\lambda\rangle \langle \lambda | U(t', t_0) | i \rangle = \sum_{\lambda} T_{f\lambda} T_{\lambda i}. \quad (9.2)$$

In quantum mechanics we might take a position basis with $\lambda = x$ in one spatial dimension or $\lambda = \mathbf{x} = (x_1, \dots)$ in several spatial dimensions. If also the initial and final states are position eigenstates, $|i\rangle = |\mathbf{x}_i\rangle$ and $|f\rangle = |\mathbf{x}_f\rangle$; we have rewritten the S-matrix element as the product of S-matrix elements of processes which are at the position \mathbf{x} at the time t' . Accordingly, we only consider processes along a *path* with the position $\mathbf{x}(t')$. Finally, this position $\mathbf{x}(t')$ is integrated over, hence the name path integral. If we use the insertion [\(9.2\)](#) after infinitesimal time steps as done in the derivation of the time-ordered time evolution operator, we end up with a path ordered *path integral*, which contains integrations over all positions $\mathbf{x}(t')$ for all $t' \in (t_0, t)$. After a Wick rotation this

yields an infinite-dimensional statistical integral. Similarly to the operator approach we then take the many-body limit of the quantum mechanical path integral, thus arriving at the path integral formulation of quantum field theory.

9.1. Path integral in quantum mechanics

We now put this idea of writing the S-matrix element (9.1) in terms of products of S-matrix elements, integrated over the patching position, to work in quantum mechanics. The S-matrix element or transition amplitude of a one-dimensional quantum mechanical system is given by

$$\langle q_f | U(t, t_0) | q_i \rangle, \quad (9.3)$$

where q is the position, \hat{q} is the position operator and $|q\rangle$ is the position eigenstate with

$$\hat{q}|q\rangle = q|q\rangle, \quad \hat{p}|p\rangle = p|p\rangle, \quad (9.4)$$

with continuous spectra $q, p \in \mathbb{R}$ and the commutation relation

$$[\hat{q}, \hat{p}] = i \quad (9.5)$$

In (9.4) we also defined the momentum operator \hat{p} and the respective eigenstate for later use. The time evolution operator satisfies the equation of motion,

$$i\partial_t U(t, t_0) = \hat{H} U(t, t_0), \quad \text{with} \quad U(t_0, t_0) = 1, \quad (9.6)$$

where \hat{H} is the Hamilton operator of the (static) system. This is reminiscent of (3.27) in the interaction picture, but $U(t, t_0)$ in (9.6) evolves with the full Hamiltonian. In the following we restrict ourselves to separable Hamiltonians of the form

$$\hat{H}(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q}), \quad (9.7)$$

The separation of the Hamiltonian in a kinetic part that only depends on the momentum operator and a potential term that only depends on the position operator leads to the following simple identity,

$$\langle p | \hat{H}(\hat{p}, \hat{q}) | q \rangle = \left[\frac{p^2}{2m} + V(q) \right] \langle p | q \rangle = H(p, q) \langle p | q \rangle, \quad \text{with} \quad H(p, q) = \frac{p^2}{2m} + V(q), \quad (9.8)$$

the $p - q$ matrix element of the Hamiltonian operator is simply the Hamiltonian functional, multiplied with the matrix element $\langle p | q \rangle$. The latter is simply a phase, which is readily seen from the following considerations. To begin with, the Hilbert space \mathcal{H} of the above system with the algebra (9.5) is the space of square-integrable functions

$$\psi : q \rightarrow \psi(q) \quad \text{with} \quad \int_{\mathbb{R}} dq |\psi|^2(q) < \infty. \quad (9.9)$$

This is accompanied with the position space representation of momentum and position operators,

$$\hat{q} : \quad \hat{q}\psi(q) = q\psi(q), \quad \hat{p} : \quad \hat{p}\psi(q) = -i\frac{\partial\psi}{\partial q}(q), \quad \text{with} \quad p = -i\frac{\partial}{\partial q}, \quad (9.10)$$

and \hat{p}, \hat{q} in (9.10) satisfy (9.5). We emphasise that the above is only a possible representation of the Heisenberg algebra (9.5) and the Hilbert space \mathcal{H} . For example, the dual one is that in momentum space, where \hat{p} is represented by multiplication and \hat{q} is a p -derivative.

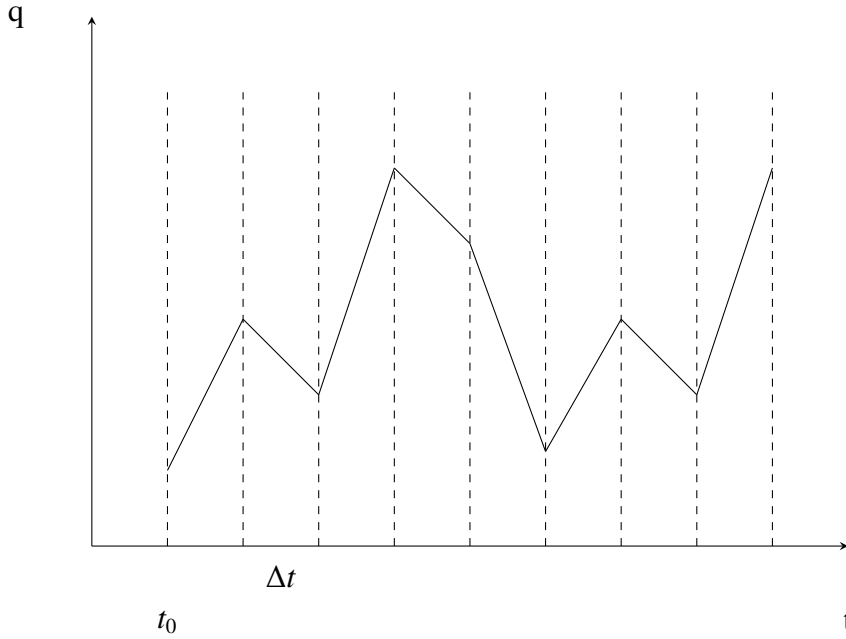


Figure 9.1.: Discretisation of the variables' values.

In the position space and momentum space representations the basis functions are plane waves (plane-wave normalisable), that are normalised to Dirac δ -functions. It is readily shown that

$$\langle q|q'\rangle = \delta(q - q') , \quad \text{from completeness:} \quad \int_{\mathbb{R}} dq' |q'\rangle \langle q'| = 1 \quad \rightarrow \quad \int_{\mathbb{R}} dq' \langle q|q'\rangle |q'\rangle = |q\rangle . \quad (9.11)$$

With a Fourier transformation it also follows that

$$\langle p|p'\rangle = 2\pi\delta(p - p') , \quad \text{and} \quad \langle p|q\rangle = e^{-ipq} . \quad (9.12)$$

This sets the stage for the derivation of the path integral for the S-matrix element or transition amplitude (9.3). With (9.12) the p-q matrix element of the Hamiltonian, (9.8), reduces to

$$\langle p|\hat{H}(\hat{p}, \hat{q})|q\rangle = H(p, q) e^{-ipq} . \quad (9.13)$$

Now we slice the time evolution in (9.3) into small intervals

$$\Delta t = \frac{t - t_0}{n} , \quad t_j = t_0 + j \Delta t , \quad (9.14)$$

and $t_n = t$, see Figure 9.1. If we alternate insertions of complete sums of position and momentum states, we convert (9.3) in products of infinitesimal S-matrix elements, expressed in terms of $H(p_j, q_j)$, where j labels the j th insertion with $j = 1, \dots, n$. This product is augmented by integrations over all p_j, q_j . This idea is now carried out in detail, and we discuss generalisations and the respective changes for specific steps in the derivation at the end. Structurally, (9.7) covers the quantum field theories we have discussed so far with the exception of fermions, which have to be discussed separately for several reasons, see Chapter 12.

In the first step of the explicit computation we split the time evolution in time steps Δt with (9.14) and

$$U(t, t'') = U(t, t') U(t', t'') \quad (9.15)$$

for $t \geq t' \geq t''$. Moreover, we only insert complete sets of position states at first. The momentum states are then inserted in a second step. This leads us to

$$\begin{aligned} \langle q_f | U(t, t_0) | q_i \rangle &= \langle q_f | U(t, t_0 + (n-1)\Delta t) \cdots U(t_0 + \Delta t, t_0) | q_i \rangle \\ &= \left[\prod_{j=1}^{n-1} \int dq_j \langle q_{j+1} | U(t_{j+1}, t_j) | q_j \rangle \right] \langle q_1 | U(t_1, t_0) | q_0 \rangle, \end{aligned} \quad (9.16)$$

with $t_n = t$, $q_n = q_f$ and $q_i = q_0$. For $\Delta t \rightarrow 0$ (that is $n \rightarrow \infty$) we can expand the time evolution operator in powers of Δt , to wit,

$$U(t_{j+1}, t_j) = \underbrace{\mathbb{1} - i\hat{H}\Delta t}_{\simeq e^{-i\hat{H}\Delta t}} + O(\Delta t^2) \quad (9.17)$$

which satisfies (9.6). In a second step we insert the complete set of momentum states to the left of each infinitesimal time evolution operator $U(t_0 + (j+1)\Delta t, t_0 + j\Delta t)$. This leads us to

$$\begin{aligned} \langle q_{j+1} | U(t_{j+1}, t_j) | q_j \rangle &\simeq \langle q_{j+1} | \left[\mathbb{1} - i\hat{H}\Delta t \right] | q_j \rangle \\ &= \int \frac{dp}{2\pi} \langle q_{j+1} | p \rangle \left[\langle p | q_j \rangle - i\langle p | \hat{H} | q_j \rangle \right] \Delta t \\ &= \int \frac{dp}{(2\pi)} e^{+ipq_{j+1}} \left[e^{-ipq_j} - i\langle p | \hat{H} | q_j \rangle \Delta t \right]. \end{aligned} \quad (9.18)$$

Now we use (9.13), only valid for separable Hamiltonians (9.7). We emphasise that so far we have not used specific properties of the system but only the product structure of the time evolution, (9.15), and the expansion of the time evolution operator for infinitesimal time steps, (9.17). With (9.13) we arrive at

$$\begin{aligned} \langle q_{j+1} | U(t_{j+1}, t_j) | q_j \rangle &= \int \frac{dp}{2\pi} e^{+ipq_{j+1}} \left[e^{-ipq_j} - i\langle p | \hat{H} | q_j \rangle \Delta t \right] \\ &\simeq \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} \left\{ 1 - i \left[\frac{p^2}{2m} + V(q_j) \right] \Delta t \right\} \\ &\simeq \int \frac{dp}{2\pi} e^{i\left\{ p(q_{j+1}-q_j) - \left[\frac{p^2}{2m} + V(q_j) \right] \Delta t \right\}} \\ &\simeq \int \frac{dp}{2\pi} e^{i\Delta t \left\{ p \frac{(q_{j+1}-q_j)}{\Delta t} - \left[\frac{p^2}{2m} + V(q_j) \right] \right\}}. \end{aligned} \quad (9.19)$$

Equation (9.19) already reveals the structure of the emerging path integral in phase space with momentum and position integrals. It hosts a momentum integral, the accompanying position integral over q_j is present in (9.18). Moreover, the integrand is a phase factor, that consists of the Hamiltonian functional and an additional phase, that is proportional to the difference of the positions q_j and q_{j-1} . This is nothing but the discrete (left) derivative,

$$\dot{q}_j := \frac{q_{j+1} - q_j}{\Delta t}, \quad (9.20)$$

and with this definition, (9.19) turns into

$$\langle q_{j+1} | U(t_j, t_{j-1}) | q_j \rangle \simeq \int \frac{dp_j}{(2\pi)} e^{i\Delta t \left\{ p_j \dot{q}_j - \left[\frac{p_j^2}{2m} + V(q_j) \right] \right\}}, \quad (9.21)$$

where we have relabelled the integration variable $p \rightarrow p_j$ for the sake of convenience. The transition amplitude is nothing but the product of (9.21) for all $j = 1, \dots, n-1$. We are led to

$$\langle q_f | U(t, t_0) | q_{in} \rangle \simeq \int \left[\prod_{j=1}^{n-1} dq_j dp_j \right] e^{i\Delta t \sum_j [\dot{q}_j p_j - H(p_j, q_j)]}, \quad (9.22)$$

with $H(p, q) = p^2/2m + V(q)$, see (9.8). Finally, we take the (formal) limit $\Delta t \rightarrow 0$ with $n \rightarrow \infty$. We arrive at

$$\langle q_f | U(t, t_0) | q_i \rangle \simeq \int \mathcal{D}q \mathcal{D}p e^{i \int_{t_0}^t dt' [\dot{q}(t') p(t') - H(p(t'), q(t'))]}, \quad (9.23)$$

where we have used that $\Delta t \sum_j \rightarrow \int dt'$ with $t_j \rightarrow t'$. The path integral measure $\mathcal{D}q 1/(p^2)^{1-\eta/2} \mathcal{D}p$ is given by the product of all position and momentum integrations in this limit,

$$\mathcal{D}q = \prod_{j=1}^{n-1} dq_j = \prod_j dq(t_j) \Big|_{\substack{q(t_0)=q_{in} \\ q(t)=q_f}}, \quad \mathcal{D}p = \prod_{j=1}^{n-1} dp_j = \prod_j dp(t_j). \quad (9.24)$$

Equation (9.23) is the final result for the path integral in quantum mechanics in its phase space form. A further step concerns the momentum integration. For separable Hamiltonians of the form (9.7) with a quadratic kinetic term we can perform the Gaussian momentum integration in (9.23). To that end we use that the integrand is diagonal in momentum space, see (9.22). Using dt' instead of Δt and p instead of p_j , we get

$$\int_{\mathbb{R}} \frac{dp}{2\pi} e^{i \text{idt}' \left(\frac{pq - p^2}{2m} \right)} = \left[\int \frac{dp}{2\pi} e^{-i \text{idt}' \left(\frac{(p - qm/2)^2}{2m} \right)} \right] e^{i \text{idt}' \frac{mq^2}{2}} = \sqrt{\frac{1}{2\pi} \frac{m}{dt'}} e^{i \text{idt}' m \frac{q^2}{2}}. \quad (9.25)$$

The q, \dot{q} independent overall normalisation diverges in the limit $n \rightarrow \infty$: the factor $\sqrt{m/dt'}$ diverges as does the power $(\sqrt{m/(2\pi dt')})^{n-1}$. We will see shortly that this normalisation factors drops out of computation of correlation functions and observables and it is dropped accordingly. Performing the Gaussian integration (9.25) for each time, we arrive at

$$\int \mathcal{D}q \mathcal{D}p e^{i \int_{t_0}^t dt' \left[\left(\dot{q}(t) p(t) - \frac{p^2}{2m} \right) - V(q) \right]} \simeq \frac{1}{\mathcal{N}} \int \mathcal{D}q e^{iS[q]}, \quad (9.26)$$

with a normalisation \mathcal{N} and the action

$$S[q] = \int_{t_0}^t dt' \underbrace{\left\{ \frac{1}{2} m \dot{q}^2 - V(q) \right\}}_{L(q, \dot{q})}. \quad (9.27)$$

In summary we are led to an integral for the transition amplitude, that only contains integrals over the position variable $q(t')$ for all times $t' \in (t_0, t)$,

$$\langle q_f | U(t, t_0) | q_{in} \rangle \simeq \int \mathcal{D}q \mathcal{D}p e^{i \int_{t_0}^t dt' \left[\left(\dot{q}(t) p(t) - \frac{p^2}{2m} \right) - V(q) \right]} \simeq \frac{1}{\mathcal{N}} \int \mathcal{D}q e^{iS[q]}, \quad (9.28)$$

Equation (9.28) is the final expression for the *path integral* in quantum mechanics. As it only contains the integral over the position variable $q(t')$, it averages over all possible paths $q(t')$ from $t' = t_0$ to $t' = t$ with the weight $\exp\{iS[q]\}$.

9.2. Correlation Functions

In the last section we have derived the path integral representation of the transition amplitude from a fixed position q_i at the initial time t_0 to a fixed position q_f at the final time t . We would like to extend this computation to time-ordered correlation functions,

$$\langle \Omega | T \hat{q}(t_n) \cdots \hat{q}(t_0) | \Omega \rangle, \quad (9.29)$$

in the Heisenberg picture with the time-dependent position operators $\hat{q}(t)$, see (9.31). We have used these correlation functions so far for accessing observables such as S-matrix elements.

For obtaining the general correlation functions (9.29), we first consider (9.28) in the presence of additional position operators \hat{q} at the times t_j with $j = 1, \dots, n$ at the times t_1, \dots, t_n with $t \geq t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq t_0$,

$$\langle q_f | U(t, t_n) \hat{q} U(t_{n-1}, t_{n-2}) \cdots U(t_2, t_1) \hat{q} U(t_1, t_0) | q_i \rangle = \langle q_f, t | \hat{q}(t_{n-1}) \cdots \hat{q}(t_1) | q_i, t_0 \rangle, \quad (9.30)$$

with the definition

$$|q, t\rangle = U(0, t)|q\rangle, \quad \hat{q}(t) = U(0, t)\hat{q}U(t, 0) \quad (9.31)$$

Before we discuss (9.30) for general $n \in \mathbb{N}$, we evaluate the general structure at the cases $n = 0, 1$. Evidently, for $n = 0$, (9.30) reduces to (9.28),

$$n = 0 : \quad \langle q_f, t | q_i, t_0 \rangle = \langle q_f | U(t, t_0) | q_i \rangle \simeq \int \mathcal{D}q e^{iS[q]}. \quad (9.32)$$

The case $n = 1$ is the first non-trivial one with a single position operator insertion $\hat{q}(t_1)$ at the time t_1 . For its evaluation we insert a complete set of states $\int dq |q, t_1\rangle \langle q, t_1|$ in (9.30), left or right of the position operator $\hat{q}(t_1)$. Then, the correlation function is converted to a product of transition amplitudes from t_0 to t_1 and t_1 to t , multiplied by the position $q(t_1)$, which is integrated over,

$$\begin{aligned} n = 1 : \quad \langle q_f, t | \hat{q}(t_1) | q_i, t_0 \rangle &= \int dq \langle q_f, t | q, t_1 \rangle q \langle q, t_1 | q_i, t_0 \rangle \\ &= \int dq q \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_1)=q}} e^{iS[q]} \int \mathcal{D}q \Big|_{\substack{q(t_1)=q \\ q(t_0)=q_i}} e^{iS[q]} \\ &= \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_0)=q_i}} q(t_1) e^{iS[q]}, \end{aligned} \quad (9.33)$$

where we have used that the two integrations do not overlap, the times of the positions $q(t')$ of the left integral are in the interval $t' \in (t, t_1)$ and the times t'' in the right integral are in the interval $t'' \in (t_1, t)$.

In essence, the insertion of the complete set of states has reduced the $n = 1$ correlation function to the product of two $n = 0$ correlation functions. Hence, if we have already computed the path integral for $n = 0$, the one-point function as well as higher correlation functions follow suit. The two $n = 0$ parts of the path integral are ‘glued’ together at t_1 with a further integration over all positions q (at t_1), multiplied with $q(t_1)$. This resolves the fixed boundary condition of the two transition amplitudes at t_1 , and the two path integrals with a fixed upper and lower boundary $q(t_1)$ can be combined into one path integral with fixed boundaries $q(t)$ and $q(t_0)$,

$$\int dq q \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_1)=q}} \int \mathcal{D}q \Big|_{\substack{q(t_1)=q \\ q(t_0)=q_i}} \mathcal{F}[q(t)] = \int \mathcal{D}q \Big|_{\substack{q(t)=q_f \\ q(t_0)=q_i}} q(t_1) q(t_1) \mathcal{F}[q(t)], \quad (9.34)$$

for all functionals $F[q]$. For (9.33) we have $\mathcal{F}[q(t)] = \exp\{iS[q]\}$. As in (9.33) we have used the time-ordering of the two integrals: they do not overlap. In this sense, time ordering is natural *but not necessary* for the path integral.

With the two cases $n = 0, 1$ we readily evaluate the general case $n \in \mathbb{N}$ with $t \geq t_n \geq \cdots \geq t_1 \geq t_0$. For each operator insertion $\hat{q}(t_j)$ with $j = 1, \dots, n$ we insert a complete set of states to the left or the right as done for $n = 1$. This leads us to

$$\langle q_f, t | \hat{q}(t_n) \cdots \hat{q}(t_1) | q_{in}, t_0 \rangle = \int \mathcal{D}q \Big|_{\substack{q(t_0)=q_{in} \\ q(t_f)=q_f}} q(t_n) \cdots q(t_1) e^{iS[q]}, \quad (9.35)$$

where we have used (9.34) or rather (9.33) for all times t_j . For a specific t_j we use (9.33) for the one-point function

$$\langle q_f, t_{j+1} | \hat{q}(t_j) | q_{j-1}, t_0 \rangle. \quad (9.36)$$

This concludes the derivation of the path integral representation of general correlation functions with boundary conditions $q(t_0) = q_i$ and $q(t) = q_f$ for the initial and final time.

However, in QFT, in most physics applications we are more interested in vacuum correlation functions

$$\langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle, \quad (9.37)$$

see also (3.78) in Chapter 3. The correlation functions (9.37) are obtained from (9.35) by sending the initial and final times t_0, t to $\mp\infty$. As discussed in Chapter 3, in these limits only the vacuum state survives. This projection is achieved in a controlled way by introducing an infinitesimal damping factor in (9.22) or (9.23) with

$$e^{-i\Delta t H} \rightarrow e^{-i\Delta t(1-i\epsilon)H} = e^{-i\Delta t H} e^{-\Delta t H \epsilon}. \quad (9.38)$$

With (9.38), all states are suppressed relative to the ground state with $e^{-(E-E_0)\Delta t \epsilon}$ in each time step. Hence, in the limit $t_0 \rightarrow -\infty$ and $t \rightarrow \infty$, these suppression factors add up and the contribution of all higher energy states vanish. This leads us to

$$\lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow \infty}} \langle q_f, t | \hat{q}(t_n) \cdots \hat{q}(t_1) | q_i, t_0 \rangle \simeq \langle 0 | \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle. \quad (9.39)$$

Note that strictly speaking (9.39) only holds if $\langle q_f, T | 0 \rangle \neq 0$. Assuming this property we take the limit (9.39) in (9.35),

$$\langle 0 | T \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \sim \int \mathcal{D}q q(t_n) \cdots q(t_1) e^{iS_\epsilon[q]} \quad (9.40)$$

with

$$S_\epsilon[q] = \int dt \left(\frac{1}{2} m \dot{q}^2 - V(q) + i\epsilon q^2 \right) \quad (9.41)$$

The derivation of (9.40) was done with $t_n \geq t_{n-1} \geq \dots \geq t_1$. However, the *path integral* or *functional integral* on the rhs of (9.40) can be written down for general times t_1, \dots, t_n without time ordering,

$$\langle 0 | T \hat{q}(t_n) \cdots \hat{q}(t_1) | 0 \rangle \simeq \int \mathcal{D}q q(t_1) \cdots q(t_n) e^{iS_\epsilon[q]} \quad (9.42)$$

where the ϵ prescription of (9.41) is implicitly understood. In a final step we have to fix the normalisation of the correlation functions. This normalisation is fixed, as in (3.78), with the condition, that properly normalised states are normalised to one. Put differently the expectation value of the unity operator $\mathbb{1}$ is one. This expectation value is obtained for $n = 0$, and we arrive at our final path integral expression for normalised correlation functions,

$$\langle \Omega | T \hat{q}(t_1) \cdots \hat{q}(t_n) | \Omega \rangle = \frac{\langle 0 | T \hat{q}(t_1) \cdots \hat{q}(t_n) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\int \mathcal{D}q q(t_1) \cdots q(t_n) e^{iS[q]}}{\int \mathcal{D}q e^{iS[q]}} \quad (9.43)$$

This concludes the derivation of the path integral expression for general normalised correlation functions. We close this Section with two remarks on the regularisation and ordering implicit in (9.43). In the derivation above we have introduced a regularising damping factor in the Hamiltonian, see (9.38). This led to a respective damping in the action, see (9.41). The quadratic part of the action S_ε in (9.41) is the action of the harmonic oscillator. It reads

$$S_\varepsilon[q] = \frac{1}{2} \int dt q(t) [-\partial_t^2 - \omega^2 + i\varepsilon] q(t) \quad (9.44)$$

where we have put the mass to one, $m \rightarrow 1$ and the frequency of the harmonic oscillator is given by $\pm\omega$. Its propagator or covariance C follows from (9.44) in momentum space with momentum p ,

$$C(p) = \frac{1}{p^2 - \omega^2 + i\varepsilon} = C. \quad (9.45)$$

Equation (9.45) is the time-ordered (Feynman) propagator. Hence, the path integral in (9.43) provides time-order correlation function. In turn, changing the covariance or propagator (9.45) in the path integral leads to correlation functions that carry the ordering or causality structure of the respective covariance,

$$\text{Choice of covariance } C \Leftrightarrow \text{choice of operator ordering} \quad (9.46)$$

This result is very apparent in terms of generating functionals, that are discussed in the next Section.

9.3. Generating Functionals

In the derivation of the path integral expression for expectation values (9.43) we have used that these expectation values can be transformed into products of transition amplitudes with operator insertions $\hat{q}(t_j)$ for $j = 1, \dots, n$. Moreover, (9.43) are simply the normalised n th moments of the path integral. This suggests to define a generating functional $Z[j(t)]$ for these moments, whose n th derivatives with respect to the currents are these moments.

9.3.1. Generating functional in a toy example

The following considerations necessitate the use of functional derivatives, as well as being at ease with the notion of infinite-dimensional integrals. Indeed, while the latter will be treated in most applications as their finite-dimensional counterparts, this should be done with caution as there are a few notable exceptions to this 'phenomenological' rule. One of them is the axial anomaly (or general anomalies) discussed in Chapter 8 and we shall rediscuss it later in the functional integral approach.

Apart from these exceptions we shall use the 'phenomenological' rule. Hence, we start the evaluation of the generating functional in the present Section with a finite-dimensional toy example. This allows us to discuss the structural properties in a familiar setting in terms of a generating function, before we transport the respective results to generating functionals in quantum mechanics and quantum field theory.

We consider the integral one-dimensional $\int_{\mathbb{R}} dq e^{iS(q)}$ with the moments,

$$\langle q^n \rangle := \frac{1}{\mathcal{N}} \int_{\mathbb{R}} dq q^n e^{iS(q)}, \quad \text{with} \quad \mathcal{N} = \int_{\mathbb{R}} dq e^{iS(q)}. \quad (9.47)$$

These moments are generated by the generating function $Z(j)$ with

$$Z(j) = \frac{1}{\mathcal{N}} \int_{\mathbb{R}} dq e^{i\{S(q)+jq\}}. \quad (9.48)$$

The generating functional is nothing but the Fourier transform of the path integral 'measure' $dq \exp\{iS(q)\}$. Its n th derivatives with respect to the current j generate the moments (9.47),

$$\langle q^n \rangle = (-i)^n \frac{\partial^n Z[j]}{\partial j^n} \Big|_{j=0}. \quad (9.49)$$

We can already discuss some of the advantageous properties of the present formulation within the toy example (9.48). First of all the path integral is readily performed in the free, Gaußian, theory with the free action

$$S_0(q) = \frac{1}{2} q \alpha q, \quad (9.50)$$

The factor α is nothing but the dispersion in a field theory which is obtained with $\alpha \rightarrow -\partial^2 - m^2$, $q \rightarrow \phi(x)$ and the respective d -dimensional space-time integral. We insert (9.50) in (9.48) and obtain

$$Z_0(j) = \frac{1}{\mathcal{N}_0} \int_{\mathbb{R}} dq e^{i\{\frac{1}{2}q\alpha q + jq\}} = \frac{1}{\mathcal{N}_0} \int_{\mathbb{R}} dq e^{i\{\frac{\alpha}{2}(q+\frac{1}{\alpha}j)^2 - \frac{1}{2}j\frac{1}{\alpha}j\}} = e^{-\frac{1}{2}j\frac{1}{\alpha}j}, \quad \text{and} \quad \mathcal{N}_0(\bar{\alpha}) = \sqrt{\frac{\pi}{\bar{\alpha}}}. \quad (9.51)$$

with the covariance $C = 1/\alpha$ and $\bar{\alpha} = -i\alpha$. In a free scalar field theory this is the free propagator, and with its choice the operator ordering in the correlation functions is determined. The explicit Gaußian form on the right hand side of (9.51) allows us to determine the correlation functions in a closed form,

$$\langle q^{2n} \rangle = \frac{2n!}{n!} \frac{1}{2^n} \frac{1}{\alpha^n}, \quad \text{and} \quad \langle q^{2n+1} \rangle = 0, \quad (9.52)$$

where the odd correlation functions follow from the symmetry of the free action under $q \rightarrow -q$, see also (9.56) for the general case. We also note that the even correlation functions can also be obtained from the respective power of α -derivatives of the normalisation \mathcal{N}_0 , which is simply the integral in the absence of the source term. Then, an $\bar{\alpha}$ -derivative pulls down $q^2/2$, and we get

$$\langle q^{2n} \rangle = \frac{1}{\mathcal{N}_0(\bar{\alpha})} \left(-2 \frac{\partial}{\partial \bar{\alpha}} \right)^n \mathcal{N}_0(\bar{\alpha}). \quad (9.53)$$

The factor $1/\alpha^n$ in (9.52) entails that the even correlation functions in the free theory are nothing but the products of the covariance or propagator. The combinatoric prefactor suggests that the derivation of Feynman rules including the combinatorial factors may be simpler in the path integral approach: the denominator $1/(n!)$ originates in the fact, that the $2n$ th derivative with respect to the current (at $j = 0$) only acts on the n th order expansion term of the exponential. The power $(2n)!$ is generated by $2n$ derivatives of j^{2n} . Finally, the factor $1/2^n$ is the product of n factors $1/2$ in the Gaußian exponential.

This simple access to the combinatorics is further elucidated by extending the present structural analysis in a toy model to the interacting case with the action,

$$S(q) = \frac{1}{2} q \alpha q - V(q), \quad \text{with e.g.} \quad V(q) = \frac{\lambda}{4!} q^4. \quad (9.54)$$

With (9.54), the integral (9.48) cannot be performed analytically. However, for the sake of completeness we note that the integral can be performed for $j = 0$ and we obtain the normalisation as a modified Bessel function,

$$\mathcal{N} = -\frac{(-1)^{3/4} \sqrt{\alpha} e^{-\frac{\alpha^2}{8\lambda}} K_{\frac{1}{4}}\left(-\frac{\alpha^2}{8\lambda}\right)}{2\sqrt{\lambda}}, \quad (9.55)$$

where we have assumed $\text{Im}(\alpha) > 0$, $\text{Im}(\lambda) < 0$ and $\text{Re}(\lambda) > 0$. Equation (9.55) comes handy for some analytical and numerical checks. Note also that the odd expectation values vanish as $S(-q) = S(q)$ and hence

$$\langle (-q)^{2n+1} \rangle = \langle q^{2n+1} \rangle \quad \longrightarrow \quad \langle q^{2n+1} \rangle = 0 \quad (9.56)$$

As in the free case this leaves us with the even correlation functions, which can be obtained from (9.53) by substituting the free normalisation with the interacting one, $\mathcal{N}_0 \rightarrow \mathcal{N}$ in (9.55). Accordingly, the even correlation functions are simply combinations of derivatives of the Bessel function and $\sqrt{\alpha}$.

Beyond the present toy model, the normalisation cannot be computed analytically and we have to either resort to numerical methods or to perturbation theory (or other analytic resummation schemes). In perturbation theory the path integral is expanded about the free path integral. This leaves us to

$$\begin{aligned} Z(j) &= \frac{1}{\mathcal{N}} \int dq e^{i\{S(q)+jq\}} = \frac{1}{\mathcal{N}} \int dq e^{i\{S_0(q)+jq\}} e^{-iV(q)} = \frac{1}{\mathcal{N}} \int dq \sum_n \frac{[-iV(q)]^n}{n!} e^{i\{S_0(q)+jq\}} \\ &= \frac{1}{\mathcal{N}} \sum_n \frac{[-iV(-i\frac{\partial}{\partial j})]^n}{n!} \int dq e^{i\{S_0(q)+jq\}} = \frac{\mathcal{N}_0}{\mathcal{N}} \sum_n \frac{[-iV(-i\frac{\partial}{\partial j})]^n}{n!} Z_0(j). \end{aligned} \quad (9.57)$$

The series in the last line of (9.57) is the exponential series of the operator $-iV(-i\partial/\partial j)$ and we arrive at

$$Z(j) = \frac{\mathcal{N}_0}{\mathcal{N}} e^{-iV(-i\frac{\partial}{\partial j})} Z_0(j). \quad (9.58)$$

In perturbation theory, the generating functional or the respective correlation functions are expanded in powers of the interaction operator $-iV(-i\partial/\partial j)$. All topologies of the 'diagrams' and the combinatorial factors follow from the combination of the prefactors of the n th powers of j and the j -derivatives.

A final remark concerns the quantum equations of motion and quantum symmetry identities. For the symmetry identities we consider a complex variable, $q \rightarrow z = x_1 + iy_2 \in \mathbb{C}$ with polar coordinates $z = r \exp\{i\theta\}$. The generating path now reads

$$Z(j) = \frac{1}{\mathcal{N}} \int d^2x e^{i\{S(z\bar{z})+jz\}}, \quad (9.59)$$

where we have restricted ourselves to actions that only depend on the radial coordinate $z\bar{z} = r^2 = x_1^2 + x_2^2$. This action has rotational symmetry, as has the measure,

$$d^2x' = d^2x, \quad \text{and} \quad S(z'\bar{z}') = S(z\bar{z}). \quad (9.60)$$

Hence, changing the integration variable to z' , the generating functional has the representation

$$Z(j) = \frac{1}{\mathcal{N}} \int d^2x e^{i\{S(z\bar{z})+jre^{i\varphi}\}}, \quad \text{with} \quad \frac{\partial^n Z(j)}{\partial \varphi^n} = 0. \quad (9.61)$$

The φ -derivatives only hit the current term and lead to 'quantum' symmetry identities,

$$\langle z^n \rangle = 0. \quad (9.62)$$

In the present toy theory this simply entails that the expectation value of powers of the phase is vanishing. Splitting the measure into a radial integration and the phase integration,

$$\int d^2x = \int_0^\infty r dr \int_0^{2\pi} d\theta, \quad (9.63)$$

the above correlation functions in (9.62) reduce to

$$\langle z^n \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{in\theta} \langle r^n \rangle = 0. \quad (9.64)$$

In short, for phase-independent actions the expectation values of all monomial 'operators', that have a phase dependence, vanish.

Finally, the 'quantum' equation of motion is readily derived from

$$\frac{1}{\mathcal{N}} \int dq \left[\frac{\partial}{\partial q} e^{i[S(q)+jq]} \right] = i \left[\left\langle \frac{\partial S(q)}{\partial q} \right\rangle + j \right] = 0. \quad (9.65)$$

Equation (9.65) entails, that the expectation value of the classical equation of motion in the presence of a current, $\partial S(q)/\partial q + j = 0$, also vanishes, it is the path integral form of Ehrenfest's theorem or the quantum equation of motion, the Dyson-Schwinger equation (DSE). With this remark we conclude our discussion of the properties of (path) integrals. The chosen examples elucidate, that we can expect significant simplifications for both, symmetry and structural considerations as well as explicit analytic and numerical computations.

9.3.2. Generating functional for quantum mechanics

Most of the results derived in the toy model in Section 9.3.1 carry directly over to quantum mechanics and quantum field theory. Structurally this can be understood by undoing the limit $\Delta t \rightarrow 0$ in (9.23). Then we have to deal with a high but finite number of standard integrals as in Section 9.3.1. Moreover, for finite Δt the source term reads

$$\exp \left\{ i\Delta t \sum_l^n q_l j_l \right\} \xrightarrow{\Delta t \rightarrow 0} \exp \left\{ i \int_{t_0}^t dt' j(t') q(t') \right\}, \quad (9.66)$$

where a derivative on the right hand side with respect to j_l pulls down the position variable q_l at the position $t_0 + l\Delta t$.

$$-i \frac{1}{\Delta t} \frac{\partial}{\partial j_m} e^{i\Delta t \sum_l^n q_l j_l} = q_m e^{i\Delta t \sum_l^n q_l j_l}. \quad (9.67)$$

The cautious reader may stay in this discrete formulation of the path integral and only consider the convergence towards continuous times at the end. However, it is more convenient to consider directly the limit $\Delta t \rightarrow 0$ with

$$(q_l, j_l) \rightarrow (q(t), j(t)) \quad \text{and} \quad \frac{1}{\Delta t} \left(\frac{\partial}{\partial q_l}, \frac{\partial}{\partial j_l} \right) \rightarrow \left(\frac{\delta}{\delta q(t)}, \frac{\delta}{\delta j(t)} \right). \quad (9.68)$$

The functional derivative obeys the rule

$$\frac{\delta j(t')}{\delta j(t)} = \delta(t-t'), \quad \longrightarrow \quad \frac{\delta \partial_{t'}^n j(t')}{\delta j(t)} = \partial_{t'}^n \delta(t-t'), \quad (9.69)$$

and satisfies the Leibniz rule. This suffices to derive other rules such as the product and the chain rules. More details on functional derivatives can be found in [Appendix E](#). With (9.69), the analogue of (9.67) follows as

$$-i \frac{\delta}{\delta j(t)} e^{i \int_{\mathbb{R}} dt' j(t') q(t')} = \int_{\mathbb{R}} dt' \frac{\delta j(t')}{\delta j(t)} q(t') e^{i \int_{\mathbb{R}} dt' j(t') q(t')} = i q(t) e^{i \int_{\mathbb{R}} dt' j(t') q(t')}. \quad (9.70)$$

More details on functional derivatives are provided in [Appendix E](#). In summary we are led to

Generating functional

$$Z[j] = \frac{1}{\mathcal{N}} \int \mathcal{D}q e^{i\{S[q] + \int dt j(t)q(t)\}} \quad \text{with} \quad \mathcal{N} = \int \mathcal{D}q e^{iS[q]}, \quad (9.71)$$

and the normalised correlation functions are determined by

Correlation functions

$$\langle T \hat{q}(t_1) \cdots \hat{q}(t_n) \rangle = \frac{\langle 0 | T \hat{q}(t_1) \cdots \hat{q}(t_n) | 0 \rangle}{\langle 0 | 0 \rangle} = (-i)^n \frac{\delta^n Z[j]}{\delta j(t_1) \cdots \delta j(t_n)} \Big|_{j=0}. \quad (9.72)$$

[Equations \(9.71\) and \(9.72\)](#) are the final functional integral expressions for the generating functional and the correlation functions in quantum mechanics. We close this Section with a discussion of the free Gaussian generating functional in quantum mechanics, and the analogue of the toy example (9.58), set-up for the perturbative expansion.

The free generating functional is obtained from (9.71) with the (free) action of the harmonic oscillator,

$$S_0[q] = \frac{1}{2} \int dt q(t) (-\partial_t^2 - \omega^2) q(t). \quad (9.73)$$

With (9.73), the generating functional is given by

$$Z_0[j] = \frac{1}{\mathcal{N}_0} \int \mathcal{D}q e^{i\{S_0[q] + \int dt j(t)q(t)\}} = \frac{1}{\mathcal{N}_0} \int \mathcal{D}q e^{iS_0[q']} e^{-\frac{1}{2} \int dt dt' j(t) G(t, t') j(t')}, \quad (9.74)$$

with the covariance or free propagator $G(t, t')$,

$$(-\partial_t^2 - \omega^2) G(t, t') = i \delta(t - t'), \quad (9.75)$$

and the shifted position variable $q'(t)$

$$q'(t) = q(t) - i \int dt' G(t, t') j(t'). \quad (9.76)$$

While the normalisation drops out in the final expression, it is a simple Gaussian integral. Its explicit computation is deferred to [Appendix F](#) and we quote the result

$$\mathcal{N}_0 = \int \mathcal{D}q e^{iS_0[q]} \simeq \frac{1}{\sqrt{\det(-\partial_t^2 - \omega^2)}} = \sqrt{\det G(t, t')}, \quad (9.77)$$

where we dropped an (infinite) constant normalisation factor $\lim_{n \rightarrow \infty} (2\pi)^{n/2}$ present in (F.4). In contradistinction to the determinant this factor does not depend on the parameters of the theory, and can be safely dropped.

In (9.74) we have completed the square as in the one-dimensional toy example. In quantum mechanics this includes the inversion of the dispersion instead of simply taking the inverse of a number. Now we use, that the path integral measure is translation invariant

$$\mathcal{D}q = \mathcal{D}q', \quad (9.78)$$

which follows directly from the translation invariance of $dq = dq'$ and $\mathcal{D}q = \prod_k dq(t_k)$. This allows us to shift the integration variable to q' and perform the functional integral over q' . Consequently we find

Free generating functional

$$Z_0[j] = e^{-\frac{1}{2} \int dt dt' j(t) G(t, t') j(t')}. \quad (9.79)$$

with the propagator $G(t, t')$ defined in (9.75). Equation (9.79) makes it apparent, that the choice of the covariance $G(t, t')$ determines the operator ordering implemented in the functional integral, see (9.46). We also emphasise that while (9.79) looks like a damped exponential, it is actually a phase as the propagator G includes an imaginary unit i , see (9.75).

The result (9.79) for the free theory sets the stage for the derivation of the perturbative expansion in the interacting theory with the action

$$S[q] = S_0[q] - \int dt V(q), \quad (9.80)$$

with the free action S_0 defined in (9.73). In analogy to (9.58) we find

$$Z[j] = \frac{\mathcal{N}_0}{\mathcal{N}} e^{-i \int dt V(-i \frac{\delta}{\delta j(t)})} Z_0[j]. \quad (9.81)$$

We close this Section with a brief evaluation of perturbation theory and the simple generation of all diagrams and combinatorial factors in the present functional integral approach. The respective derivation of the Feynman rules will be detailed in Section 10.2. For the sake of simplicity we restrict ourselves to a quartic potential

$$V(a) = \frac{\lambda}{4!} \int dt q(t)^4. \quad (9.82)$$

The n -point correlation functions or n th moments are obtained by the n th derivatives of the generating functional with respect to the current, see (9.72). The contribution to a given n -point function proportional to λ^l is generated by Z_l with

$$Z_l[j] = \frac{\mathcal{N}_0}{\mathcal{N}} \frac{1}{l!} \left[-i \int dt V \left(-i \frac{\delta}{\delta j(t)} \right) \right]^l Z_0[j]. \quad (9.83)$$

We remark that $Z_l[j]$ does not generate l -loop diagrams as the power of the coupling does not coincide with the number of loops. As $Z_0[j]$ only contains even powers of j and the powers of the potential carry $4l$ j -derivatives, the odd order $(2n + 1)$ correlation functions vanish. Consequently we only consider even order $(2n)$ correlation functions. Applying $2n$ derivatives with respect to $j(t_k)$ with $k = 1, \dots, 2n$ to (9.83), and finally setting $j = 0$ leads us to

$$\begin{aligned} & \left. \prod_{k=1}^{2n} (-i) \frac{\delta}{\delta j(t_k)} Z_l[j] \right|_{j=0} \\ &= \frac{\mathcal{N}_0}{\mathcal{N}} (-\lambda)^l \left(-\frac{1}{2} \right)^{2l+n} \frac{1}{(2l+n)!} \frac{1}{l!} \left[\prod_{k=1}^{2n} \frac{\delta}{\delta j(t_k)} \right] \left[\int dt \left(\frac{\delta}{\delta j(t)} \right)^4 \right]^l \left[\int dt dt' j(t) G(t, t') j(t') \right]^{2l+n}. \end{aligned} \quad (9.84)$$

Equation (9.84) elucidates the fact that the different classes of diagrams as well as the combinatorial factors are simply generated by the prefactors of the two exponential series, the factorial factors due to the differentiation and the different possibilities to generate the same diagram. For more details see **Section 10.2**.

10. Scalar field theories

The derivation of the functional integral for quantum field theories is now put forward in a completely analogous way to that in quantum mechanics in [Section 9.1](#), and we will refer to the respective analogous steps on the way.

10.1. Functional integral for a real scalar field

We consider a d -dimensional scalar quantum field theory with the action

$$S[\varphi] = \int d^d x \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right], \quad (10.1)$$

with the scalar field

$$\varphi = \varphi(x) \in \mathbb{R} \quad \text{with} \quad x \in \mathbb{R}^d \quad (10.2)$$

The corresponding Hamiltonian operator is given by

$$\hat{H} = \int d^{d-1} x \left[\frac{1}{2} \hat{\pi}^2 + \frac{1}{2} (\nabla \hat{\varphi})^2 + V(\hat{\varphi}) \right]. \quad (10.3)$$

As in quantum mechanics we restrict ourselves to separable Hamiltonians with a quadratic dependence on the field momentum $\hat{\pi}(x)$. The field and momentum operators in the Hamiltonian satisfy the canonical commutation relations

$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}). \quad (10.4)$$

The functional integral approach is based on the functional integral representation of the transition amplitude

$$\langle \phi_f | U(t, t_0) | \phi_{in} \rangle, \quad (10.5)$$

with the initial and final states $|\phi_i\rangle$ and $|\phi_f\rangle$ in our quantum field theory. Now we split the time evolution into infinitesimal steps with times $t_j = t_0 + j\Delta t$, see [\(9.14\)](#). Then the time evolution operator can be expanded in powers of Δt , to wit,

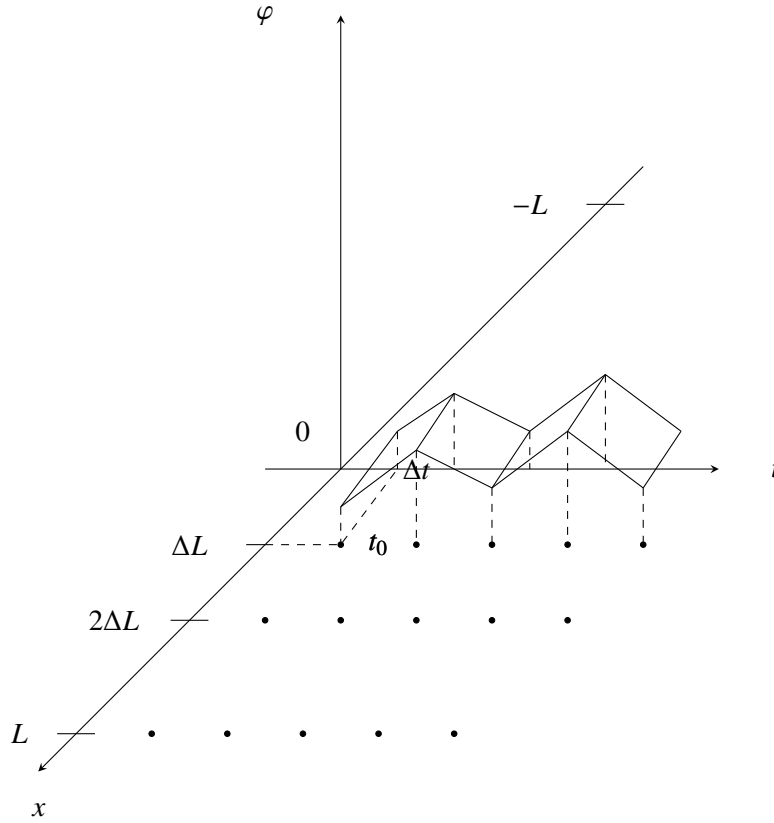
$$U(t_j, t_{j+1}) = \mathbb{1} - i\hat{H}\Delta t + O(\Delta t^2), \quad (10.6)$$

with the Hamiltonian operator [\(10.3\)](#). For the insertion of a complete set of field and field-momentum states these states are required. They are defined in complete analogy to the definition in quantum mechanics in [\(9.10\)](#),

$$\hat{\varphi}|\varphi\rangle = \varphi(\vec{x})|\varphi\rangle, \quad \hat{\pi}|\pi\rangle = \pi(\vec{x})|\pi\rangle, \quad (10.7)$$

which satisfy the completeness relations

$$\begin{aligned} \mathbf{1} &= \int \mathcal{D}\varphi |\varphi\rangle\langle\varphi|, & \langle\varphi|\varphi'\rangle &= \delta[\varphi(\vec{x}) - \varphi'(\vec{x})], \\ \mathbf{1} &= \int \mathcal{D}\pi |\pi\rangle\langle\pi|, & \langle\pi|\pi'\rangle &= \delta[\pi(\vec{x}) - \pi'(\vec{x})], \end{aligned} \quad (10.8)$$


 Figure 10.1.: Discretisation of the function $\varphi(t, x)$ on a space-time lattice (grid).

However, in contradistinction to quantum mechanics, the completeness relation (10.8) in quantum field theory is already functional: it involves functional integrals $dq \rightarrow \mathcal{D}\varphi$ and $dp \rightarrow \mathcal{D}\pi$ as well as functional δ -functions. These relations can be derived in the many-body limit of quantum mechanics, which has been used in QFTI to derive quantum field theory in the first place. To that end we consider

$$\psi(q) \rightarrow \psi(q_1, \dots, q_n) \rightarrow \Psi[\varphi], \quad \delta(q - q') \rightarrow \prod_j \delta(q_j - q'_j) \rightarrow \delta[\varphi(x) - \varphi'(x)]. \quad (10.9)$$

Equation (10.9) implies the formulation of the field theory on a spatial grid at the locations q_j , together with the temporal grid used in the derivation of the path integral in (10.6). This is depicted in Figure 10.1, which is the basis for the lattice formulation of quantum field theory. This functional integral approach is used both for formal considerations and proofs as well as most prominently for numerical simulations. We will discuss it in more detail in Chapter 15.

In summary we have collected all the necessary tools for repeating all steps for the derivation of the transition amplitude in quantum mechanics. We cut this derivation short, and present the final result for correlation functions,

Correlation functions

$$\langle T \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) \rangle = \frac{\langle 0 | T \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\int \mathcal{D}\varphi \varphi(x_1) \cdots \varphi(x_n) e^{iS[\varphi]}}{\int \mathcal{D}\varphi e^{iS[\varphi]}} \quad (10.10)$$

with the action

$$S[\varphi] = S_0[\varphi] - \int d^d x V_{\text{int}}(\varphi), \quad \text{with} \quad S_0[\varphi] = \int d^d x \left\{ \frac{1}{2} \varphi(x) [-\Delta - m^2 + i\varepsilon] \varphi(x) \right\}, \quad (10.11)$$

and the functional integral measure

$$\mathcal{D}\varphi = \prod_{j \in \text{lattice}} d\varphi_j, \quad \text{with} \quad \mathbf{j} = (j_0, \vec{j}). \quad (10.12)$$

Equation (10.11) leads to time-ordered correlation functions due to the $+i\varepsilon$, other choices are possible as discussed before. This concludes our discussion of the derivation of the functional integral for scalar field theories.

We close this Section with a brief discussion of alternative representations of the path integral. The derivations so far relied explicitly or implicitly on limits of integrals on an underlying space-time grid, and the respective basis are functions that live on these space-time points. While this emerged naturally in our derivation of the path integral of the transition amplitude, it is not a natural basis for functions $\varphi(x)$ on \mathbb{R}^d or some (compact) submanifold of it. These functions may also be expanded in some countable complete set of orthonormal square integrable basis functions,

$$\varphi(x) = \sum_{n \in \mathbb{N}} c_n \varphi_n(x), \quad \text{with} \quad (\varphi_n, \varphi_m) = \int d^d x \varphi_n(x) \varphi_m(x) = \delta_{nm}. \quad (10.13)$$

Note that the functions $\varphi_n(x)$ are normalised to unity, and not to δ -functions such as the plane waves. Integrating over all functions $\varphi(x)$ is identical to integrating over all values of the coefficients $c_n \in \mathbb{R}$. This leads us to

$$\int \mathcal{D}\varphi e^{iS[\varphi]} \simeq \int \left[\prod_n dc_n \right] e^{iS[\varphi]}, \quad (10.14)$$

where we have used

$$\mathcal{D}\varphi \simeq \prod_n dc_n. \quad (10.15)$$

Note that the Jacobian of this transformation is field (or c_n)-independent. Moreover, the equivalence of the path integral in terms of $\int \prod_n dc_n$ is readily shown at the example of the free theory as we can use the representation (9.83) also in quantum field theory. Accordingly, it suffices to show that the normalisation \mathcal{N}_0 takes the same form as in (9.77).

In the free theory the action (10.11) reduces to its kinetic part, $S[\varphi] = S_0[\varphi]$. We make a convenient choice for the set of basis functions used in 10.1: the eigenfunctions of the kinetic operator itself,

$$(-\Delta - m^2) \varphi_n = \lambda_n \varphi_n, \quad \text{with} \quad \lambda_n \in \mathbb{R}^+ \quad (10.16)$$

We emphasise that the results for the path integral does not depend on the chosen basis as the rotation from one orthonormal basis to another one has the Jacobian $J = 1$. Note also that in \mathbb{R}^d the eigenfunctions are the plane waves and obviously we are back to plane-wave normalisable functions. We assume in the following, that the theory is formulated in a finite space-time box (hypercube) which reflects the common experimental situation: the lab has a finite extend and the experiment takes place in a finite time interval. Then the spectrum is discrete and the eigenfunctions satisfy (10.13). For the path integral we are led to

$$\int \mathcal{D}\varphi e^{iS_0[\varphi]} \simeq \int \left[\prod_n dc_n \right] e^{\frac{i}{2} \sum_m c_m^2 (\lambda_m + i\varepsilon)}, \quad (10.17)$$

with the exponential damping factor $\exp\{-\varepsilon c_m^2\}$ for each c_m -integration. In (10.17) we have used that the action is diagonal in the chosen basis (10.16),

$$\begin{aligned} S_0[\varphi] &= \frac{1}{2} \sum_{n,m} \int d^d x c_n c_m \varphi_n(x) [-\Delta - m^2 + i\varepsilon] \varphi_m(x) \\ &= \frac{1}{2} \sum_{n,m} c_n c_m (\lambda_m + i\varepsilon) \int d^d x \varphi_n(x) \varphi_m(x) = \frac{1}{2} \sum_m c_m^2 (\lambda_m + i\varepsilon). \end{aligned} \quad (10.18)$$

Collecting all these results we arrive at

$$\int \mathcal{D}\varphi e^{iS_0[\varphi]} \simeq \prod_{n \in \mathbb{N}} \lambda_n^{-1/2} = \det^{-\frac{1}{2}}(-\Delta - m^2), \quad (10.19)$$

where again we dropped an (infinite) constant normalisation factor $\lim_{n \rightarrow \infty} (2\pi)^{n/2}$ present in (F.4) as we did in the derivation of (9.77). The functional integral of the free theory gives us the square root of the determinant of the kinetic operator. This reproduces the result in quantum mechanics and proves explicitly that the relation is well justified. As discussed before, with the representation (9.83) in the present quantum field theory the above result for the free theory suffices. Strictly speaking it is only left to show that the measure in terms of the c_n is invariant under a shift of the field with a function $c(x)$,

$$\varphi(x) \rightarrow \varphi'(x) = \varphi(x) + c(x), \quad \text{with} \quad c(x) = \sum \Delta c_n \varphi_n. \quad (10.20)$$

Accordingly, the above shift in the field shifts all coefficients $c_n \rightarrow c'_n = c_n + \Delta c_n$ and hence $dc'_n = dc_n$. We conclude

$$\mathcal{D}\varphi' \simeq \prod_n dc'_n = \prod_n dc_n \simeq \mathcal{D}\varphi, \quad (10.21)$$

which finalises our equivalence proof. In short, the path integral can be formulated in any convenient basis. While in most numerical applications the space-time grid basis is used, it is more efficient to use different bases in specific applications.

10.2. Generating functional and Feynman rules

In the last Section we have derived the path integral representation of correlation functions in the quantum field theory of a real scalar field. With the introduction of a source term we readily obtain the generating functional $Z[J]$, which already in quantum mechanics allowed us an easy access to the Feynman rules, see the brief discussion around (9.81). In the present Section we use the generating functional for a derivation of the Feynman rules. The generating functional is given by

Generating functional of a scalar QFT

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi e^{i\{S[\varphi] + \int d^d x J(x)\varphi(x)\}}, \quad (10.22)$$

and the correlation functions (10.10) follows as the respective derivatives of $Z[J]$,

Correlation functions derived from $Z[J]$

$$\langle T \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) \rangle = (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0}. \quad (10.23)$$

where we have used the functional derivative

$$\frac{\delta J(x)}{\delta J(y)} = \delta(x - y) \quad (10.24)$$

for a d -dimensional function $J(x)$ with $x = (x_1, \dots, x_d)^T$.

For setting up perturbation theory we rewrite the generating functional in terms of the generating functional of the free theory, $Z_0[J]$, and the exponential of the interaction term, $\exp\{-i \int d^d x V_{\text{int}}(-i\delta/\delta J)\}$ similar to (9.81). For the generating functional of the free theory we find

$$Z_0[J] = e^{-\frac{1}{2} \int d^d x d^d y J(x)G(x,y)J(y)}, \quad \text{with} \quad (-\Delta - m^2)_x G(x, y) = i\delta(x - y), \quad (10.25)$$

see (9.81) for $Z_0[j]$ for the harmonic oscillator. As there the exponential is a phase due to the imaginary unit in the definition in the propagator.

As simple example for the use of (10.23) we use it for the computation of $2n$ -point functions in the free theory. Evidently, the $2n + 1$ -point functions vanish as discussed before. For the two-point function we find

$$\langle T\varphi(x)\varphi(y) \rangle = - \frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)} \Big|_{J=0} = \frac{\delta^2}{\delta J(x)\delta J(y)} \left[\frac{1}{2} \int d^d x' d^d y' J(x')G(x', y')J(y') \right] = G(x, y). \quad (10.26)$$

Similarly we find for the four-point function,

$$\langle T\varphi(x_1) \dots \varphi(x_n) \rangle = \frac{\delta^4 Z_0[J]}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0} = G(x_1, x_2)G(x_3, x_4) + \text{permutations}. \quad (10.27)$$

This is the well-known results for the free four-point function with the depiction

$$\langle T\varphi(x_1) \dots \varphi(x_4) \rangle = \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \cdot \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \quad (10.28)$$

These results readily extend to the general $2n$ -point function, to wit,

$$\begin{aligned} \langle T\varphi(x_1) \dots \varphi(x_{2n}) \rangle &= (-i)^{2n} \frac{\delta^{2n} Z[J]}{\delta J(x_1) \dots \delta J(x_{2n})} \Big|_{J=0} \\ &= \frac{1}{2^n} \frac{1}{n!} \frac{\delta^{2n}}{\delta J(x_1) \dots \delta J(x_{2n})} \left[\int d^d x d^d y J(x)G(x, y)J(y) \right]^n \\ &= G(x_1, x_2)G(x_3, x_4) \dots G(x_{2n-1}, x_{2n}) + \text{permutations}. \end{aligned} \quad (10.29)$$

The above results iterate the fact, that all correlation functions or moments of the free theory are simply products of the second moment, the two-point function. This points at the redundancy of the generating functional $Z[J]$: it generates full correlation functions of the theory including the disconnected terms. In the logarithm of $Z[J]$, the *Schwinger functional*, this redundancy is absent. We define

Schwinger functional

$$W[J] = \ln Z[J]. \quad (10.30)$$

We remark that the logarithm in (10.30) entails that the normalisation \mathcal{N} drops out of all derivatives, as it occurs in (10.30) as the J -independent additive term $\ln \mathcal{N}$.

The use and content of the Schwinger functional can be already understood within the free theory. We have already discussed, that the only non-trivial moment or correlation function of the free theory is the two-point function. Indeed we get for the free Schwinger functional,

$$W_0[J] = \ln Z_0[J] = -\frac{1}{2} \int d^d x d^d y J(x)G(x, y)J(y), \quad (10.31)$$

it is proportional to the (classical) propagator, sandwiched between two currents J . Note that strictly speaking it is only the symmetric part of the propagator, that enters (10.31) due to the symmetry of exchanging the currents $J(x)$ and $J(y)$. However, the propagator is symmetric in the first place, $G(x, y) = G(y, x)$, so no projection takes place.

The second J -derivative of the free Schwinger functional is the free propagator,

$$(-i)^2 \frac{\delta^2 W_0[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} = G(x_1, x_2), \quad (10.32)$$

and all other derivatives at vanishing current $J = 0$ vanish: we start with the one-point function,

$$\langle \varphi(x_1) \rangle = (-i) \frac{\delta W_0[J]}{\delta J(x_1)} \Big|_{J=0} = (-i) \frac{1}{Z_0[J]} \frac{\delta Z_0[J]}{\delta J(x_1)} \Big|_{J=0} = \int dx G(x_1, x) J(x) \Big|_{J=0} = 0. \quad (10.33)$$

Note that the expectation value of the field only vanishes for $J = 0$. For a non-vanishing field it simply is the current, contracted with the propagator,

$$\langle \varphi(x) \rangle_J := (-i) \frac{\delta W_0[J]}{\delta J(x_1)} = \int dx G(x_1, x) J(x). \quad (10.34)$$

This is the solution to the equation of motion of the action in the presence of a background, $S[\varphi]_0 + \int d^d x J(x) \varphi(x)$, indicated by the subscript J . This expectation value is normalised by the path integral in the presence of the source term. The respective equation of motion reads

$$\frac{\delta S_0[\varphi]}{\delta \varphi(x)} + J(x) = (-\Delta - m^2 + i\varepsilon)\varphi(x) + J(x) = 0, \quad (10.35)$$

and the solution of the EoM (10.35) is given by (10.33) for $J \neq 0$.

We proceed with the two-point function derived from the Schwinger functional. We already know from (10.32) that this is simply the propagator. In terms of Z_0 the second J -derivative of $W[J]$ reads

$$\begin{aligned} (-i)^2 \frac{\delta^2 W_0[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} &= \left[(-i)^2 \frac{1}{Z_0[J]} \frac{\delta^2 Z_0}{\delta J(x_1) \delta J(x_2)} - (-i) \frac{1}{Z_0} \frac{\delta Z_0}{\delta J(x_1)} (-i) \frac{1}{Z_0} \frac{\delta Z_0}{\delta J(x_2)} \right] \Big|_{J=0} \\ &= \langle T \varphi(x_1) \varphi(x_2) \rangle - \langle \varphi(x_1) \rangle \langle \varphi(x_2) \rangle. \end{aligned} \quad (10.36)$$

Equation (10.36) is the connected part of the two-point function, already discussed in QFT I. We emphasise that this property holds true in the interacting theory as well as in the presence of a source term. This concludes our discussion of the structure of correlation functions at the example of the free theory.

We proceed with the interacting scalar theory with the action $S[\varphi] = S_0[J] - \int d^d x V_{\text{int}}(\varphi)$, see (10.11). This is split into the part $S_0[\varphi]$, that includes the full quadratic part in the field, and the higher order interaction term is designed for perturbation theory, the expansion of the full theory about the free theory. While the following arguments are not restricted to a specific interaction term, our working example will be the φ^4 -theory with

$$V_{\text{int}}(\varphi) = \frac{\lambda}{4!} \int d^d x \varphi^4(x). \quad (10.37)$$

In analogy to the derivation in quantum mechanics we get for a general interaction potential,

$$\begin{aligned} Z[J] &\simeq \int \mathcal{D}\varphi e^{i\{S_0[\varphi] - \int d^d x V_{\text{int}}(\varphi) + \int d^d x J(x) \varphi(x)\}} = e^{-i \int d^d x V_{\text{int}}(-i \frac{\delta}{\delta J(x)})} \int \mathcal{D}\varphi e^{i\{S_0[\varphi] + \int d^d x J(x) \varphi(x)\}} \\ &\simeq e^{-i \int d^d x V_{\text{int}}(-i \frac{\delta}{\delta J(x)})} Z_0[J]. \end{aligned} \quad (10.38)$$

With the analytic Gaussian expression (10.25) for the generating functional $Z_0[J]$ of the free theory we arrive at our final parametrisation for the generating functional of the interacting theory, set-up for perturbation theory,

$$Z[J] \simeq e^{-i \int d^d x V_{\text{int}}(-i \frac{\delta}{\delta J(x)})} e^{-\frac{1}{2} \int_{x,y} J(x) G(x,y) J(y)}, \quad \text{with} \quad \int_x = \int d^d x, \quad \int_{x,y} = \int d^d x d^d y. \quad (10.39)$$

For the derivation of the Feynman rules in perturbation theory we expand (10.39) in powers of the interaction potential. For the explicit example of the φ^4 -theory this is an expansion in powers of the quartic coupling λ in (10.37). We iterate the comment made below (9.83) in the context of perturbation theory in quantum mechanics, that the expansion in powers of the coupling is not an expansion in loops. For example, the order λ^l hosts all l -loop diagrams for the two-point function, but $l - (n - 1)$ -loop diagrams for $2n$ -point functions. An explicit example is given by the two- and four-point functions at the order λ^2 . This is the $[l - (n - 1) = 2]$ -loop approximation for the $2n = 2$ -point function, but the $[l - (n - 1) = 1]$ -loop approximation for the $2n = 4$ point function.

The structure is already seen clearly in terms of the lowest orders of the expansion of $Z[J]$ in powers of V_{int} and the source term. As we are only interested in the structure, we use the notational abbreviation

$$S_{\text{int}}[\varphi] = - \int_x V_{\text{int}}[\varphi], \quad J \cdot G \cdot J := \int_{x,y} J(x) G(x,y) J(y), \quad (10.40)$$

With (10.40) we get

Generating functional in the perturbative expansion

$$Z[J] \simeq \left[1 + i S_{\text{int}} \left[\frac{\delta}{\delta J} \right] - \frac{1}{2} \left(S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \right)^2 + \dots \right] \left[1 - \frac{1}{2} J \cdot G \cdot J + \frac{1}{8} (J \cdot G \cdot J)^2 + \dots \right]. \quad (10.41)$$

For $J = 0$, the double expansion in (10.41) terminates for each power of S_{int} : the latter contains powers of the field, φ^n with $n > 2$. In most cases the interaction terms is even a monomial such as in the φ^4 -theory. In (10.41) these powers of the fields are powers in the J -derivatives. Evaluated at $J = 0$ the number of derivatives have to match precisely that of the powers of J .

We elucidate this fact within the φ^4 -theory with $V_{\text{int}}(\varphi) = 1/4! \varphi(x)^4$, see (10.37). Then the interaction part of the action reads

$$S_{\text{int}} \left[\frac{\delta}{\delta J} \right] = -\frac{1}{4!} \int_x \left(\frac{\delta}{\delta J(x)} \right)^4. \quad (10.42)$$

We start the discussion of the diagrammatic expansion with the vacuum diagrams that make up the normalisation $\mathcal{N} = Z[0]$. It follows from (10.39), that the vacuum functional reads

$$Z[0] \simeq Z_0[0] e^{i S_{\text{int}} \left[\frac{\delta}{\delta J} \right]} e^{-\frac{1}{2} J \cdot G \cdot J} \Big|_{J=0} = \left[\underbrace{1 + i S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \frac{1}{2} \left(-\frac{1}{2} J \cdot G \cdot J \right)^2}_{(\alpha)} - \frac{1}{2} \left(S_{\text{int}} \left[\frac{\delta}{\delta J} \right] \right)^2 \frac{1}{4!} \left(-\frac{1}{2} J \cdot G \cdot J \right)^4 + \dots \right]_{J=0}, \quad (10.43)$$

where we have used condensed notation introduced in (10.40). We start with the vacuum generating functional of the free theory, $Z_0[0]$, which produces a λ^0 -term. With (10.19) the free vacuum functional reads

$$Z_0[0] \simeq \det^{-\frac{1}{2}} (-\Delta - m^2) = \det^{\frac{1}{2}} G, \quad (10.44)$$

with the propagator G defined in (10.25). We now rewrite this in a form that is closer to a loop diagram. For that purpose we use that the determinant of an operator or matrix \mathcal{M} with (positive) eigenvalues λ_n can be rewritten in terms of a trace,

$$\det \mathcal{M} = \prod_n \lambda_n = \prod_n e^{\ln \lambda_n} = e^{\sum_n \ln \lambda_n} = e^{\text{Tr} \ln \mathcal{M}}. \quad (10.45)$$

Using (10.45) in (10.44), we are led to

$$Z_0[0] \simeq e^{\frac{1}{2} \text{Tr} \ln G}, \quad \text{with} \quad \text{Tr} \ln G = \int_x \ln G(x, x), \quad (10.46)$$

where we have used that $\ln G^{1/2} = 1/2 \ln G$ (for positive G). For elucidating the relation to vacuum diagrams, we take a mass derivative of the exponent in (10.46),

$$\frac{1}{Z_0[0]} \partial_{m^2} Z_0[0] = \frac{i}{2} \int_x G(x, x). \quad (10.47)$$

As the source is finally set to zero, the power of field derivatives in the expansion of S_{int} in (10.42) has to match the number of currents from the expansion of the free generating functional $Z_0[J]$. Each of the field derivatives can hit all the currents, which accounts for the different diagrammatic topologies. Note also that the symmetry of $G(x, y)$ under commuting the arguments x, y leads to

$$\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \frac{1}{2} \int_{x,y} J(x) G(x, y) J(y) = G(x_1, x_2). \quad (10.48)$$

This allows us to generate all diagrams including the combinatorial factors by simply performing the functional derivatives in (10.43). This can be easily put on a computer. The Feynman rules are derived from the two building blocks, the vertex

$$-i \frac{\lambda}{4!} \int_x \left(\frac{\delta}{\delta J(x)} \right)^4 \sim \text{Diagram: a central vertex with four external lines meeting at a point.}$$

and the propagator

$$-\frac{1}{2} \int_{x,y} J(x) G(x, y) J(y) \sim \text{Diagram: a horizontal line connecting two points labeled } x \text{ and } y. \quad (10.49)$$

We now proceed with the computation of the first term (α) in the expansion of $Z[0]$ in (10.43), the order λ^1 diagram. This leads us to

$$(\alpha) : \quad \frac{1}{8} \frac{1}{4!} 4! \text{ Diagram: a figure-eight loop with four external lines labeled 1, 2, 3, 4.} \quad (10.50)$$

where the factor $4!$ comes from the different permutations for generating the vacuum eight. Multiplying out the prefactors leads us to

$$\frac{1}{8} \text{Diagram: a figure-eight loop with four external lines labeled 1, 2, 3, 4.} = -\frac{i\lambda}{8} \int d^d x G(x, x) G(x, x). \quad (10.51)$$

Equation (10.51) is divergent and has to be renormalised as has been already discussed in QFTI. Here we are only interested in the emergence of perturbation theory, and proceed with the second class of diagrams, (β) , the order λ^2 diagrams.

$$(\beta) : \frac{1}{4!} \frac{1}{2^4} \frac{1}{2} \left(\frac{1}{4!}\right)^2 \quad (10.52)$$

The analysis above for the normalisation factor $\mathcal{N} = (1 + \text{vac diags})$ readily extend to correlation functions. We get

$$\begin{aligned} \langle T\varphi(x_1) \dots \varphi(x_n) \rangle &= \frac{(-i)^n}{Z[0]} \frac{\partial^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \Big|_{J=0} = \frac{(-i)^n}{1 + \text{vac diags}} \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} e^{iS_{\text{int}}[\frac{\delta}{\delta J}]} e^{-\frac{1}{2}J \cdot G \cdot J} \\ &= \left[\frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} e^{iS_{\text{int}}[\frac{\delta}{\delta J}]} e^{-\frac{1}{2}J \cdot G \cdot J} \right]_{\text{no vacuum diagrams}} \end{aligned} \quad (10.53)$$

It follows directly from (10.53), that all diagrams with external lines are multiplied by $(1 + \text{vac diags})$, which is cancelled by the denominator. In summary this leads us to the Feynman rules

Feynman rules

- Propagator:

$$x \text{ --- } y = G(x, y) \quad (10.54)$$

- Vertex:

$$\begin{array}{c} \diagup \\ \times \\ \diagdown \end{array} x = -i\lambda \int d^4x \quad (10.55)$$

a) Write down all diagrams in a given order N of λ of $2n$ -point correlation function.

b) Combinatorial factor of diagrams:

$$\left(\frac{1}{4!}\right)^N \frac{1}{N!} \frac{1}{(2N+n)!} \frac{1}{2^{2N+n}} (\text{permutations}) \quad (10.56)$$

derived from

$$\Rightarrow \left(\frac{1}{4!}\right)^N \frac{1}{N!} \left[\lambda \int_x \left(\frac{\delta}{\delta J}\right)^4 \right]^N \cdot \frac{1}{(2N+n)!} \left[\frac{1}{2} \int_{x,y} J \cdot G \cdot J \right]^{2N+n} \quad (10.57)$$

We exemplify this analysis with a brief re-evaluation of the perturbative expansion of the two-point function. To that end

$$\langle T\varphi(x) \varphi(y) \rangle = x \text{ --- } y + \frac{1}{2} \text{ tadpole} + \dots \quad (10.58)$$

For the tadpole term we get schematically for the derivatives,

$$\begin{aligned} \int \left(\frac{\delta}{\delta J}\right)^4 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} (J \cdot G \cdot J)^3 &= \int \left(\frac{\delta}{\delta J}\right)^4 3 \cdot 2 \cdot (2J \cdot G)(x_1)(2J \cdot G)(x_2) (J \cdot G \cdot J) \\ &\rightarrow 3!4!2^2 G^3 \end{aligned} \quad (10.59)$$

This has to be multiplied by the combinatorial factors of the double expansion and the factor $1/2$ in $1/2(J \cdot G \cdot J)$,

$$\frac{1}{4!} \frac{1}{3!} \left(\frac{1}{2}\right)^3. \quad (10.60)$$

In combination this yields the well-known symmetry factor $1/2$.

We close this Section on the Feynman rules from the functional integral with the observation that the above results extend straightforwardly to general scalar theories. A simple example is provided by a theory of two scalar fields, coupled with a Yukawa coupling,

$$S[\varphi, \phi] = \int_x \left\{ \frac{1}{2} \varphi (-\Delta - m_\varphi^2) \varphi + \frac{1}{2} \phi (-\Delta - \mu_\phi^2) \phi - h \int_x \phi \varphi^2 \right\}. \quad (10.61)$$

The respective generating functional has the following form,

$$Z[J_\varphi, J_\phi] = \int \mathcal{D}\varphi \mathcal{D}\phi e^{\{S[\varphi, \phi] + \int_x (J_\varphi \varphi + J_\phi \phi)\}} \simeq e^{-ih \int_x \frac{\delta}{\delta J_x} \frac{\delta^2}{\delta J_x^2}} Z_0[J_\varphi, J_\phi], \quad (10.62)$$

with

$$Z_0[J_\varphi, J_\phi] \simeq e^{-\frac{1}{2}(J_\varphi \cdot G_\varphi \cdot J_\varphi + J_\phi \cdot G_\phi \cdot J_\phi)}, \quad \text{and} \quad (-\Delta_x - m_{\varphi/\phi}^2) G_{\varphi/\phi}(x, y) = i\delta(x - y). \quad (10.63)$$

The Feynman rules for the theory with (10.61) to (10.63) follow from the combinatorial factors of the expansion and the application of the functional derivatives.

10.3. Wick rotation & statistical interpretation

The functional integrals and the generating functionals were derived from transition amplitudes, that are matrix elements of the time evolution operator $U(\infty, -\infty)$ with the infinitesimal element $U(t + \Delta t, t) = \exp\{-i\Delta t \hat{H}\}$. The latter unitary operator leads to the phase $\mathcal{D}\varphi e^{iS[\varphi]}$ in the measure of the functional integral. For practical purposes, both perturbative as well as analytic computations, it is in most cases convenient or even necessary, to perform a Wick rotation. This is done with

$$x_M^0 \rightarrow -ix_E^0, \quad \text{with} \quad e^{-i\Delta t \hat{H}} \rightarrow e^{-\Delta t \hat{H}}, \quad (10.64)$$

which converts the unitary phase into a damping factor. Hence, we also expect $\exp\{iS_M[\varphi]\} \rightarrow \exp\{-S_E[\varphi]\}$, where we have introduced the notation S_M and S_E for the Minkowski and Euclidean space-time actions respectively. For the derivation of this rotation we use

$$i \int d^d x_M \rightarrow \int d^d x_E, \quad -\partial_\mu \partial^\mu \rightarrow +\partial_\mu \partial_\mu, \quad \varphi(x_M) \rightarrow \varphi(x_E), \quad (10.65)$$

the scalar field is invariant under this rotation, only its argument changes. In combination this leads us to

$$\underbrace{i \int d^d x \left\{ \frac{1}{2} \varphi (\partial_\mu \partial^\mu) \varphi - V(\varphi) \right\}}_{S_M[\varphi]} \rightarrow - \underbrace{\int d^d x \left\{ \frac{1}{2} \varphi (-\partial_\mu \partial_\mu) \varphi + V(\varphi) \right\}}_{S_E[\varphi] \geq 0}. \quad (10.66)$$

The Euclidean generating functional takes the form

Generating Functional

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi e^{-S[\varphi] + \int d^d x J(x)\varphi(x)}, \quad (10.67)$$

with the Euclidean action

$$S[\varphi] = \int_x \frac{1}{2} (\partial_\mu \varphi)^2 + V(\varphi), \quad (10.68)$$

where we have dropped the subscript E . Euclidean correlation functions are derived from (10.67) with

$$\langle T\varphi(x_1) \cdots \varphi(x_n) \rangle = \left. \frac{\partial^n Z}{\partial \varphi(x_1) \cdots \partial \varphi(x_n)} \right|_{J=0}. \quad (10.69)$$

For the perturbative expansion we first discuss the free theory with the free action

$$S_0[\varphi] = \frac{1}{2} \int d^d x \varphi (-\Delta_E + m^2) \varphi, \quad \Delta_E = \partial_\mu^2, \quad (10.70)$$

with the positive kinetic operator $-\Delta_E$. The free generating functional follows as

$$Z_0[J] = e^{\frac{1}{2} \int d^d x J(x) G(x,y) J(y)}, \quad \text{with} \quad (-\Delta_E + m^2) G(x,y) = \delta^d(x-y), \quad (10.71)$$

with the normalisation $\mathcal{N}_0 = \det^{-1/2}(-\Delta_E + m^2)$. The free dispersion $(-\Delta_E + m^2)$ is a positive definite operator and so is the propagator G . Equation (10.71) has to be compared with the free generating functional in (10.25). The Feynman rules follow similarly to that in Minkowski space-time summarised in (10.54) to (10.57): the combinatorial factors (10.56) and (10.57) do not change and the Minkowski propagator is substituted by the Euclidean propagator,

$$G_M(p) = \frac{i}{p_M^2 - m^2 + i\varepsilon} \longrightarrow G_E(p) = \frac{1}{p^2 + m^2}, \quad (10.72)$$

and the Minkowski vertex s substituted by the Euclidean vertex,

$$-i\lambda \rightarrow -\lambda \quad (10.73)$$

In conclusion, the Euclidean functional integral is a statistical integral with statistical measure $\mathcal{D}\varphi e^{-S[\varphi]}$ (not normalised). The Schwinger functional is nothing but the free energy F of the Euclidean quantum field theory. Importantly, Euclidean correlation functions can be Wick-rotated back to Minkowski space (on real time), when they provide the real-time correlation functions,

$$\langle T\varphi(x_{1E}) \cdots \varphi(x_{nE}) \rangle \xrightarrow{t_E \rightarrow it_M} \langle T\varphi(x_{1M}) \cdots \varphi(x_{nM}) \rangle, \quad (10.74)$$

where $t_E \rightarrow it_M$ stands for the Wick rotation of all Euclidean times in (10.74) back to Minkowski times. If the Euclidean correlation function on the right hand side of (10.74) is known analytically, this transformation is readily done. However, in most cases the Euclidean correlation function is only known numerically on some data points with a given numerical precision. Then, the Wick rotation has to be done numerically, which in general is an ill-conditioned inverse problem.

11. Functional Methods

The discussion of the generating functional $Z[J]$ and perturbation theory has already impressively demonstrated, that the functional integral approach, or more generally, the functional formulation of quantum field theory, facilitates the discussion of many conceptual but also practical aspects. Indeed, functional approaches offer a non-perturbative diagrammatic approach to quantum field theory amiable to numerical computations, which complements the standard numerical non-perturbative approach, lattice field theory discussed later in the lecture course, [Chapter 15](#). In the present Chapter we will discuss different generating functionals and their use for perturbative and non-perturbative applications. A very interesting and important example is the computation of the one-loop effective potential of a scalar field theory and its application. It allows us to discuss fluctuation-induced spontaneous symmetry breaking (Coleman-Weinberg). This mechanism has to be contrasted with the Higgs mechanism in the Standard Model. There, the classical potential already carries the seed of spontaneous symmetry breaking. In the present scalar case with a Z_2 -symmetry the latter case is given by a classical potential $V(\varphi)$ with non-trivial minima $\pm\varphi_0$, that survive in the quantum effective potential V_{eff} , see the left hand side of [Figure 11.1](#).

While this discussion and the respective computations can also be performed with $Z[J]$ or the Schwinger functional $W[J]$, the most convenient access is via the effective action $\Gamma[\phi]$, the Legendre transform of $W[J]$. This is the generating functional of one-particle-irreducible correlation functions and has already defined in [Section 7.1.2](#) in the Chapter about renormalisation, see [\(7.20\)](#). There it has been used for facilitating the renormalisation programme, as it allows to set up the renormalisation conditions in terms of 1PI correlation functions.

11.1. Effective Action

The effective action Γ is the generating function of the one-particle-irreducible (1PI) Green/Correlation function. This property can be proven within the functional integral by its definition as a Legendre transformation of the Schwinger functional $W[J] \simeq \ln Z[J]$. We refrain from discussing this proof here, and defer it to the discussion of the Dyson-Schwinger equation (or quantum equation of motion) in [Section 13.3.1](#).

The effective action $\Gamma[\phi]$ follows from W via a Legendre transformation, see [\(7.20\)](#) in [Section 7.1.2](#). We recall the definition there,

Effective action

$$\Gamma[\phi] = \sup_J \left(\int d^d x J(x) \phi(x) - W[J] \right). \quad (11.1)$$

Commonly, one simply writes $\Gamma[\phi] = \int d^d x J(x) \phi(x) - W[J]$, and the supremum is understood. Moreover, in general quantum field theories the supremum in [\(11.1\)](#) is needed, as the right hand side not necessarily admits a global maximum. Moreover, differentiability with respect to J cannot be assumed in general, as the Schwinger functional may have cusps or even jumps in J . While these general cases include some relevant and interesting cases, we shall assume in the following that [\(11.1\)](#) has a unique global maximum at J_{max} and is differentiable with respect to J and ϕ . If these properties are present, we conclude that the J -derivative of the right hand side

of (11.1) vanishes at J_{\max} , to wit,

$$\frac{\delta}{\delta J(y)} \left[\int_x J \cdot \phi - W[J] \right]_{J=J_{\max}} = 0, \quad \rightarrow \quad \phi(x) = \left. \frac{\delta W[J]}{\delta J(x)} \right|_{J=J_{\max}} = \frac{1}{Z[J_{\max}]} \left. \frac{\delta Z[J]}{\delta J(x)} \right|_{J=J_{\max}}. \quad (11.2)$$

This implies, that the J -derivative of the Schwinger functional for $J = J_{\max}$ is the argument ϕ of the effective action. Hence the latter is the mean field in the background J_{\max} ,

$$\phi(y) = \langle \varphi(x) \rangle_{J_{\max}} \quad (11.3)$$

As the double Legendre transformation is the identity on convex functionals, the Legendre transform of the effective $\Gamma[\phi]$ is the Schwinger functional $W[J]$, or strictly speaking the convex hull of $W[J]$. This is also apparent from the short hand notation for (11.1) given by $\Gamma[\phi] = \int d^d x J(x) \phi(x) - W[J]$. Resolving the latter equation for $W[J]$ gives the desired relation. This entails that both sides can be either read as functionals of J and $\phi[J]$ or of ϕ and $J[\phi]$. The fact, that the Schwinger functional is the Legendre transformation of the effective action, implies readily the 'dual' relations

$$J(x) = \frac{\delta \Gamma}{\delta \phi(x)}, \quad \phi(x) = \frac{\delta W[J]}{\delta J(x)}. \quad (11.4)$$

We close this brief introduction to the effective action with a discussion of the respective two-point functions of $Z[J]$, $W[J]$ and $\Gamma[\phi]$. This also elucidates the nature of the generating functionals: Generating of all parts (disconnected and connected) of a given correlation function ($Z[J]$), generating the connected parts ($W[J]$) and only the 1PI parts ($\Gamma[\phi]$). Let us start with the two-point function of $Z[J]$,

$$Z^{(2)}[J](x_1, x_2) = \langle \varphi(x_1) \varphi(x_2) \rangle_J, \quad (11.5)$$

where the right hand side is taken in the presence of the current J , indicated by the subscript J . In (11.5) we have introduced the short hand notation

$$F^{(n)}[f](x_1, \dots, x_n) = \frac{\delta^n F[f]}{\delta f(x_1) \cdots \delta f(x_n)}, \quad (11.6)$$

for n th derivatives of a general functional $F[f]$ with respect to the argument $f(x)$. The two-point function $Z^{(2)}[J]$ contains connected and disconnected parts, and the latter is simply

$$\langle \varphi(x_1) \rangle_J \langle \varphi(x_2) \rangle_J = \phi(x_1) \phi(x_2). \quad (11.7)$$

Now we proceed to the two-point function of the Schwinger functional, which is given by

$$\begin{aligned} W^{(2)}[J](x_1, x_2) &= \frac{1}{Z[J]} \frac{\delta^2 Z}{\delta J(x_1) \delta J(x_2)} - \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x_1)} \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x_2)} \\ &= \langle \varphi(x_1) \varphi(x_2) \rangle_J - \phi(x_1) \phi(x_2) = \langle \varphi(x_1) \varphi(x_2) \rangle_c. \end{aligned} \quad (11.8)$$

We conclude that $W^{(2)}[J]$ is the connected two-point function in a general background current J . The relation of $W^{(2)}[J]$ to the 1PI two-point function $\Gamma^{(2)}[\phi]$ is derived as follows. We consider a δ -function that can be obtained from the derivative of a current J with respect to the current itself,

$$\delta(x_1 - x_2) = \frac{\delta J(x_1)}{\delta J(x_2)} = \frac{\delta}{\delta J(x_1)} \frac{\delta \Gamma}{\delta \phi(x_2)} = \int_z \frac{\delta \phi(z)}{\delta J(x_1)} \frac{\delta^2 \Gamma[\phi]}{\delta \phi(z) \delta \phi(x_2)} = \int_z \frac{\delta^2 W[J]}{\delta J(x_1) \delta J(z)} \frac{\delta^2 \Gamma[\phi]}{\delta \phi(z) \delta \phi(x_2)} \quad (11.9)$$

where we have used both relations in (11.4). In short this reads

$$\int_z W^{(2)}[J](x_1, z) \Gamma^{(2)}[\phi](z, x_2) = \delta(x_1 - x_2), \quad (11.10)$$

the 1PI two-point function is the inverse of the propagator or connected two-point function,

$$\langle \varphi(x_1) \varphi(x_2) \rangle_c = \frac{1}{\Gamma^{(2)}[\phi]}(x_1, x_2), \quad (11.11)$$

where the expectation value on the left hand side of (11.11) is evaluated in the background $J[\phi]$.

11.2. Effective action & spontaneous symmetry breaking

With the preparations in the last Section, Section 11.1, we compute the one-loop effective potential $V_{\text{eff}}^{(1\text{loop})}$ for a real scalar theory. This computation is done in Euclidean space, and we first consider the generating functional $Z[J]$ defined in (10.67). For the evaluation of the functional integral in (10.67) we use a saddle-point expansion: we assume that the neighbourhood of the (assumedly) unique maximum of $\exp\{-S[\varphi] + \int_x J\varphi\}$ dominates the integral and expand the exponent about the solution ϕ of the equation of motion in the presence of a background current J ,

$$\left. \frac{\delta S[\varphi]}{\delta \varphi(x)} \right|_{\varphi=\phi} = J(x). \quad (11.12)$$

For any given background ϕ we can use the shift invariance of the path integral measure, $\mathcal{D}\varphi = \mathcal{D}(\varphi + \phi)$ and hence

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi e^{-S[\phi+\varphi] + \int d^d x J(x)[\phi(x)+\varphi(x)]}, \quad (11.13)$$

If ϕ satisfies (11.12), the first derivative of the exponent in (11.13) vanishes in the Taylor expansion about ϕ and we arrive at

$$\begin{aligned} S[\phi + \varphi] - \int_x J(\phi + \varphi) &= S[\phi] + \int_x \frac{\delta S[\phi]}{\delta \phi(x)} \varphi(x) + \frac{1}{2} \int_{x,y} \varphi(x) S^{(2)}[\phi](x,y) \varphi(y) + O(\varphi^3) - \int_x J(\varphi + \phi) \\ &= S[\phi] + \frac{1}{2} \int_{x,y} \varphi(x) S^{(2)}[\phi](x,y) \varphi(y) - \int_x J\varphi + O(\varphi^3), \end{aligned} \quad (11.14)$$

where we have used the short hand notation (11.6) for the classical action,

$$S^{(n)}[\phi](x_1, \dots, x_n) = \frac{\delta^n S[\phi]}{\delta \phi(x_1) \cdots \delta \phi(x_n)}. \quad (11.15)$$

Now we exploit the key assumption of the saddle-point expansion, the subleading nature of the higher order terms $\propto \varphi^3, \varphi^4$ in (11.14). Dropping these higher order terms in the functional integral (11.13), we are left with a Gaussian integral of a free theory with the kinetic operator $S^{(2)}[\phi]$. This integral is readily performed and leaves us with one over the square root of the determinant of $S^{(2)}[\phi]$. Taking the logarithm of this expression leads us to the Schwinger functional at one-loop,

$$W[J] = \int_x J\phi - S[\phi] + \ln \int \mathcal{D}\varphi e^{-\frac{1}{2} \int_{x,y} \varphi(x) S^{(2)}[\phi](x,y) \varphi} = \int_x J\phi - S[\phi] - \frac{1}{2} \text{Tr} \ln S^{(2)}[\phi]. \quad (11.16)$$

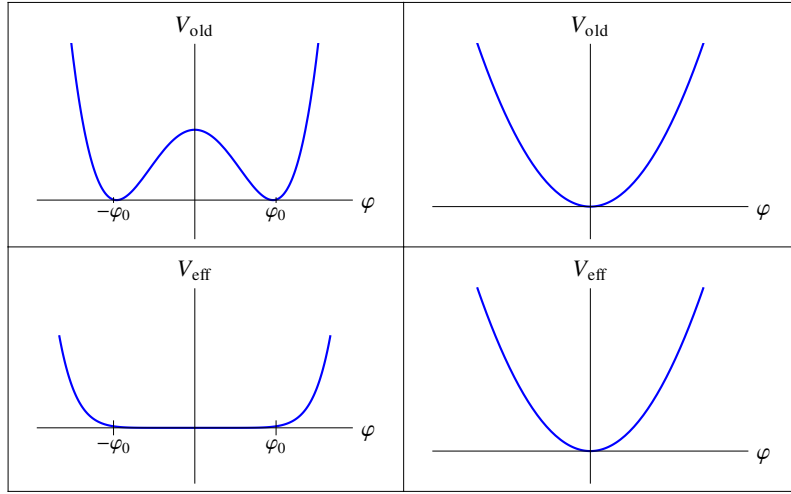


Figure 11.1.: Upper two figures: classical potentials with non-trivial minima at $\varphi = \pm\varphi_0$ (left), and minimum at vanishing field $\varphi = 0$. Lower two figures: corresponding effective potentials (convex hull)

Now we use the definition of the effective action $\Gamma[\phi]$ in (11.1) as the Legendre transform of the Schwinger functional, see (7.20). In the condensed notation this reads $\Gamma[\phi] = J \cdot \phi - W[J]$ where we have to take the maximum (strictly speaking the supremum) of all background currents J . With (11.1) we arrive at the concise expression

$$\Gamma_{1\text{-loop}}[\phi] = S[\phi] + \frac{1}{2} \text{Tr} \ln S^{(2)}[\phi], \quad (11.17)$$

where the second trace log term includes all one loop quantum corrections: taking n derivatives with respect to the scalar field ϕ leads to the one-loop diagrams contributing to the 1PI part of the n -point correlation function. However, the use of (11.17) or (11.16) goes beyond generating the perturbative loop expansion. The effective action is the quantum analogue of the classical action and reduces to the latter in the classical approximation. Consequently, the loop term in (11.17) comprises the one-loop quantum correction to the classical potential. We project on the effective potential $V_{\text{eff}}(\phi)$ by evaluating $\Gamma[\phi]$ for constant fields ϕ_c ,

$$V_{\text{eff}} = \frac{1}{\mathcal{V}_d} \Gamma[\phi_c], \quad \text{with} \quad \mathcal{V}_d = \int_{\mathcal{M}} d^d x, \quad \text{and} \quad \partial_\mu \phi_c = 0. \quad (11.18)$$

Here, $\mathcal{M} \rightarrow \mathbb{R}^d$ is the manifold, the scalar theory lives on. In most cases the limit from a compact manifold \mathcal{M} to the space-time manifold \mathbb{R}^d is not referred to and one simply writes $\mathcal{V}_d = \mathbb{R}^d$. A trivial example for (11.18) is its application to the classical action, where (11.18) leads to the classical action.

We will apply this set-up to the scalar ϕ^4 -theory in $d = 3, 4$ and evaluate whether quantum fluctuations can lead us from a classical potential with a trivial minimum (right column in Figure 11.1) to an effective potential with a non-trivial minimum (left column in Figure 11.1). If the effective potential feature a non-trivial minimum, this can give rise to spontaneous symmetry breaking. Note also, that the converse situation is equally interesting: Can quantum fluctuations wash-out a non-trivial minimum.

Bearing in mind the potential necessity of renormalising the effective potential, we start with the bare action of the ϕ^4 -theory in $3d = 3, 4$ Euclidean space-time dimensions,

$$S[\phi_0] = -\frac{1}{2} \int d^d x \phi_0 \Delta \phi_0 + \int d^d x V_0(\phi_0), \quad \text{with} \quad V_0(\phi_0) = \frac{1}{2} m_0^2 \phi_0^2 + \frac{\lambda}{4!} \phi_0^4, \quad (11.19)$$

with the bare (mean) field ϕ_0 , the mass squared m_0^2 and the bare coupling λ_0 . The bare quantities depend on the regularisation and for the present computation we chose a momentum cutoff: all loop integrals are restricted to momenta squared smaller than this cutoff scale: $p^2 \leq \Lambda^2$. Necessarily, the bare parameters also depend on this cutoff and in combination the computations yield finite, cutoff-independent results, see [Chapter 7](#).

Now we use that the one-loop wave function renormalisation of the scalar field is trivial, see [Section 7.1.3](#). This entails $\phi_0 = \phi$ at one loop. In contrast to perturbation theory we do not use the split into the quadratic part of the action and the interaction part. It is far more convenient to split the action and the effective action into the kinetic (derivative) terms and the effective potential. For the computation of (11.18) with (11.17) we need the second derivative of the bare $S[\phi]$ with respect to the fields,

$$S^{(2)}[\phi](x, y) = \left[-\Delta + m_0^2 + \frac{\lambda_0}{2} \phi^2 \right] \delta(x - y). \quad (11.20)$$

The computation is conveniently done in momentum space. To that end we perform the Fourier transformation of (11.20) and take the trace in (11.17) in momentum space. As the fields are constant, the Fourier transform of (11.20) takes the simple form

$$S^{(2)}[\phi](p, q) = \left[p^2 + m_0^2 + \frac{\lambda_0}{2} \phi^2 \right] (2\pi)^d \delta(p + q). \quad (11.21)$$

The trace in (11.17) can be written in a general basis with orthonormal states $|\lambda\rangle$: We take the matrix element $\langle \lambda | \ln S^{(2)}[\phi] | \lambda \rangle$, fo to the diagonal $\lambda' = \lambda$ and integrate/sum over all states. This yields

$$\text{Tr} \ln S^{(2)}[\phi] = \sum_{\lambda} \left[\langle \lambda | \left[\ln S^{(2)}[\phi] \right] (\lambda, \lambda') | \lambda' \rangle \right]_{\lambda'=\lambda}. \quad (11.22)$$

In momentum space the orthogonality relation gives us $(2\pi)^d \delta(p + q)$ and

$$\sum = \int \frac{d^d p}{(2\pi)^d}. \quad (11.23)$$

We note in passing that (11.23) is finite in the presence of a momentum cutoff and the volume factors drops out in a mathematically well-controlled way.

A final intricacy is given by the fact that the diagonal matrix element $\langle p | \ln S^{(2)}[\phi](p, q) | q \rangle_{q \rightarrow -p}$ includes a δ -function in momentum space, evaluated at vanishing argument, $(2\pi)^d \delta(0)$. However, this is simply the space-time volume,

$$(2\pi)^d \delta(p + q) \Big|_{q \rightarrow -p} = \int_{\mathcal{M}} d^d x e^{ix(p+q)} \Big|_{q \rightarrow -p} = \int_{\mathcal{M}} d^d x = \mathcal{V}_d, \quad (11.24)$$

which is cancelled by the normalisation $1/\mathcal{V}_d$ in (11.18). In summary we arrive at the simple momentum integral for the one-loop effective potential,

$$V_{\text{eff}}(\phi) = V_0(\phi) + \frac{1}{2} \int_{p \leq \Lambda^2} \frac{d^d p}{(2\pi)^d} \ln \left(p^2 + m_0^2 + \frac{\lambda_0}{2} \phi^2 \right) + \ln \mathcal{N}, \quad (11.25)$$

were the momentum cutoff Λ regularises the loop integral and the terms with positive powers in Λ are cancelled by respective ones in the bare action. We write the bare mass and coupling as

$$m_0^2 = m^2 + \Delta m_0^2, \quad \lambda_0 = \lambda + \Delta \lambda_0, \quad \text{with} \quad \Delta m_0^2 = O(\lambda), \quad \Delta \lambda_0 = O(\lambda^2), \quad (11.26)$$

where m^2, λ are the renormalised mass squared and coupling respectively. $\Delta m_0^2, \Delta \lambda_0$ are the counterterm parameters and encode the subtractions of the loop diagrams and hence carry the respective powers of the coupling. Moreover, we have also resurrected (or allowed for) an overall field-independent normalisation $\ln \mathcal{N}$ which can be adjusted such that the vacuum divergence, the first divergence we encounter in QFT I, is also cancelled. In [Chapter 7](#) it has been discussed in terms of vacuum loops.

In short, the right hand side is a sum of the bare potential, the one-loop momentum integral. All terms on the right hand side are separately divergent, but the sum is finite. It yields the *finite* renormalised effective potential $V(\phi)$ on the left hand side of [\(11.25\)](#) which does not depend on Λ , but only on the renormalised mass squared and coupling m^2, λ and the renormalisation scale μ .

[Equation \(11.25\)](#) is the starting point of the standard renormalisation programme discussed in [Chapter 7](#):

- (1) Compute the momentum integral in the presence of the cutoff Λ .
- (2) Isolate the terms with a positive power in Λ within a Taylor expansion in the fields.
- (3) Adjust the counterterm parameters in [\(11.26\)](#) such that all positive powers in Λ are removed.
- (4) Take the limit $\Lambda \rightarrow \infty$.

Instead of following this standard perturbative route of renormalisation, we use a different procedure, which is already introducing the structure of the functional renormalisation group that allows for a non-perturbative renormalisation procedure as well as the successive (numerical) computation of correlation functions. In the present context we simply realise that the UV degree of divergence of V_{eff} in [\(11.25\)](#) is lowered by taking a derivative with respect to the renormalised mass parameter m^2 . At one loop this yields

$$\partial_{m^2} V_{\text{eff}}(\phi) = \partial_{m^2} V_0(\phi) + \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left[\frac{1}{p^2 + m^2 + \frac{1}{2}\phi^2} - \frac{1}{p^2 + m^2} \right], \quad (11.27)$$

where we have used the convenient choice

$$\partial_{m^2} \ln \mathcal{N} = -\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + m^2}, \quad (11.28)$$

for the overall normalisation \mathcal{N} . On the right hand side of [\(11.27\)](#) we have used that the subtraction terms $\Delta m_0^2, \Delta \lambda_0$ are of higher order in the coupling. Hence, they only contribute from the two-loop level onwards and can be dropped within the present one-loop computation.

The subtraction with [\(11.28\)](#) removes the UV divergence of degree d that originates from the vacuum terms. Then, the m^2 -derivative has lowered the UV degree of divergence further by two. Consequently, [\(11.27\)](#) has the UV degree of divergence $d - 4$. [Equation \(11.27\)](#) also reveals that a further reduction of the UV degree of divergence is readily achieved by further m^2 -derivatives. After a sufficiently high power n of m^2 -derivatives all terms on the right hand side are separately finite and we can safely remove the cutoff. This seemingly also removes the necessity of counterterms and of the renormalisation conditions. However, after applying n derivatives with respect to m^2 , the effective potential is then obtained by successive n integration of the finite result with respect to m^2 , leaving us with n integration constants, tantamount to the renormalisation conditions: we have converted the task of adjusting renormalisation conditions in a choice of initial conditions of a *finite* functional differential equation, and one of the (many) appealing features of such a procedure lies in the inherent finiteness of all terms at all steps of the computation without the necessity of cutoff terms.

We close this discussion of the differential renormalisation procedure with the remark, that such an m^2 -derivative can also be applied to the effective action. If doing so, we are led to a functional equation, which is the functional Callan-Symanzik equation, first written down by Symanzik in 1970. This is the first (of many) functional renormalisation group equations.

In both cases, the $d = 3$ dimensional and $d = 4$ dimensional ϕ^4 -theory, the integration can be performed analytically and we obtain an analytic effective potential $V_{\text{eff}}(\phi)$. Interestingly, for a classical potential $V(\phi)$ with a minimum at $\phi = 0$, the effective potential exhibits non-trivial minima ϕ_0 (within a restricted parameter range of m, λ) This fluctuation-induced minimum leads to spontaneous symmetry breaking and a first order phase transition. Below we discuss both theories and their phase structure in more detail, and we start with $d = 3$ in [Section 11.2.1](#) and discuss the Coleman-Weinberg phase transition in $d = 4$ in [Section 11.2.2](#).

11.2.1. One-loop effective potential and first order phase transition in $d = 3$ dimensions

In the three-dimensional scalar theory only the vacuum diagrams (degree of divergence three) and the mass requires renormalisation (linear divergent) and we have $\lambda_0 = \lambda$. Hence, the subtraction of the vacuum divergence with (11.28) lowers the degree of divergence to one, and the additional m^2 -derivative of the effective potential (11.27) is finite: in terms of loop contributions to the mass this is readily understood as the only one loop contribution is the tadpole diagram, which has a linear UV divergence. The m^2 -derivative of the tadpole is finite and hence all divergences are eliminated,

$$\left[\partial_{m^2} \quad \text{---} \bigcirc \text{---} \right] \text{ finite in } d = 3. \quad (11.29)$$

After performing the remaining momentum integral we are led to

$$\partial_{m^2} V_{\text{eff}}(\phi) = \partial_{m^2} V_0(\phi) - \frac{1}{8\pi} \left[\sqrt{m^2 + \frac{\lambda}{2}\phi^2} - \sqrt{m^2} \right], \quad (11.30)$$

where the m^2 -derivative of the bare potential only hits the mass term,

$$\partial_{m^2} V_0(\phi) = \frac{1}{2}\phi^2 + \frac{1}{2} \frac{\partial \Delta m_0^2}{\partial m^2} \phi^2, \quad (11.31)$$

where the *finite* coefficient $\partial \Delta m_0^2 / \partial m^2$ of the second term is at our disposal and encodes the renormalisation condition: We use this flexibility and choose it such that it cancels the entire ϕ^2 -contribution in the second term in (11.30). This leads us to

$$\partial_{m^2} V_{\text{eff}}(\phi) = \frac{1}{2}\phi^2 - \frac{1}{8\pi} \sqrt{m^2 + \frac{\lambda}{2}\phi^2} + \frac{1}{32\pi} \frac{\lambda}{m} \phi^2 + \frac{1}{8\pi} m, \quad \text{with} \quad \frac{\partial \Delta m_0^2}{\partial m^2} = \frac{1}{16\pi} \frac{\lambda}{m}, \quad (11.32)$$

where now we have restricted ourselves to $m^2 > 0$. We note that the extension to $m^2 < 0$ can readily be done with $m \rightarrow \sqrt{m^2}$ in (11.32), if restricting ourselves to $\phi^2 > -2m^2/\lambda$ (and choosing another normalisation $\ln \mathcal{N}$). This restriction is necessary as the one-loop effective potential is non-convex for $\phi^2 < -2m^2/\lambda$. This already entails that a perturbative expansion does not maintain the convexity of the effective potential. We also remark that the renormalisation condition is mass-dependent due to the $1/m$ factor in the m^2 -derivative of Δm_0^2 .

The last two terms on the right hand side of (11.32) encode the renormalisation of the vacuum contribution (last term) and the mass renormalisation (penultimate term). The renormalisation condition for the mass and the renormalisation scale μ are given as follows: as we only discuss the effective potential or rather constant fields with $\partial_\mu \phi = 0$, the respective momentum is $p^2 = 0$. This relates to a renormalisation condition with $\mu = 0$. If we remove all quantum corrections to the mass at $p = 0$, we identify the mass parameter in the effective potential with the physical mass at $p = 0$, if we evaluate the correlation functions at $\phi = 0$. Note that for vanishing mass such a choice may introduce further (infrared) divergences. Moreover, for a non-vanishing solution ϕ_0 to the equation of motion we may use ϕ_0 as the expansion point, and this relates to choosing $\mu = \phi_0$.

For now we assume $m > 0$ and $\mu = 0$ and $\phi = 0$. Then, in terms of correlation functions the renormalisation condition for the mass reads

$$\Gamma^{(2)}(p^2 = 0) = m^2, \quad (11.33)$$

to be compared with the on-shell renormalisation condition (7.10a). In terms of the 1PI effective action it reads $\Gamma^{(2)}(p^2 = -m^2) = 0$. This concludes the discussion of the renormalisation conditions and the related fixing of coupling parameters.

It is left to perform the mass squared integration of (11.30), that leads to the full effective potential for a given mass m ,

$$V_{\text{eff}}(\phi; m^2) = V_{\text{eff}}(\phi; m_{\text{in}}^2) + \int_{m_{\text{in}}^2}^{m^2} dm'^2 \partial_{m'^2} V_{\text{eff}}(\phi; m'^2). \quad (11.34)$$

The simple form (11.32) of the m^2 -derivative of the effective potential allows for an analytic integration. We also choose a simple initial condition for the effective potential or mass m_{in}^2 with $m_{\text{in}}^2 \rightarrow \infty$. For $m^2 \rightarrow \infty$ the theory reduces to the classical theory with the renormalised classical potential $V(\phi; m^2)$ with

$$V(\phi, m^2) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (11.35)$$

The choice of $\ln \mathcal{N}$ or rather that of its m^2 -derivative in (11.28) eliminates the constant term at vanishing field, $V(0) = 0$. The renormalisation condition (11.33) adjusts the coefficient of the full ϕ^2 term to be m^2 . In summary this leads us to

One-loop Effective Potential in $d = 3$

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{1}{12\pi} \left(m^2 + \frac{\lambda}{2} \phi^2 \right)^{3/2} + \frac{1}{16\pi} m \lambda \phi^2 + \frac{1}{12\pi} m^3. \quad (11.36)$$

The effective potential (11.36) shows non-trivial minima and hence spontaneous symmetry breaking for small m/λ . This minimum is located at ϕ_0 with

$$\bar{\phi}_0^2 = \frac{9}{64\pi^2} \left[\bar{\lambda} + \left(\bar{\lambda}^2 - 32\pi\bar{\lambda} - \frac{512\pi^2}{3} \right)^{1/2} \right] - \frac{6}{\bar{\lambda}} - \frac{3}{4\pi}, \quad \text{with} \quad \bar{\phi}_0^2 = \frac{\phi_0^2}{m}, \quad \bar{\lambda} = \frac{\lambda}{m}, \quad (11.37)$$

with the dimensionless $\bar{\phi}_0$ and $\bar{\lambda}$, measured in the mass m . The rewriting in terms of the dimensionless field and coupling emphasises the fact, that the theory has one coupling parameter $\bar{\lambda}$. This entails that the large coupling limit is either achieved by $\lambda \rightarrow \infty$ or $m \rightarrow 0$. Note that the minimum $\bar{\phi}_0$ has to be real (or $\bar{\phi}_0^2 > 0$), which is achieved for

$$\bar{\lambda} > \bar{\lambda}_{\text{min}}, \quad \text{with} \quad \bar{\lambda}_{\text{min}} = 16\pi \left(1 + \sqrt{\frac{5}{3}} \right). \quad (11.38)$$

For $\bar{\lambda} < \bar{\lambda}_{\text{min}}$, (11.37) is complex and the non-trivial minimum is absent. As expected, this includes the limit $\bar{\lambda} \rightarrow 0$, in which perturbation theory is valid. In turn, for $\bar{\lambda} \rightarrow \infty$, the theory is strongly coupled. Importantly, for $\bar{\lambda} > \bar{\lambda}_*$, the minimum at ϕ_0 is the global minimum. In turn, for $\bar{\lambda} < \bar{\lambda}_*$, the global minimum is located at $\phi = 0$. Here, $\bar{\lambda}_*$ is given by

$$\bar{\lambda}_* \approx 0.0205. \quad (11.39)$$

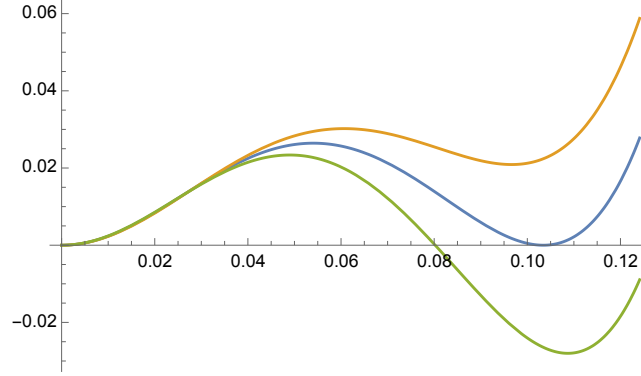


Figure 11.2.: Dimensionless effective potential $V_{\text{eff}}(\bar{\phi})/m^3$ as a function of $\bar{\phi}$ for $\bar{\lambda} = (0.97\bar{\lambda}_*, \bar{\lambda}_*, 1.03\bar{\lambda}_*)$.

For $\bar{\lambda} = \bar{\lambda}_*$, the non-trivial minimum is located at $\bar{\phi}_0 \approx 0.103 \neq 0$. Accordingly, the expectation value of the field jumps from $\langle \varphi \rangle = \phi_0$ to $\langle \varphi \rangle = 0$ at $\bar{\lambda} = \bar{\lambda}_*$. This constitutes a first order phase transition, depicted in [Figure 11.2](#).

We close this Section with a cautionary remark. It is well-known that the ϕ^4 -theory in three dimensions exhibits a 2nd order phase transition, and the present one-loop computation does not do justice to this fact. Still, it captures the presence of a phase transition. However, we used a perturbative ordering in powers of λ including dropping higher order terms in the loop while adjusting the full ϕ^2 -term. Evidently, we could have chosen another RG-condition. Indeed, while the chosen one is seemingly natural, it does not fix the correlation function on the equation of motion for $\phi_0 \neq 0$.

11.2.2. One-loop effective potential and first order phase transition in $d = 4$ dimensions

The $d = 3$ case evaluated in [Section 11.2.1](#) serves as a warm-up for the $d = 4$ case discussed now. Also in this case, the one-loop effective potential shows fluctuation-induced spontaneous symmetry breaking: the effective potential has non-trivial minima $\phi_0 \neq 0$ even for classical potentials with $\phi_{0,\text{cl}} = 0$. This is the celebrated Coleman-Weinberg mechanism for spontaneous symmetry breaking, discussed in [\[5\]](#) at one loop.

In four dimensions the UV degree of divergence is higher than in three dimensions and [\(11.27\)](#) is not finite, but logarithmically divergent. This remaining divergence is that of the mass, as can be seen from the m^2 -derivative of the tadpole, [\(11.29\)](#). This divergence can be resolved by either taking a further m^2 -derivative, or using the standard renormalisation with the help of the bare mass parameter m_0^2 in [\(11.26\)](#). Here we follow the second more standard route and choose the counterterms $\ln \mathcal{N}$ (vacuum terms), Δm_0^2 (mass counterterm) and $\Delta \lambda_0$ (coupling counterterm) such that we satisfy the following renormalisation conditions,

$$V_{\text{eff}}(0) = 0, \quad V_{\text{eff}}^{(2)}(0) = m^2, \quad V_{\text{eff}}^{(4)}(0) = \lambda. \quad (11.40)$$

The first condition for the vacuum energy is already adjusted with [\(11.28\)](#) used in [\(11.27\)](#). We emphasise again, that this choice is pure convenience as $V_{\text{eff}}(0)$ does not enter observables.

For the second condition we take the second derivative of [\(11.27\)](#) with respect to ϕ at vanishing $\phi = 0$, assuming $m \neq 0$. This leads us to

$$V_{\text{eff}}^{(2)}(0) = m^2 + \Delta m_0^2 + \frac{\lambda}{2} \int_{p^2 \leq \Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2}, \quad (11.41)$$

and the comparison with [\(11.40\)](#) leads us to

$$\Delta m_0^2 = -\frac{\lambda}{2} \int_{p^2 \leq \Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2}. \quad (11.42)$$

Two further field derivatives lead us to $V_{\text{eff}}^{(4)}(0)$ and a comparison of the explicit expression derived from (11.27) and the renormalisation condition for the coupling in (11.40) leads us to

$$\Delta\lambda_0 = \frac{3\lambda^2}{2} \int_{p^2 \leq \Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2}. \quad (11.43)$$

In summary this leads us to

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \left[\ln \left(1 + \frac{\frac{\lambda}{2} \phi^2}{p^2 + m^2} \right) - \frac{\frac{\lambda}{2} \phi^2}{p^2 + m^2} + \frac{1}{2} \left(\frac{\frac{\lambda}{2} \phi^2}{p^2 + m^2} \right)^2 \right], \quad (11.44)$$

with the renormalised classical potential $V(\phi)$ defined in (11.35). In (11.44), the limit $\Lambda \rightarrow \infty$ has been taken as the momentum integral is finite due to the subtractions. Note however, that the last term in (11.44) contains a logarithmic infrared divergence for $m \rightarrow 0$. This is not surprising as in massless theories the renormalisation conditions for logarithmically divergent couplings cannot be taken naïvely at $\mu = 0$: an logarithmic divergence is both, ultraviolet and infrared divergence and a subtraction at all scales vanishing, $p = 0$ and $\phi = 0$ is ill-defined. Note that in $d = 3$ we only faced a linear divergence for the mass and this problem was not present. As we finally also want to discuss the massless limit of the scalar theory in $d = 4$, we generalise the RG condition for the coupling in (11.40): instead of renormalising the coupling at $\phi = 0$, we choose $\phi = \mu$. This leads us to

$$\Delta\lambda_0 = \frac{3\lambda^2}{2} \int_{p^2 \leq \Lambda^2} \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2} + c\lambda^2, \quad (11.45)$$

where c is chosen such that

$$V_{\text{eff}}^{(4)}(\mu) = \lambda. \quad (11.46)$$

The constant $c\lambda^2$ in the counterterm parameter $\Delta\lambda_0$ only leads to the additional term $c/(4!)\lambda^2 \phi^4$ in the effective potential in (11.44). The momentum integral in (11.44) can be performed analytically, which leads to our final expression for the one-loop effective potential

One-loop Effective Potential in $d = 4$

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{1}{4!} c\lambda^2 \phi^4 + \frac{1}{64\pi^2} \left(m^2 + \frac{\lambda}{2} \phi^2 \right)^2 \ln \left(1 + \frac{\frac{\lambda}{2} \phi^2}{m^2} \right) - \frac{1}{64\pi^2} \left[\frac{1}{2} m^2 \lambda \phi^2 + \frac{3}{2} \left(\frac{\lambda}{2} \phi^2 \right)^2 \right]. \quad (11.47)$$

V_{eff} obeys the mass renormalisation condition in (11.40) and the coupling renormalisation condition in (11.46) for the choice

$$c = -\frac{3}{32\pi^2} \ln \left(1 + \frac{\lambda}{2} \bar{\mu}^2 \right) - \frac{1}{4\pi^2} \frac{\frac{3}{2} + \frac{\lambda}{2} \bar{\mu}^2}{\left(1 + \frac{\lambda}{2} \bar{\mu}^2 \right)^2} \frac{\lambda}{2} \bar{\mu}^2, \quad \text{with} \quad \bar{\mu}^2 = \frac{\mu^2}{m^2}. \quad (11.48)$$

Equation (11.48) reduces to $c = 0$ for $\bar{\mu} = 0$. Before we discuss the Coleman-Weinberg potential for $m = 0$, we check the $m \rightarrow \infty$ limit, where the effective potential (11.47) should reduce to the classical renormalised potential $V(\phi)$. We first notice, that the constant c in (11.48) tends towards zero independent of the chosen renormalisation scale μ , $c(m \rightarrow \infty) = 0$. All terms with positive powers in m^2 stemming from the last two terms in (11.47) and we are left with the desired result,

$$\lim_{m \rightarrow \infty} V_{\text{eff}}(\phi) = V(\phi). \quad (11.49)$$

We note that this result also follows readily from the renormalisation condition that fix the first three terms in the Taylor expansion in ϕ^2 to be that in the classical potential. The higher order terms necessarily carry inverse powers of m^2 for dimensional reasons and hence vanish for $m \rightarrow 0$. Moreover, in this limit we find a potential with a trivial global minimum at $\phi_0 = 0$. Note that strictly speaking the limit in (11.49) can only be taken relative to another scale, and the only other dimensionful quantities is the field ϕ . Accordingly, the limit in (11.49) should rather be in the dimensionless field $\phi/m \rightarrow 0$, see also our discussion around (11.37). Accordingly, we briefly discuss the dimensionless potential with

$$u(\bar{\phi}) = \frac{V_{\text{eff}}(\phi)}{m^4}, \quad \bar{\phi}^2 = \frac{\phi^2}{m^2}, \quad (11.50)$$

and $\bar{\mu}^2$ defined in (11.48). The field can take any value and the only mass dependence left is that in $c(\bar{\mu})$, see (11.48). Inserting the definitions (11.50) into the potential (11.47) leads us to

$$u(\bar{\phi}, \lambda, \bar{\mu}) = \frac{1}{2}\bar{\phi}^2 + \frac{\lambda}{4!} [1 + c(\bar{\mu})] \bar{\phi}^4 + \frac{1}{64\pi^2} \left(1 + \frac{\lambda}{2}\bar{\phi}^2\right)^2 \ln\left(1 + \frac{\lambda}{2}\bar{\phi}^2\right) - \frac{1}{64\pi^2} \left[\frac{1}{2}\lambda\bar{\phi}^2 + \frac{3}{2}\left(\frac{\lambda}{2}\bar{\phi}^2\right)^2\right]. \quad (11.51)$$

Finally, we discuss the massless limit as computed in [5]. In our dimensionless variables this is the limit $\bar{\mu} \rightarrow \infty$. We first note that in this limit the constant c in (11.48) tends towards

$$c(\bar{\mu}) \xrightarrow{\bar{\mu}^2 \rightarrow \infty} -\frac{1}{4\pi} - \frac{3}{32\pi^2} \ln\left(1 + \frac{\lambda}{2}\bar{\mu}^2\right), \quad (11.52)$$

and the effective potential (11.47) reduces to the Coleman-Weinberg potential. As in the three-dimensional case we find non-trivial minima for $V_{\text{eff}}(\phi)$ for sufficiently small m , and for $m = 0$ we are led to

$$\ln \frac{\phi_0^2}{\mu^2} = -\frac{32\pi^2}{3} \frac{1}{\lambda} + \frac{11}{3}, \quad \longrightarrow \quad \phi_0 = \mu e^{-\frac{1}{3\lambda}(16\pi^2 - \frac{11}{2}\lambda)}. \quad (11.53)$$

In terms of the dimensionless variables one easily sees that the above massless limit is concisely defined with

$$\frac{\lambda}{2}\bar{\phi}_0^2 = \frac{\lambda}{2}\bar{\mu}^2 e^{-\frac{1}{3\lambda}(32\pi^2 - 11\lambda)} \rightarrow \infty, \quad (11.54)$$

which also implies $(\lambda/2)\bar{\mu}^2 \rightarrow \infty$ as the exponential factor is bounded from above by its $\lambda \rightarrow \infty$ limit. This result relates the scale of the (physical) expectation value ϕ_0 to the RG scale μ . Note however, that a full analysis also involves the one-loop running of the coupling, and μ is simply substituted by a physical scale, $\mu \rightarrow \Lambda_{\text{phys}}$. In the present case of a ϕ^4 theory in four space-time dimensions this can be the position of the Landau pole or the physical cutoff scale Λ_{phys} , beyond which the scalar theory ceases to be valid. Naturally, in the vanishing mass limit with $m \rightarrow 0$, $\bar{\mu}$ or rather

$$\bar{\mu}^2 = \bar{\Lambda}_{\text{phys}}^2 = \frac{\Lambda_{\text{phys}}^2}{m^2}, \quad (11.55)$$

tends towards infinity. Hence, we can take the Coleman-Weinberg limit as (11.54) holds. This leaves us with the Coleman-Weinberg scenario of fluctuation-induced spontaneous symmetry breaking: it is based on a large physical UV cutoff Λ_{phys} with $\mu = \Lambda_{\text{phys}}$. With a sufficiently small coupling, $3\lambda \ll 16\pi^2$, the scale of symmetry breaking is far below the UV cutoff scale Λ_{phys} due to the exponential suppression, leaving us with a huge hierarchy between the explicit scales such as Λ_{phys} and the symmetry breaking scale and the size of the condensate ϕ_0 .

One-loop Effective Potential in $d = 4$

$$V_{\text{eff}}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2} \left[\ln \frac{\phi^2}{\mu^2} - \frac{25}{6} \right]. \quad (11.56)$$

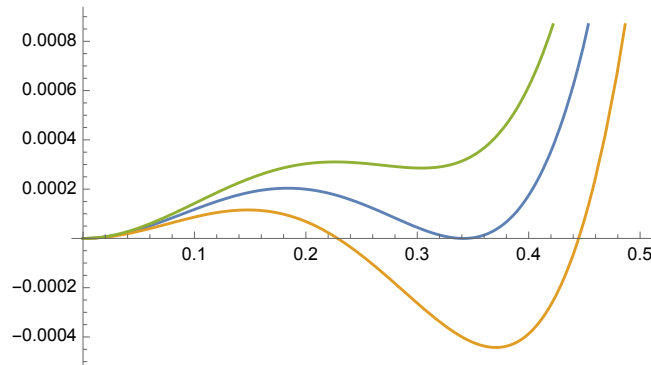


Figure 11.3.: Dimensionless Coleman-Weinberg potential $u(\bar{\phi})$ for the parameters $\lambda = 20$ as a function of ϕ/μ . We depict the potential for $m = 0.19\mu$ in the symmetric phase, $m = 0.1\mu$ in the broken phase and $m \approx 0.17737\mu$ at the 1st order phase transition.

We close the discussion of the Coleman-Weinberg potential with depicting the first order phase transition as a function of the mass m . For the sake of convenience we use a large coupling $\lambda = 20$ and measure all quantities in units of the RG-scale μ . Then, for $m \rightarrow 0$, the minimum settles at

$$\lim_{m \rightarrow 0} \phi_0 \approx 0.45\mu, \quad \text{for } \lambda = 20. \quad (11.57)$$

In turn, for $m > m_*$, the theory settles in the symmetric phase with the global minimum at $\phi_0 = 0$. For the parameter choice in (11.57) the phase transition happens at

$$m_* \approx 0.17737\mu. \quad (11.58)$$

In [Figure 11.3](#) we depict the dimensionless potential $u(\bar{\phi})$ as a function of ϕ/μ for the masses $m = 0.19\mu$ (symmetric phase), $m = 0.1\mu$ (broken phase) and $m = m_*$ in (11.58). One clearly sees the first order phase transition.

11.3. Functional Relations

In the last Section we have applied the functional integral within a one-loop computation to the non-perturbative question of phase transitions in quantum field theory. While this simple analysis already yielded qualitative results, it cannot fully capture the non-perturbative nature of this specific problem. To begin with, the nature of the phase transition (1st order, 2nd order with infinite correlation length, crossover) cannot be predicted. Moreover, even its presence can only be predicted within a fully non-perturbative computation: in the vicinity of a phase transition fluctuations typically grow strong and quantum fluctuations beyond one-loop may dominate the physics. This is evident for a 2nd order, where the correlation length is (by definition) infinite and the theory develops a massless mode. Accordingly, the resolution of these interesting question (and many others) requires a non-perturbative formulation of the theory, that is amiable to analytic *and* numerical computation. As discussed in the introduction, two main proponents are the lattice field theory approach, see [Chapter 15](#) and functional approaches, discussed here at the example of the Dyson-Schwinger equations and in [Chapter 16](#) in terms of the functional renormalisation group. In the latter approaches one derives coupled diagrammatic relations for all correlation functions. However, instead of classical propagators and vertices these relations are cast in form of full propagators (two-point function) and vertices (n -point functions). In specific cases, the Dyson-Schwinger equations and functional renormalisation group equations, these relations are closed. Moreover, while they yield diagrammatic relations for correlation functions, they can be cast into a functional form as closed relations for generating functionals.

We now elucidate these statements at the example of the functional Dyson-Schwinger equation for the ϕ^4 -theory, the quantum equations of motion. We start this analysis with the observation, that the path integral measure $\mathcal{D}\varphi$ is invariant under space-time dependent translations $c(x)$ of the field,

$$\varphi(x) \rightarrow \varphi'(x) = \varphi(x) + c(x), \quad \longrightarrow \quad \mathcal{D}\varphi' = \mathcal{D}\varphi. \quad (11.59)$$

Accordingly, this is a symmetry of the path integral, similarly to that discussed in QFT I in the context of the Noether theorem, see [Section 2.2](#) and its global version is just a space-time independent shift of the field. As it deforms a given configuration φ into a general other one, φ' , the respective 'symmetry' equation resolves the theory. Its generator is simply the derivative with respect to the fields. Let us now evaluate this symmetry for the classical theory. As discussed in [Section 2.2](#), the local variation of the classical action vanishes on the solution of the equation of motion, ϕ_{EoM} . In the present case this leads us to

$$\left. \frac{\delta S[\phi]}{\delta \phi(x)} \right|_{\phi=\phi_{\text{EoM}}} = 0, \quad (11.60)$$

which is nothing but the equation of motion itself, valid for the generating functional for classical correlation functions, the classical action. Now we proceed to quantum generating functionals. There we expect, that the functional relations that follow from the translation invariance of the path integral measure capture the quantum equations of motion of the theory. Moreover, while this relation can be written in terms of all the generating functionals, the analogue of [Equation \(11.60\)](#) is that for the effective action. For the derivation we first rewrite the path integral in terms of a shifted field $\varphi'(x)$,

$$Z[J] \simeq \int \mathcal{D}\varphi' e^{-S[\varphi'] + \int_x J(x) \varphi'(x)} = \int \mathcal{D}\varphi e^{-S[\varphi+c] + \int_x J(x) [\varphi(x)+c(x)]}, \quad (11.61)$$

where we have used [\(11.59\)](#) and we have assumed the absence of boundary terms in the path integral. Now we evaluate [\(11.61\)](#) in the linear order of infinitesimal shifts $\delta c(x)$. Then, the translation in [\(11.61\)](#) is given by a functional derivative with respect to $\varphi(x)$ and we arrive at

$$\frac{1}{2Z[J]} \int \mathcal{D}\varphi \frac{\delta}{\delta \varphi(x)} \left[e^{-S(\varphi) + \int_x J(x) \varphi(x)} \right] = 0. \quad (11.62)$$

For a one-dimensional integral this reduces to $\int dq d/dq[\exp\{-S(q) + Jq\}] = 0$, see also the discussion in [Section 9.3.1](#) around [\(9.62\)](#). Performing the derivative in [\(11.62\)](#), we get

$$J(x) = \left\langle \frac{\delta S[\varphi]}{\delta \varphi(x)} \right\rangle_J. \quad (11.63)$$

[Equation \(11.63\)](#) is called the Dyson-Schwinger equation (DSE) of the theory. Note, that [\(11.63\)](#) is simply the expectation value of the classical equation of motion for a given background J ; see [\(11.12\)](#). We have used the latter for the saddle point expansion of the effective action, and [\(11.63\)](#) is the quantum equation of motion in a given background J .

In its form [\(11.63\)](#) it can be written in terms of a sum of first, second and third powers of J -derivatives of $Z[J]$ or $W[J]$, as $\langle \delta S[\varphi]/\delta \varphi(x) \rangle$ is a sum of $\langle \varphi(x) \rangle$, $\langle \varphi(x)^2 \rangle$ and $\langle \varphi(x)^3 \rangle$. More explicitly we have

$$\frac{\delta S}{\delta \varphi(x)} = (-\Delta + m^2) \varphi(x) + \frac{\lambda}{3!} [\varphi(x)]^3 \quad (11.64)$$

and its expectation value is given by

$$\left\langle \frac{\delta S}{\delta \varphi(x)} \right\rangle_J = (-\Delta + m^2) \phi(x) + \frac{1}{3!} \langle \varphi(x)^3 \rangle_J = \frac{\delta S}{\delta \phi(x)} + \frac{\lambda}{3!} [\langle \varphi(x)^3 \rangle_J - \phi(x)^3]. \quad (11.65)$$

The expectation value $\langle \varphi(x)^3 \rangle_J$ can be written in terms of J -derivatives of $Z[J]$ or $W[J]$. We first illustrate the general derivation at the example of $\langle \varphi^3 \rangle_J$, relevant for the ϕ^4 -theory. We use that the three-point function in the presence of a background current can be written as a J -derivative of the two-point function,

$$\begin{aligned} \langle \varphi(x)^3 \rangle_J &= \frac{1}{Z[J]} \int \mathcal{D}\varphi \varphi(x)^3 e^{-S[\varphi] + \int_x J\varphi} = \left(\frac{\delta}{\delta J(x)} + \phi(x) \right) \frac{1}{Z[J]} \int \mathcal{D}\varphi \varphi(x)^2 e^{-S[\varphi] + \int_x J\varphi} \\ &= \left(\frac{\delta}{\delta J(x)} + \phi(x) \right) Z^{(2)}[J](x, x) = \left(\frac{\delta}{\delta J(x)} + \phi(x) \right) [W^{(2)}[J](x, x) + \phi(x)^2], \end{aligned} \quad (11.66)$$

where $\phi[J] = \langle \varphi \rangle_J$ is a functional of J . Iterating the steps in (11.66) leads us to the general functional relation

$$\left\langle \prod_i \varphi(x_i) \right\rangle_J = \prod_i \left[\frac{\delta}{\delta J(x_i)} + \phi(x_i) \right]. \quad (11.67)$$

Inserting (11.67) in (11.63) leads us to the final DSE in terms of J -dependent generating functionals,

$$J(x) = \frac{\delta S}{\delta \varphi(x)} \left[\varphi(x) = \frac{\delta}{\delta J(x)} + \phi(x) \right], \quad (11.68)$$

with $\phi = W^{(1)}[J]$. In summary, (11.63) in its form (11.68) is the promised closed functional relation for all correlation functions of the theory. The respective relations for all n -point correlation functions are then derived by taking further J -derivatives of (11.63) and evaluate them at $J = 0$.

We now proceed by rewriting (11.68) as a functional relation for the effective action $\Gamma[\phi]$. A first step is done by using the relation between J and the first ϕ -derivative of the effective action, (11.4). This leads us to

$$\frac{\delta \Gamma}{\delta \phi(x)} = \frac{\delta S}{\delta \varphi(x)} \left[\varphi(x) = \frac{\delta}{\delta J(x)} + \phi(x) \right], \quad (11.69)$$

where the background current on the right hand side is the field-dependent maximum current in the Legendre transformation. Equation (11.69) elucidates the fact, that quantum effective action $\Gamma[\phi]$ is the quantum analogue of the classical action. It encodes the quantum equations of motion (QEoM) in the vacuum,

$$\left. \frac{\delta \Gamma}{\delta \phi(x)} \right|_{\phi_{\text{EoM}}} = 0 = \left\langle \frac{\delta S}{\delta \phi} \right\rangle, \quad (11.70)$$

and for $\phi \neq \phi_{\text{EoM}}$ it constitutes the QEoM in the background $J = \delta \Gamma / \delta \phi$. It is left to recast the J -derivative in terms of a ϕ -derivative. This has already been used in the derivation of the relation between $\Gamma^{(2)}$ and $W^{(2)}$ in (11.9) and we derive analogously

$$\frac{\delta}{\delta J(x)} = \int_y \frac{\delta \phi(y)}{\delta J(x)} \frac{\delta}{\delta \phi(y)} = \int_y W^{(2)}[J](x, y) \frac{\delta}{\delta \phi(y)} = \int_y \frac{1}{\Gamma^{(2)}[\phi]}(x, y) \frac{\delta}{\delta \phi(y)}. \quad (11.71)$$

Inserting (11.71) in (11.69) leads us to the final form of the DSE,

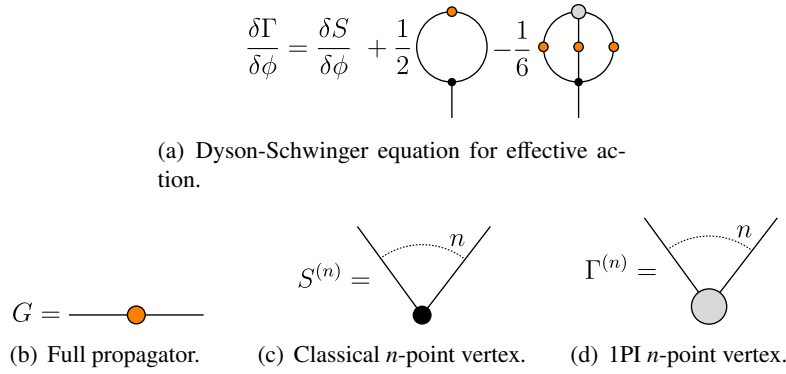


Figure 11.4.: Functional Dyson-Schwinger equation

Dyson-Schwinger equation

$$\frac{\delta\Gamma}{\delta\phi(x)} = \frac{\delta S}{\delta\phi(x)} \left[\varphi(x) = \int_y \frac{1}{\Gamma^{(2)}(\phi)}(x,y) \frac{\delta}{\delta\phi(y)} + \phi(x) \right], \quad (11.72)$$

which is a closed functional equation for all 1PI correlation functions $\Gamma^{(n)}[0](x_1, \dots, x_n)$. Diagrammatic relations for the latter are derived by taking $n - 1$ - ϕ -derivatives of (11.72) and evaluate these relations at $\phi = 0$ or, more generally, at ϕ_{EoM} . Then the left hand side is simply $\Gamma^{(n)}[0]$. In the ϕ^4 -theory the right hand side is a sum of classical terms and one- and two-loop diagrams with full propagators, classical vertices and full vertices. This is depicted in Figure 11.4. More generally the right hand side contains up to $n - 1$ -loop terms for a classical action with maximally n th powers of the field.

This concludes our discussion of functional relations and we close this chapter with two applications. The first one concerns a simple proof of the 1PI nature of the effective action and the second one concerns the derivation of perturbation theory from Figure 11.4:

We want to show, that $\Gamma[\phi]$ generates 1PI correlation functions. This is done by induction. Obviously, the classical action is 1PI. Inserting it on the right hand side of (11.72) provides us with functional one-loop and two-loop diagrams, that are 1PI, see Figure 11.4 with classical propagators and vertices on the right hand side. Moreover, taking $n - 1$ ϕ -derivatives generates 1PI n -point functions in a general background ϕ . The next step in our iteration is then obtained by re-inserting this result into the right hand side, or rather $\Gamma^{(2)}[\phi]$ and $\Gamma^{(3)}[\phi]$ as the right hand side of (11.72) only depends on these correlation functions, see also Figure 11.4. Let us now assume that we have proven the 1PI nature of $\Gamma[\phi]$ for the i th iteration. Now we use the respective $\Gamma_i^{(n)}$ with $n = 2, 3$ on the right hand side. However, as we insert a sum of 1PI diagrams into a 1PI expression, this only leads to 1PI diagrams. Hence, every order i of the iteration is 1PI and assuming convergence of this iteration procedure leads to the desired result.

We emphasise that these iterations do not reflect the perturbative ordering, they involve resummations of whole classes of 1PI diagrams due to the fact, that the propagator is the inverse of $\Gamma^{(2)}$. However, in the first iteration (using the classical propagator and action on the right hand side) all one-loop diagrams are included, in the second iteration all two-loop diagrams are included and so on. Hence, the i th iteration generates the i th order perturbation theory and resummations. Dropping the latter higher order diagrams leaves us with i th order perturbation theory.

As a corollary of this proof it follows that the Schwinger functional generated connected correlation functions, and a simple way to see this is via the relation (11.9). This concludes the discussion of functional relations.

12. Fermionic field theories

In the present Chapter we extend the functional integral approach to fermionic quantum field theories. It follows from the structure of the derivations and results so far that most of the steps can be taken over to fermionic theories, after the anti-commutation relations of fermionic fields are properly taken into account. Instead of the commutation relations (10.4) of the scalar field we have anti-commutation relations,

$$\{\psi(\mathbf{x}), \bar{\psi}(\mathbf{y})\} = \gamma^0 \delta(\mathbf{x} - \mathbf{y}), \quad (12.1)$$

with the fermionic field ψ and its Dirac conjugate $\bar{\psi} = \psi^\dagger \gamma^0$. This leaves us with the question of defining field eigenstates and momentum operator eigenstates, for quantum mechanics see (9.4), the respective completeness relations (9.11) as well as the matrix element between position and momentum states, (9.12). As in the scalar theory we first discuss these topics at the example of (fermionic) quantum mechanics in Section 12.1, before discussing fermionic quantum field theories in Section 12.2.

12.1. Quantum Mechanics

We start our analysis with considering a simple free fermionic Hamiltonian with

$$H = \omega a^\dagger a \quad (12.2)$$

with the creation/annihilation operators a^\dagger, a respectively. These operators satisfy the anti-commutation relation

$$\{a, a^\dagger\} = aa^\dagger + a^\dagger a = 1 \quad \text{and} \quad a^2 = a^{\dagger 2} = 0, \quad (12.3)$$

where the second pair of relations states that the anti-commutator of a with itself and a^\dagger with itself vanishes. In contradistinction to the respective commutator relation these are non-trivial relations that signal the Grassmann nature of the fermionic creation and annihilation operators. Similar to the position space representation of the position operator \hat{q} as multiplication with q and the momentum operator $\hat{p} \simeq \partial_q$, the algebra (12.3) can be represented in terms of Grassmann variables

$$c^2 = \bar{c}^2 = \{c, \bar{c}\} = 0. \quad (12.4)$$

Similarly to the position space representation of the Heisenberg algebra $[\hat{q}, \hat{p}] = i$, we can represent (12.3) by the Grassmann number \bar{c} and its derivative $\frac{\partial}{\partial \bar{c}}$,

$$\left\{ \frac{\partial}{\partial \bar{c}}, \bar{c} \right\} = 1, \quad (12.5)$$

where we have use that the derivative with respect to a Grassmann variable anti-commutes with another Grassmann variable.

12.1.1. Properties of Grassmann variables

This property is but one of many relations that follow from the Grassmann property $c^2 = 0$. Accordingly, we now summarise some basic facts about Grassmann algebras, differentiation and integration. When repeating step by step the derivation of the functional integral of scalars in the present fermionic set-up, we will need many different Grassmann numbers c_i and hence we consider n Grassmann variables $c_i, i = 1, \dots, n$. They anti-commute with each other,

Grassmann algebra

$$c_i c_j + c_j c_i = 0, \quad \forall i, j. \quad (12.6)$$

as well as

$$\frac{\partial}{\partial c_i} c_j = \delta_{ij}. \quad (12.7)$$

It follows directly from (12.6) and (12.7) by considering $\frac{\partial}{\partial c_i} \frac{\partial}{\partial c_j} c_n c_m$, that

$$\frac{\partial}{\partial c_i} \frac{\partial}{\partial c_j} + \frac{\partial}{\partial c_j} \frac{\partial}{\partial c_i} = 0, \quad c_i \frac{\partial}{\partial c_j} + \frac{\partial}{\partial c_j} c_i = \delta_{ij}, \quad (12.8)$$

Note also that $c_i c_j$ is not a Grassmann variable, as $(c_i c_j) c_m = c_m (c_i c_j)$ with (12.6). Accordingly, the product of two different Grassmann numbers behaves as a number, it commutes with a Grassmann number as well as with a number. However, it is not a number itself, as $(c_i c_j)^2 = 0$.

In summary this leads us to (12.5), where we also can exchange the Grassmann number and its derivative due to the anti-commutator,

$$\left\{ \bar{c}, \frac{\partial}{\partial \bar{c}} \right\} = \left\{ \frac{\partial}{\partial \bar{c}}, \bar{c} \right\} = 1 \quad (12.9)$$

These preparations enable us to provide the 'position space' representation of the fermionic anti-commutation relation (12.3) of the fermionic harmonic oscillator,

$$a^\dagger \rightarrow \bar{c}, \quad a \rightarrow \frac{\partial}{\partial \bar{c}}. \quad (12.10)$$

The functional integrals will require Grassmann integrations in the completeness relations. First we note that a general function $f(c)$ is given by

$$f(c) = f_0 + f_1 c, \quad (12.11)$$

where higher order terms do not exist due to $c^2 = 0$. Hence, for the Grassmann integrations we only have to define the Grassmann integral of a constant and that of c itself. The integral of a constant vanishes, as follows readily with

$$\int dc 1 = \int dc \frac{\partial}{\partial c} c = 0, \quad (12.12)$$

with the assumption that boundary terms are absent. The Grassmann integral of c is non-vanishing and we normalise it to unity,

$$\int dc c := 1. \quad (12.13)$$

It follows from (12.12) and (12.13), that the Grassmann integration is equivalent to Grassmann differentiation,

$$\int dc_i c_j = \frac{\partial}{\partial c_i} c_j = \delta_{ij}. \quad (12.14)$$

This allows us to discuss general coordinate transformation. To that end we consider $\int dc f(c)$, and the change of the integration variable to the new variable $c = ac' + b$. It follows from (12.14) that

$$\int dc f(c) = \frac{1}{a} \int dc' f(ac' + b), \quad \text{with } \frac{\partial}{\partial c} = \frac{1}{a} \frac{\partial}{\partial c'}. \quad (12.15)$$

Equation (12.15) provides us with a relation for the Jacobian $J(c')$ of this transformation, defined by

$$\int dc = \int dc' J(c'), \quad \longrightarrow \quad J = \frac{1}{a}. \quad (12.16)$$

Equation (12.16) has to be contrasted with the Jacobian in standard integrals, where $\int dq = \int dq' J(q')$ and $J = a$ for $q = aq' + b$. Note that the proof of (12.15) used (12.14). We could also compute both integrals explicitly using (12.12) and (12.13). For a given function $f(c) = ac + b$ we obtain

$$\int dc(ac + b) = a, \quad \text{and} \quad \int dc' J(c') [a(ac' + b) + b] = J(c')a^2. \quad (12.17)$$

Comparing both integrals we are led to (12.15). The derivations and relations above extend readily to similar ones for n Grassmann numbers. Let us consider a transformation

$$c_i = a_{ij}c'_j + b_i. \quad (12.18)$$

In the straightforward generalisation of (12.15) we find

$$dc_1 \cdots dc_n = dc'_1 \cdots dc'_n J(c'), \quad \text{with} \quad J^{-1} = \det \frac{\partial c_i}{\partial c'_j} = \det a. \quad (12.19)$$

Equation (12.19) can also be derived from the derivative nature of the Grassmann integrations,

$$\int dc_1 \cdots dc_n f(c) = \prod_i \frac{\partial}{\partial c_i} f(c) = \prod_{i,j} \frac{\partial c'_j}{\partial c_i} \frac{\partial}{\partial c'_j} f(c) = \det \frac{\partial c'_j}{\partial c_i} \prod_k \frac{\partial}{\partial c'_k} f(c) = \int dc'_1 \cdots dc'_n J(c') f(c). \quad (12.20)$$

Finally we consider Gaussian integrals as required in the derivation of the functional integral. To that end we consider complex c_i with $\bar{c}_i = c_i^*$. It follows

$$\int dc_1 d\bar{c}_1 \cdots dc_n d\bar{c}_n e^{\bar{c}_i a_{ij} c_j} = \det a \int \prod_l dc_l d\bar{c}_l e^{\bar{c}_l c_l} = \det a, \quad (12.21)$$

with $c'_i = a_{ij}c_j$ and (12.19). The last step in (12.21) follows from

$$\int \prod_{l=1}^N (dc_l d\bar{c}_l) \prod_{i=1}^N (\bar{c}_i c_i) = 1. \quad (12.22)$$

Equation (12.22) is derived from the integration rules (12.12) and (12.14). They entail that only terms of the form $\prod_i (\bar{c}_i c_i)$ contribute to the integral. All other terms either lack a specific c_i , or are proportional to $c_i^2 = 0$ after anti-commuting some Grassmann variables.

Finally, similarly to the derivation of the scalar path integral we will use states that are normalised to Grassmann δ -functions. Similarly to the standard δ -function the Grassmann δ -function is defined as an integral kernel,

$$\int dc \delta(c) f(c) = f(0) \quad \longrightarrow \quad \delta(c) = c = \int d\bar{c} e^{\bar{c}c}, \quad (12.23)$$

where the last relation introduces the Fourier representation of the Grassmann δ -function. In the path integral derivation we also need the scalar product of general functions $f(c) = f_0 + f_1 c$ with general functions $g(c) = g_0 + g_1 c$ of complex Grassmann variables c . This scalar product is reminiscent of the scalar product of square integrable functions. We have the orthogonal constant basis functions 1 and c and we define

$$(f, g) = \bar{f}_0 g_0 + \bar{f}_1 g_1, \quad \longrightarrow \quad \|f\|^2 = |f_0|^2 + |f_1|^2. \quad (12.24)$$

In terms of Grassmann integrals, (12.24) is represented by

$$(f, g) = \int dcd\bar{c} e^{\bar{c}c} \overline{f(c)} g(c). \quad (12.25)$$

We close the derivation and collection of Grassmann differentiation and integration rules with a remark on (real) Gaußian Grassmann integrations. The Gaußian integral with an antisymmetric matrix $a_{ij} = -a_{ji}$ is the Pfaffian $\text{Pf}(a)$ of the matrix a ,

$$\text{Pf}(a) = \int dc_{2n} \dots dc_1 e^{1/2 c_i a_{ij} c_j} = \frac{1}{2^n n!} \int dc_{2n} \dots dc_1 (c_i a_{ij} c_j)^n = \frac{1}{2^n n!} \sum_{\rho} \varepsilon(\rho) a_{i_1 i_2} \dots a_{i_{2n-1} i_{2n}}, \quad (12.26)$$

with the permutations ρ of $\{i_1, \dots, i_{2n}\}$ and the signature $\varepsilon(\rho)$ of the permutations ρ . Note also that the integrals in (12.26) can also be performed for a general matrix $a = a^+ + a^-$, where $a_{ij}^{\pm} = \pm a_{ji}^{\pm}$ are the symmetric and anti-symmetric parts of the matrix a . We still arrive at $\text{Pf}(a) = \text{Pf}(a^-)$. It follows

$$(\text{Pf}(a))^2 = \det a^-. \quad (12.27)$$

Pfaffians play a crucial role in the path integral quantisation of the Weyl and Majorana fermions.

12.1.2. Fermionic path integral in quantum mechanics

With the collection of rules and relations in Section 12.1.1 we can derive the path integral for a quantum mechanical fermionic system. We consider the Grassmann representation of the Hamiltonian H in (12.2) with

$$H = \omega \bar{c} \frac{\partial}{\partial \bar{c}}. \quad (12.28)$$

In analogy to bosonic quantum mechanics we introduce position states $|c\rangle$ with $\hat{c}|c\rangle = c|c\rangle$. These states are normalised to delta-functions,

$$\langle c|c'\rangle = \delta(c - c'), \quad (12.29)$$

with the Grassmann δ -function $\delta(c - c')$ and its Fourier representation defined in (12.23).

The momentum state in fermionic quantum mechanics is given by $|\bar{c}\rangle$, which is already evident from (12.28). Note also in this context that the field momentum of a Dirac fermion is $\bar{\psi}$, see also (12.1). The state $|\bar{c}\rangle$ is given by the Fourier transformation of a position state $|c\rangle$, to wit

$$|\bar{c}\rangle = \int dc e^{\bar{c}c} |c\rangle, \quad (12.30)$$

and hence

$$\langle c|\bar{c}\rangle = e^{\bar{c}c}. \quad (12.31)$$

With these preparations we start the derivation of the path integral representation of the *Euclidean* fermionic transition amplitude, simply repeating the steps for that in the bosonic quantum mechanical system done in Section 9.1,

$$\langle c_i|U(t_i, t_{i-1})|c_{i-1}\rangle = \int d\bar{c} \langle c_i|\bar{c}\rangle \langle \bar{c}|U(t_i, t_{i-1})|c_{i-1}\rangle = \int d\bar{c} e^{\bar{c}c_i} \langle \bar{c}|U(t_i, t_{i-1})|c_{i-1}\rangle, \quad (12.32)$$

with $U(t_i, t_{i-1}) \simeq e^{-\Delta t H}$ and $\Delta t = t_i - t_{i-1}$. With the Hamiltonian in (12.28) we get for the infinitesimal transition amplitude

$$\langle \bar{c}|U(t, t - \Delta t)|c\rangle = \left[1 - \Delta t \omega \bar{c} \frac{\partial}{\partial \bar{c}} + O(\Delta t^2) \right] e^{-\bar{c}c} = e^{-\bar{c}c(1 - \Delta t \omega)} + O(\Delta t^2). \quad (12.33)$$

Hence, in the limit $t_0 = -T$ to $t = T$, $T \rightarrow \infty$ we get

$$\langle \bar{c}|U(t, t_0)|c\rangle = \lim_{n \rightarrow \infty} \int \prod_{i=1}^{n-1} dc_i d\bar{c}_i e^{-S(\vec{c}, \vec{\bar{c}})}. \quad (12.34)$$

with the action

$$S(\vec{c}, \vec{\bar{c}}) = - \sum_{i=1}^{n-1} \bar{c}_i (c_i - c_{i-1}) - \bar{c}_n c_{n-1} + \omega \frac{2T}{n} \sum_{i=1}^n c_i c_{i-1} \quad (12.35)$$

Finally we take the continuum limit, leading to the analogous expression to (9.28),

$$\langle 0|U(\infty, -\infty)|0\rangle = \int \mathcal{D}c \mathcal{D}\bar{c} e^{-S[c(t), \bar{c}(t)]} \quad (12.36)$$

with the action

$$S[c, \bar{c}] = - \int dt [\bar{c}(t) \dot{c}(t) + \omega \bar{c}(t) c(t)]. \quad (12.37)$$

The respective fermionic generating functional is given by

Fermionic generating functional

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}c \mathcal{D}\bar{c} e^{-S[c, \bar{c}] + \int_x (\bar{\eta} c - \bar{c} \eta)}, \quad (12.38)$$

where the ordering in the source terms, $\bar{\eta}c - \bar{c}\eta$ is one of the common conventions. Another one is $\bar{\eta}c + \bar{c}\eta$. The correlation functions, including their disconnected terms, read

Fermionic correlation functions

$$\langle T c(t_1) \cdots c(t_n) \bar{c}(t_{n+1}) \cdots \bar{c}(t_{2n}) \rangle = \left[\frac{1}{Z[\eta, \bar{\eta}]} \frac{\delta}{\delta \bar{\eta}(t_1)} \cdots \frac{\delta}{\delta \bar{\eta}(t_n)} \frac{\delta}{\delta \eta(t_{n+1})} \cdots \frac{\delta}{\delta \eta(t_{2n})} Z[\eta, \bar{\eta}] \right]_{\eta=\bar{\eta}=0}. \quad (12.39)$$

This concludes the discussion of fermionic quantum mechanics. The Schwinger functional and the effective action are defined as in the scalar case and generate connected and 1PI fermionic correlation functions respectively.

12.2. Quantum Field Theory

The results in the last Section translate directly into fermionic quantum field theories. We consider a Euclidean Dirac theory with the action

$$S_D[\psi, \bar{\psi}] = - \int d^d x \bar{\psi} (\not{\partial} + m) \psi, \quad (12.40)$$

with $\not{\partial} = \gamma_\mu \partial_\mu$, see also [Chapter 4](#) in QFT I. [Equation \(12.40\)](#) is obtained with a Wick rotation from [\(4.70\)](#) with

$$x_{0M} \rightarrow -ix_{0E}, \quad \gamma_{0M} \rightarrow -i\gamma_{0E}, \quad \bar{\psi}_M \rightarrow -i\bar{\psi}_E. \quad (12.41)$$

The Euclidean Clifford algebra is given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \quad (12.42)$$

and γ_5 or rather γ_{2n+1} in even dimensions $d = 2n$ is defined with

$$\gamma_{2n+1} = \gamma_0 \gamma_1 \cdots \gamma_{2n}, \quad \text{with} \quad \gamma_{2n+1}^2 = \mathbb{1}. \quad (12.43)$$

A complete basis of the Euclidean $O(4)$ spin representation also includes the (Hermitian) generator

$$\sigma_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu], \quad (12.44)$$

which generates $SU(2) \times SU(2)$ rotations in $d = 4$. The derivation of the functional integral is done in complete analogy to that of the quantum mechanical case in [Section 12.1.2](#), leading to

Fermionic generating functional

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\psi, \bar{\psi}] + \int_x (\bar{\eta}\psi - \bar{\psi}\eta)}, \quad (12.45)$$

where now we allow for a general fermionic action $S[\psi, \bar{\psi}]$, one example being the free Dirac action [\(12.40\)](#). Correlation functions of a fermionic quantum field theory follow from [\(12.45\)](#) as

Fermionic correlation functions

$$\langle T \psi(x_1) \cdots \psi(x_n) \bar{\psi}(x_{n+1}) \cdots \bar{\psi}(x_{2n}) \rangle = \frac{1}{Z} \frac{\partial^{2n} Z}{\delta \bar{\eta}(x_1) \cdots \delta \eta(x_{2n})} \Big|_{\eta, \bar{\eta}} = 0. \quad (12.46)$$

Interacting fermionic theories of particular interest with fermions are of the Yukawa type with the action

$$S_{\text{Yuk}}[\psi, \bar{\psi}, \phi] = \int_x \bar{\psi}(\not{\partial} + m + h\phi)\psi + S_\phi[\phi], \quad (12.47)$$

with a real scalar field ϕ . In (12.47), $S_\phi[\phi]$ stands for the purely scalar part of the action, for example a free scalar action or a ϕ^4 action. In this case the generating functional in (12.45) has to be augmented by a path integral over ϕ .

Another important example is QED with

$$S_{\text{QED}}[\psi, \bar{\psi}, A] = \int_x \bar{\psi}(\not{\partial} - ie\not{A} + m)\psi + S_A[A], \quad (12.48)$$

where A_μ is the U(1) gauge field and S_A is the pure gauge field action. Then, the generating functional in (12.45) has to be augmented by a path integral over A_μ . Evidently, as in the discussion of QED with canonical quantisation, we have to treat the redundancy of the gauge field also in the path integral: the gauge field integration $\int \mathcal{D}A$ includes infinities due to the redundancies.

The final example is QCD with a non-Abelian SU(3) gauge group with the fermionic action

$$S_{\text{QCD}}[\psi, \bar{\psi}, A] = \int_x \bar{\psi}(\not{\partial} - ie\not{A} + m)\psi + S_A[A], \quad A_\mu = A_\mu^a t^a, \quad (12.49)$$

where t^a are the generators of the Lie algebra $\mathfrak{su}(3)$.

Perturbation theory works the same way as in the scalar theory, but we have to take care of the Grassmann nature of the field derivatives. It is an expansion about the Gaussian generating functional of the free fermionic generating functional,

$$Z_0[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_0[\psi, \bar{\psi}] + \int_x (\bar{\eta}\psi - \bar{\psi}\eta)} \quad (12.50)$$

with the free Dirac action

$$S_0[\psi, \bar{\psi}] = \int_x \bar{\psi}(\not{\partial} + m)\psi. \quad (12.51)$$

The Gaussian integral in (12.50) is solved by shifting the fermionic fields similarly to the scalar case,

$$\psi = \psi' - \frac{1}{\not{\partial} + m}\eta, \quad \bar{\psi} = \bar{\psi}' + \bar{\eta} \frac{1}{\not{\partial} + m}. \quad (12.52)$$

Inserting the shifts (12.52) in (12.50) leads us to

$$Z_0[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_0[\psi, \bar{\psi}]} e^{-\int_{x,y} \bar{\eta}(x) G_\psi(x,y) \eta(y)}. \quad (12.53)$$

with the free propagator

$$(\not{\partial}^x + m) G_\psi(x, y) = \delta(x - y). \quad (12.54)$$

The remaining Gaussian functional integral yields the fermionic determinant,

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_0[\psi, \bar{\psi}]} \simeq \det(\not{\partial} + m). \quad (12.55)$$

With these steps we are led to the final expression for the generating functional of the free Dirac theory,

Free Fermionic generating functional

$$Z_0[\eta, \bar{\eta}] = \det(\not{\partial} + m) e^{-\int_{x,y} \bar{\eta}(x) G_\psi(x,y) \eta(y)}. \quad (12.56)$$

The generating functional is a look alike of the scalar one and the only non-trivial correlation function is the two point function. This is shown by going to the Schwinger functional

$$W_0[\eta, \bar{\eta}] = Z_0[\eta, \bar{\eta}]. \quad (12.57)$$

We get the two-point function from the $\eta, \bar{\eta}$ -derivative of $Z[\eta, \bar{\eta}]$, which leads us to

$$\langle T\psi(x)\bar{\psi}(y) \rangle = \left[\frac{1}{Z[\eta, \bar{\eta}]} \frac{\delta^2 Z[\eta, \bar{\eta}]}{\delta \bar{\eta}(x) \delta \eta(y)} \right]_{\eta, \bar{\eta}=0} = G_\psi(x, y). \quad (12.58)$$

As an example of the interacting theories we consider the Yukawa theory with the action (12.47), gauge theories will be treated in the next Chapter. We first remark that the generating functional is still Gaussian in the fermions,

$$Z[\eta, \bar{\eta}, J] = \int \mathcal{D}\varphi \left[\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_D[\psi, \bar{\psi}, \varphi] + \int_x (\bar{\eta}\psi - \bar{\psi}\eta)} \right] e^{-S_\phi[\varphi] + \int_x J\varphi}. \quad (12.59)$$

Accordingly, it can be readily performed, leading to

$$Z[\eta, \bar{\eta}, J] = \int \mathcal{D}\varphi^{-S_{\text{eff}}[\varphi] + \int_x J\varphi} e^{-\int_{x,y} \bar{\eta} \frac{1}{\not{\partial} + m + h\varphi} \eta}, \quad (12.60)$$

with the 'effective' (non-local) classical action

$$S_{\text{eff}}[\varphi] = S_\phi[\varphi] - \text{Tr}(\not{\partial} + m + h\varphi) \quad (12.61)$$

$-\text{Tr}(\not{\partial} + m + h\varphi)$ is nothing but the fermionic part of the one-loop effective potential. Moreover, the relative minus sign signals the fermionic loops.

We close with the functional representation of perturbation theory of a Yukawa theory with the action (12.47), where the scalar part is given by

$$S_\phi[\varphi] = \int_x \left(\frac{1}{2} \varphi (-\Delta + m_\phi^2) \varphi + V_{\text{int}}(\varphi) \right). \quad (12.62)$$

The perturbative expansion of the generating functional involves a double expansion in Yukawa interactions and scalar self-interactions, to wit

$$Z[\eta, \bar{\eta}, J] = e^{-\int_x V(\frac{\delta}{\delta J}) + h \int_x \frac{\delta}{\delta \eta} \frac{\delta}{\delta J} \frac{\delta}{\delta \bar{\eta}}} e^{\frac{1}{2} \int_{x,y} J G_\phi J - \int_{x,y} \bar{\eta} G_\psi \eta}, \quad (12.63)$$

with the free fermionic and scalar propagators,

$$(-\Delta + m_\phi^2) G_\phi = \delta(x-y), \quad (\not{\partial} + m_\psi) G_\psi = \delta(x-y). \quad (12.64)$$

We can combine the sum of the interaction terms into

$$e^{-\int_x V(\frac{\delta}{\delta J}) + h \int_x \frac{\delta}{\delta \eta} \frac{\delta}{\delta J} \frac{\delta}{\delta \bar{\eta}}} = e^{-\int_x V(\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}, \frac{\delta}{\delta J})}, \quad \text{with} \quad V(\bar{\psi}, \psi, \varphi) = V(\varphi) + \int_x h \bar{\psi} \varphi \psi. \quad (12.65)$$

for the sake of notational brevity. In any case the above representation leads us to the well-known Feynman rules,

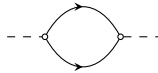
- (a) Write down all diagrams in a given order N of the coupling.
 (b) Combinatorial factors.
 (c) (-1) for closed fermionic loops.

Explanation for (c): Consider Yukawa theory:

$$\varphi\text{-prop: } \begin{array}{c} x \quad y \\ \text{---} \end{array} G_\phi(x, y) \quad (12.66)$$

$$\psi\text{-prop: } \begin{array}{c} x \quad y \\ \overline{\psi} \quad \psi \end{array} G_\psi(x, y) \quad (12.67)$$

$$\text{Vertex: } \begin{array}{c} \overline{\psi} \\ \swarrow \\ \text{---} \quad k \\ \searrow \\ \psi \end{array} \quad (12.68)$$

One-loop 2-point function: 

From QFT I we recall:

$$h^2 \left\langle T \varphi(x) \int_z \overline{\psi}_z \varphi \psi_z \int_{z'} \overline{\psi}_{z'} \varphi_{z'} \psi_{z'} \varphi(y) \right\rangle = \underset{\substack{\uparrow \\ \text{Grassmann prop}}}{-} h^2 \int_{z, z'} \langle T \varphi(x) \varphi(z) [\psi(z) \overline{\psi}(z')] [\psi(z') \overline{\psi}(z)] \varphi(z') \varphi(y) \rangle \quad (12.69)$$

Via functional integral:

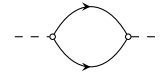
$$\langle T \varphi(x) \varphi(y) \rangle_{\text{connected}} = \frac{\delta^2}{\delta J(x) \delta J(y)} \ln Z[\eta, \bar{\eta}, J] \Big|_{\eta, \bar{\eta}, J=0} = \left[\frac{1}{Z} \frac{\delta^2 Z}{\delta J(x) \delta J(y)} - \left(\frac{1}{Z} \frac{\delta Z}{\delta J(x)} \right) \left(\frac{1}{Z} \frac{\delta Z}{\delta J(y)} \right) \right] \Big|_{\eta, \bar{\eta}, J=0} \quad (12.70)$$

$$\begin{aligned} \frac{\delta^2 Z}{\delta J(x) \delta J(y)} &= \frac{\delta^2}{\delta J(x) \delta J(y)} \\ &\times \left[1 + h \int_{x'} \frac{\delta}{\delta \eta(x')} \frac{\delta}{\delta \bar{\eta}(x')} \frac{\delta}{\delta J(x')} + \frac{h^2}{2} \int_{x', x''} \frac{\delta}{\delta \eta(x')} \frac{\delta}{\delta \bar{\eta}(x')} \frac{\delta}{\delta J(x')} \frac{\delta}{\delta \eta(x'')} \frac{\delta}{\delta \bar{\eta}(x'')} \frac{\delta}{\delta J(x'')} + \mathcal{O}(h^3) \right] \\ &\times e^{\frac{1}{2} \int_{x,y} J \cdot G_\phi \cdot J - \int_{x,y} \bar{\eta} G_\psi \eta} \Big|_{\eta, \bar{\eta}, J=0}. \end{aligned} \quad (12.71)$$

At vanishing fields the linear term (in h) vanishes, and the h^2 -term gives the 1-loop contribution.

$$\frac{\delta^2 Z}{\delta \varphi(x) \delta \varphi(y)} \Big|_{1\text{-loop}} = G_\phi(x, y) + h^2 \int_{x', x''} \left\{ G_\phi(x, x') G_\phi(x'', y) [-G_\psi(x', x'') \cdot G_\phi(x'', x')] \right\}, \quad (12.72)$$

where the minus sign in the square bracket is that for fermionic loops. \hbar^2 -term:



In momentum space:

$$G_\psi(p) = \frac{1}{i\not{p} + m} = \frac{-i\not{p} + m}{p^2 + m^2} \quad (12.73)$$

$$G_\phi(p) = \frac{1}{p^2 + m^2}, \text{ where } + \text{ in both cases comes from Euclidean space}$$

Vacuum polarisation:

$$\begin{aligned}
 \text{Diagram: } \text{---} \xrightarrow{p} \text{---} \text{---} \text{---} \xrightarrow{p} \text{---} &\approx -\hbar^2 \int \frac{d^d q}{(2\pi)^d} \text{Tr}(-i\not{q} + m)(-i\not{(p+q)} + m) \frac{1}{q^2 + m^2} \cdot \frac{1}{(p+q)^2 + m^2} \\
 &= -\hbar^2 \int \frac{d^d q}{(2\pi)^d} \frac{[-dq \cdot (p+q) + dm^2]}{(q^2 + m^2)((p+q)^2 + m^2)} \quad (12.74)
 \end{aligned}$$

13. Non-Abelian Gauge Theories

Abelian and non-Abelian gauge theories are formulated within the functional integral similarly to scalar and fermionic theory. However, one has to deal with the redundancy of the gauge field degrees of freedom that has already complicated the canonical quantisation of gauge theories, see [Chapter 5](#) for the Abelian case. There we have separated the unphysical part of the Hilbert space within Gupta-Bleuler quantisation. Note however, that the Feynman rules and hence the perturbative (and non-perturbative) involved both, the physical and non-physical part of the Hilbert space.

Roughly speaking we will follow a similar approach in the functional integral in [Sections 13.1](#) and [13.2](#) of a gauge theory with gauge group

$$G = U(1), SU(N), \dots, \quad \text{with} \quad \Omega : \text{Lie algebra of } G, \quad (13.1)$$

see also [Section 5.1](#). The gauge field A_μ lives in the Lie algebra Ω of the gauge group and transforms as

$$-ig A_\mu(x) \rightarrow -ig A_\mu^U(x) = U(x)[D_\mu U^\dagger(x)] = -ig U(x)A_\mu(x)U^\dagger(x) + U(x)\partial_\mu U^\dagger(x), \quad \text{with } U \in G, \quad (13.2)$$

with the (minimal) gauge coupling g and the covariant derivative

$$D_\mu = \partial_\mu - ig A_\mu, \quad (13.3)$$

in a given representation of the gauge group. We assume that [\(13.2\)](#) leaves the gauge field action $S_A[A]$ invariant: $S_A[A^U] = S_A[A]$. The gauge field action will be specified later, the current structural analysis applies to generic classical gauge field actions. An infinitesimal transformation $U = \exp\{i\omega\} = \mathbb{1} + i\omega + O(\omega^2)$ follows from [\(13.2\)](#) as

$$A_\mu \rightarrow A_\mu^U = A_\mu + \frac{1}{g} D_\mu \omega + O(\omega^2), \quad (13.4)$$

where D is the covariant derivative in the adjoint representation. Due to the assumed gauge invariance of the gauge field action S_A , we can separate the path integral measure into a physical part and an unphysical one over the Lie algebra with $A_\mu \rightarrow (A_\mu^{\text{phys}}, \omega)$ with $\omega \in \Omega$,

$$\mathcal{D}A_\mu = \mathcal{D}A_\mu^{\text{phys}} \mathcal{D}\omega J(A_\mu), \quad (13.5)$$

where $J(A_\mu)$ is the Jacobian determinant of the coordinate transformation in field space. Evidently, the functional integration of the Lie algebra drops out of the path integral and may be dropped. This also entails that the full functional integral with the measure $\mathcal{D}A_\mu$ carries an infinite redundancy

$$\int \mathcal{D}\omega, \quad (13.6)$$

that has to be regularised. This is done with the Fadeev-Popov trick which implements the split [\(13.5\)](#) in terms of an insertion of unity in the path integral.

Toy example: two-dimensional Faddeev-Popov trick

We close this introduction with a brief discussion of the respective idea within the example of a two-dimension integral with a rotation-invariant integrand,

$$\int_{\mathbb{R}} d^2x e^{-S(r)}, \quad \text{with} \quad r = \sqrt{x_1^2 + x_2^2}, \quad \text{and} \quad \mathbf{x} = (x_1, x_2). \quad (13.7)$$

The integral in (13.7) can be rewritten in terms of a radial integration and an angular part, which is readily performed,

$$\int_{\mathbb{R}} d^2x e^{-S(r)} = \int_{-\pi}^{\pi} d\omega \int_{\mathbb{R}} d^2x e^{-S(r)} = 2\pi \int_{\mathbb{R}} d^2x e^{-S(r)}. \quad (13.8)$$

Instead of performing the active coordinate transformation in (13.8), the Faddeev-Popov trick resorts to a passive one by inserting unity in the path integral in terms of an integration over ω ,

$$1 = \frac{1}{I_0\left(\frac{r^2}{2\xi}\right)} \int_{-\pi}^{\pi} d\bar{\omega} \int_{\mathbb{R}} dc \delta(r \cos \bar{\omega} - c) e^{-\frac{1}{2\xi}c^2} \quad (13.9)$$

with the modified Bessel function $I_0(x)$. In (13.9), the δ -function carries a 'gauge fixing condition': the coordinate x_1 is fixed to a value c , a natural value being 0. This fixes the value of ω . The integration over c is then introduced for the sake of convenience. Inserting (13.9) into the two-dimensional integral (13.7) allows us to keep the two-dimensional integral without redundancy at the expense of an additional r -dependence. To that end we use that we can rotate the coordinate in the x_1 direction,

$$\mathbf{x}' = O(\bar{\omega})\mathbf{x}, \quad \text{with} \quad \mathbf{x}' = (r \cos \bar{\omega}, r \sin \bar{\omega}), \quad (13.10)$$

which absorbs the $\bar{\omega}$ -dependence in the integration over \mathbf{x}' . This leads us to our final expression for (13.7),

$$\begin{aligned} \int_{\mathbb{R}} d^2x' e^{-S(r)} &= \int_{-\pi}^{\pi} d\bar{\omega} \left[\int_{\mathbb{R}} d^2x \frac{1}{I_0\left(\frac{r^2}{2\xi}\right)} e^{-S(r)} \int_{\mathbb{R}} dc \delta(x'_1 - c) e^{-\frac{1}{2\xi}c^2} \right] \\ &\simeq \int_{\mathbb{R}} d^2x \frac{1}{I_0\left(\frac{r^2}{2\xi}\right)} e^{-S(r)} e^{-\frac{1}{2\xi}x_1^2}. \end{aligned} \quad (13.11)$$

We have dropped the redundant integration over $\bar{\omega}$ in the second line and relabelled $\mathbf{x}' \rightarrow \mathbf{x}$. Evidently, in a functional integral over A_μ this procedure regularises the redundant part of the gauge field integration by a Gaußian damping factor. This comes at the expense of the non-trivial Jacobian, in the present case this is the inverse of the modified Bessel function, $1/I_0(r^2/(2\xi))$. In the present example the Faddeev-Popov trick does not come with any advantage, as it is in general far simpler to only perform the radial integral. In gauge theories the latter integral entails that one has to deal with projections on gauge-fixed configurations. These projections complicate practical computations tremendously, hence the Faddeev-Popov trick is commonly used.

In Section 13.2 we perform the above procedure step by step in a non-Abelian gauge theory, and the reader can compare these steps one by one to the present trivial example.

13.1. Action & Gauge Invariance

In this Section we briefly recapitulate for the sake of convenience the set-up and derivation of Abelian and non-Abelian gauge theories done in Chapter 5. To that end we consider fermions ψ that carry some representation

of a non-Abelian group G , see (13.1). In most cases we will restrict ourselves to $G=\text{SU}(N)$. Then, gauge transformations of a Dirac fermion in the fundamental representation of $\text{SU}(N)$ reads

$$\psi(x) \rightarrow \psi^U(x) = U\psi(x), \quad \text{with} \quad \psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \quad \text{and} \quad U \in \text{SU}(N) \quad (13.12)$$

The Dirac action including the gauge field, (5.23), is invariant under the local gauge transformations with $U = U(x)$. We find

$$S_D[\psi, \bar{\psi}, A] = - \int d^d x \bar{\psi}(x) (\not{D} + m) \psi(x), \quad (13.13)$$

with the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu, \quad \text{with} \quad D_\mu(A^U) = U D_\mu U^\dagger, \quad (13.14)$$

where the transformation law of the covariant derivative follows from that of the gauge field in (13.2). We emphasise that (13.14) is an operator identity and the covariant derivative acts on everything to the right. The gauge field lives in the Lie algebra of $\text{SU}(N)$ with

$$A_\mu = A_\mu^a t^a, \quad \text{with} \quad a = 1, \dots, N^2 - 1, \quad (13.15)$$

The self-adjoint generators t^a introduced in (13.15) satisfy the Lie-algebra of $\text{SU}(N)$,

$$[t^a, t^b] = i f^{abc} t^c, \quad \text{tr}_f t^a t^b = \frac{1}{2}, \quad (13.16)$$

with the structure constants f^{abc} .

Note also that in Chapter 5 we have deduced the transformation law (13.14) for the covariant derivative from the requirement of gauge invariance of the action. This leads us to (13.2). The invariance of S_D in (13.13) follows readily as

$$S_D[\psi^U, \bar{\psi}^U, A_\mu^U] = - \int d^d x \bar{\psi}^U(x) U(x)^\dagger U(x) (\not{D} + m) U^\dagger(x) U(x) \psi = S_D[\psi, \bar{\psi}, A_\mu], \quad (13.17)$$

where we have used that $U^\dagger U = \mathbb{1}$. The action for the gauge field has also been derived in Chapter 5, and follows from the requirement of gauge invariance and a standard kinetic term with a quadratic dispersion. The latter requirement is in one-to-one correspondence to a $1/r$ -potential of the gauge theory such as observed in QED. We are led to the Yang-Mills action

$$S_A[A] = \frac{1}{2} \int d^d x \text{tr}_f F_{\mu\nu} F_{\mu\nu}, \quad (13.18)$$

where the trace is in the fundamental representation of the gauge group, and the field strength tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \quad (13.19)$$

with

$$F_{\mu\nu} = F_{\mu\nu}^a t^a, \quad \text{with} \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \quad (13.20)$$

The field strength $F_{\mu\nu}$ transforms as a tensor under gauge transformations: it inherits this property from that of the covariant derivative, see (13.14). Hence we have

$$F_{\mu\nu} \rightarrow F_{\mu\nu}^U = U F_{\mu\nu} U^\dagger, \quad (13.21)$$

With (13.21) the gauge invariance of the Yang-Mills action (13.18) follows straightforwardly from the cyclicity of the trace,

$$\text{tr}_f F_{\mu\nu} F_{\mu\nu} \rightarrow \text{tr}_f U F_{\mu\nu} U^\dagger U F_{\mu\nu} U^\dagger = \text{tr}_f U^\dagger U F_{\mu\nu} F_{\mu\nu} = \text{tr}_f F_{\mu\nu} F_{\mu\nu}. \quad (13.22)$$

In summary we have introduced the full classical action of a non-Abelian gauge theory (QCD, electro-weak theory, GUT, ...) coupled to matter (leptons, quarks),

$$S[\psi, \bar{\psi}, A] = S_A[A] + S_D[\psi, \bar{\psi}, A]. \quad (13.23)$$

In the next Section, Section 13.2, we will derive the generating functional of a pure non-Abelian gauge theory with the classical action (13.18). Then, that of the full theory with the classical action (13.23) follows readily by augmenting the generating functional of the pure gauge theory with a functional integral over the fermions as introduced in Chapter 12.

We close this Section with a brief discussion of observables. They are provided by expectation values of (local) gauge invariant operators,

$$O = \langle \hat{O}[\psi, \bar{\psi}, A] \rangle, \quad (13.24)$$

with

$$\hat{O}[\psi^U, \bar{\psi}^U, A^U] = \hat{O}[\psi, \bar{\psi}, A]. \quad (13.25)$$

Evidently, the expectation values of the actions and the Lagrangians (13.13) and (13.18) are observables. However, the fields are not. The latter already holds true for QED, where the gauge field expectation value $\langle \hat{A}_\mu \rangle$ is not an observable, and the same holds true for the electron or the scalar field in scalar QED. However, due to the linear nature of the U(1) gauge symmetry the physical interpretation of the U(1) gauge field as a photon is commonly used: while the gauge field is not a physical field, the expectation values of the field strength components, the magnetic and electric fields, are observables,

$$F_{\mu\nu}^U = F_{\mu\nu}, \quad \text{E-field : } E^i = -F^{0i} = -(\partial^0 A^i - \partial^i A^0), \quad \text{B-field : } B^i = \varepsilon^{ijk} F^{jk}. \quad (13.26)$$

In a non-Abelian gauge theory (like QCD) chromo-magnetic and chromo-electric field strength components are not observables, as the field-strength is not gauge invariant, $F_{\mu\nu}^U = U F_{\mu\nu} U^\dagger \neq F_{\mu\nu}$. Moreover, in QCD the only physical asymptotic states are colorless due to the phenomenon of confinement. In turn, in QED we have charge superselection sectors. However, for high energies, QCD and more generally non-Abelian gauge theories show the phenomenon of asymptotic freedom,

$$\alpha_s(p^2 \rightarrow \infty) = \frac{g^2(p^2 \rightarrow \infty)}{4\pi} \rightarrow 0, \quad (13.27)$$

and we may consider "asymptotic" gluons due to

$$gA_\mu^U = gA_\mu + \partial_\mu \omega + \mathcal{O}(g) \xrightarrow{g \rightarrow 0} gA_\mu + \partial_\mu \omega, \quad (13.28)$$

which renders them Abelian in a crude approximation. Indeed, one can identify gluon jets or rather their decay products in the detectors of high energy colliders such as the LHC. For the computation of the respective cross sections we have to at least set-up a framework for perturbative computations in non-Abelian gauge theories. In turn, the infrared physics of QCD with confinement and dynamical spontaneous chiral symmetry breaking requires non-perturbative methods such as lattice field theory, see Chapter 15, or functional methods, see Chapters 11 and 16.

13.2. Generating Functional for gauge theories

We proceed with the formulation of the functional integral or rather the generating functional of a Non-Abelian gauge theory. As already mentioned, the functional integral is defined analogously to that of scalars and fermions, but we have to take care of the gauge redundancy. For this purpose we first consider the expectation value of a general gauge invariant operator $\hat{\mathcal{O}}$ and perform a (passive) coordinate transformation as outlined at the example of a two-dimensional radially symmetric integral at the end of the introduction in [Chapter 13](#). Formally, the expectation value is given in direct analogy to scalars and fermions,

$$\langle \hat{\mathcal{O}}[A] \rangle = \frac{\int \mathcal{D}A \hat{\mathcal{O}}[A] e^{-S_A[A]}}{\int \mathcal{D}A e^{-S_A[A]}}. \quad (13.29)$$

However, due to the gauge invariance of the operator, $\hat{\mathcal{O}}[A^U] = \hat{\mathcal{O}}[A]$, and that of the Yang-Mills action (13.18) with $S_A[A^U] = S_A[A]$, the functional integrals in (13.29) carry an infinite-dimensional redundancy, we also have $\mathcal{D}[A^U] = \mathcal{D}A$. While this redundancy occurs both in the numerator and the denominator of (13.29), it formally cancels out. However, in practical diagrammatic computations this cancellation is present at the technical level and hence has to be accommodated explicitly.

13.2.1. Generating Functional for Abelian gauge theories

A trivial example for the redundancy is provided by the Abelian U(1)-gauge theory, where we already can perform all the steps already outlined in the introduction. The gauge field action is a Gaußian free action,

$$S[A] = \frac{1}{4} \int d^d x F_{\mu\nu} F_{\mu\nu} = \frac{1}{2} \int d^d x (\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu) = -\frac{1}{2} \int d^d x A_\mu (\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu. \quad (13.30)$$

The right hand side of (13.30) makes explicit, that the longitudinal components of A_μ drop out. This suggests to split the gauge field in its transverse part with $\partial_\mu A_\mu = 0$, and its longitudinal complement, which does not occur in the action. More generally we write $A_\mu = (A_\mu^{\text{gf}})^U$, where A_μ^{gf} satisfies some gauge. In view of the discussion above we use the covariant or Landau gauge with

$$\partial_\mu A_\mu^{\text{gf}} = 0. \quad (13.31)$$

In the gauge (13.31) the longitudinal component is gauged to zero. The covariance of this gauge entails that manifest Lorentz symmetry of the correlation functions is maintained within this gauge, which is different for e.g. for the Coulomb gauge $\partial_i A_i = 0$ or axial gauges $n_\mu A_\mu = 0$. All these gauges (and more) are used in computations.

A gauge should be understood as a reparametrisation of the theory. For a given application the one or the other gauge is more convenient for the computation. A general gauge field can be obtained from the Landau gauge one with a gauge transformation,

$$A_\mu^{\text{gf}U} = A_\mu^{\text{gf}} + \frac{1}{g} \partial_\mu \omega. \quad (13.32)$$

Equation (13.32) makes the Abelian (linear) nature of the gauge field explicit. It entails that the Jacobian of the transformation $A_\mu \rightarrow (A_\mu^{\text{gf}}, \omega)$ is gauge-field independent. In turn, in non-Abelian gauge theories the shift is gauge-field dependent,

$$A_\mu^{\text{gf}U} = A_\mu^{\text{gf}} + \frac{1}{g} D_\mu \omega. \quad (13.33)$$

Equation (13.33) leads to a gauge-field dependent Jacobian in contradistinction to (13.32). This causes qualitative differences (and complications) in the quantisation of non-Abelian theories in comparison to Abelian ones.

We proceed with performing the change of variables in the Abelian case,

$$A_\mu \rightarrow (A_\mu^{\text{gf}}, \omega), \quad (13.34)$$

leading to a reparameterisation of the integration measure of the path integral,

$$\mathcal{D}A = \mathcal{D}A^{\text{gf}} \mathcal{D}\omega J, \quad (13.35)$$

with a A_μ -independent Jacobi-determinant J . Now we implement (13.35) in the functional integral. This is done by inserting unity in the path integral as done in the two-dimensional toy example, see (13.9). For a general gauge fixing condition \mathcal{F} and $U = e^{i\omega}$ we get,

$$1 = \Delta_{\mathcal{F}}[A] \int \mathcal{D}\omega \delta[\mathcal{F}(A^{U(\omega)})], \quad \text{with} \quad \Delta_{\mathcal{F}}[A] = \frac{1}{\int \mathcal{D}\omega \delta[\mathcal{F}(A^U)]}. \quad (13.36)$$

The insertion of (13.36) in the functional integral (13.29) is called the *Faddeev-Popov trick*, and $\Delta_{\mathcal{F}}[A]$ is the functional analogue of the Jacobian determinant in the two-dimensional toy example. It is called the *Faddeev-Popov determinant*. It is easily seen that it is gauge invariant,

$$\Delta_{\mathcal{F}}^{-1}[A^V] = \int \mathcal{D}\omega \delta[\mathcal{F}(A^{VU(\omega)})] = \int \mathcal{D}\omega' \delta[\mathcal{F}(A^{U(\omega')})] = \int \mathcal{D}\omega \delta[\mathcal{F}(A^{U(\omega)})] = \Delta_{\mathcal{F}}^{-1}[A], \quad (13.37)$$

where ω' is defined via

$$VU(\omega) = U(\omega'), \quad \text{with} \quad \mathcal{D}\omega = \mathcal{D}\omega'. \quad (13.38)$$

The latter property in (13.38) is that of the Haar measure $\mathcal{D}\omega$ of the group or rather the algebra, and originates in the fact that the multiplication of U with V is a unitarity rotation in the group, which is an invariance of the Haar measure.

It is left to compute $\Delta_{\mathcal{F}}$. To that end we use the property of the δ -function that $\delta(f(x)) = |1/f'(x_0)|\delta(x - x_0)$ for a function $f(x)$ with $f(x_0) = 0$ and no other zero. For the functional δ -function in (13.36) we get

$$\delta[\mathcal{F}(A^{U(\omega)})] = \frac{1}{\left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A^{U(\omega_0)})} \delta[\omega - \omega_0], \quad \text{where} \quad \mathcal{F}[A^{U(\omega_0)}] = 0. \quad (13.39)$$

As for the one-dimensional example we have assumed a unique solution ω_0 . In summary this leads us to the Faddeev-Popov determinant

$$\Delta_{\mathcal{F}}[A] = \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A^{U(\omega_0)}), \quad (13.40)$$

which can be computed for the given gauge condition. In the following we consider the covariant gauge as our example case,

$$\mathcal{F} = \partial_\mu A_\mu. \quad (13.41)$$

In the present Abelian theory this leads us to

$$\mathcal{F}[A^{U(\omega)}] = \partial_\mu A_\mu + \frac{1}{g} \partial_\mu \partial_\mu \omega, \quad (13.42)$$

whose variation with respect to the algebra element ω occurs in the Faddeev-Popov determinant (13.40),

$$g \frac{\delta \mathcal{F}}{\delta \omega}(x, y) = \partial_\mu \partial_\mu \delta(x - y). \quad (13.43)$$

Inserting (13.43) in (13.40) leads us to

$$\Delta_{\partial_\mu A_\mu} [A] \simeq \left| \det(-\partial_\mu \partial_\mu) \right| = \det(-\partial_\mu \partial_\mu), \quad (13.44)$$

where we have used $-\partial_\mu \partial_\mu \leq 0$ in the last step. we emphasise that apart from the explicit expression (13.44) every expression holds true for general Abelian and non-Abelian gauge theories.

With the above preparations we are now in the position to derive the functional integral analogue of (13.11) for expectation values of gauge invariant operators (13.29). We find

$$\langle \hat{\mathcal{O}} \rangle = \frac{\int \mathcal{D}A \mathcal{D}\omega \Delta_{\mathcal{F}}[A] \delta[\mathcal{F}(A^U)] e^{-S_A} \mathcal{O}[A]}{\int \mathcal{D}A \mathcal{D}\omega \Delta_{\mathcal{F}}[A] \delta[\mathcal{F}(A^U)] e^{-S_A}} = \frac{\int \mathcal{D}A \Delta_{\mathcal{F}}[A] \delta[\mathcal{F}(A)] e^{-S_A} \mathcal{O}[A]}{\int \mathcal{D}A \Delta_{\mathcal{F}}[A] \delta[\mathcal{F}(A)] e^{-S_A}}, \quad (13.45)$$

where we have used the gauge invariance of the Faddeev-Popov determinant $\Delta_{\mathcal{F}}$, the classical action S_A and the operator $\hat{\mathcal{O}}$ in the last step in order to absorb the gauge transformation U in the δ -function $\delta[F(A^U)] \rightarrow \delta[F(A)]$. In a final step we introduce the Gaußian averaging we have also introduced in (13.9) in the two-dimensional toy example. This amounts to using

$$\text{const} = \int \mathcal{D}\omega \mathcal{D}C \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A^U) \delta[\mathcal{F}(A^U) - C] e^{-\frac{1}{2\xi} \int d^d x C(x)^2} \quad (13.46)$$

instead of (13.36). The derivation leading to (13.45) is unchanged except of a C -dependent ω_0 with $\omega_0 = \omega_0(C)$. However, ω_0 drops out anyway, and we arrive at at

$$\begin{aligned} \langle \hat{\mathcal{O}} \rangle &= \frac{1}{\mathcal{N}} \int \mathcal{D}A \mathcal{D}C \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A) \delta[\mathcal{F}(A) - C] e^{-\frac{1}{2\xi} \int C^2} e^{-S_A[A]} \hat{\mathcal{O}}[A] \\ &= \frac{1}{\mathcal{N}} \int \mathcal{D}A \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A) e^{-[S_A[A] + S_{\text{gf}}[A]]} \hat{\mathcal{O}}[A], \end{aligned} \quad (13.47)$$

with the gauge fixing action S_{gf} and the normalisation \mathcal{N} ,

$$S_{\text{gf}}[A] = \frac{1}{2\xi} \mathcal{F}^2(A), \quad \text{and} \quad \mathcal{N} = \frac{1}{\mathcal{N}} \int \mathcal{D}A \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right| (A) e^{-[S_A[A] + S_{\text{gf}}[A]]}. \quad (13.48)$$

Equation (13.45) is the final expression for expectation values of gauge invariant operators in an Abelian gauge theory.

We close the discussion of the Abelian example with a few remarks: to begin with, the linear nature of the Abelian gauge transformation entails, that the Faddeev-Popov determinant does not depend on the Abelian gauge field for linear gauges,

$$\mathcal{F} = \ell_\mu A_\mu. \quad (13.49)$$

Prominent Examples for (13.49) are the covariant gauge with $\ell_\mu = \partial_\mu$, the Coulomb gauge with $\ell_i = \partial_i$, $\ell_0 = 0$, and general axial gauges with $\ell_\mu = n_\mu$ with a fixed vector (n_μ). Then, the Faddeev-Popov determinant can be dropped as it occurs as a constant factor in both the numerator and the denominator in (13.47).

Moreover, the gauge-fixed action with the covariant gauge (13.41) with $\ell_\mu = \partial_\mu$,

$$S[A] = S_A[A] + \frac{1}{2\xi} \int d^d x (\partial_\mu A_\mu)^2, \quad (13.50)$$

with S_A in (13.30) has been used within the Gupta-Bleuler quantisation, see in particular the discussion around (5.39) in Section 5.2. The present functional integral approach with Faddeev-Popov quantisation is identical to Gupta-Bleuler quantisation.

Finally, the generating functional of gauge-fixed correlation functions and gauge-invariant observables follows straightforwardly from (13.47) as

$$Z[J] = \int \mathcal{D}A \left| \frac{\delta \mathcal{F}}{\delta \omega} \right| e^{-S_A[A] - S_{\text{gf}}[A]} e^{\int d^d x J_\mu A_\mu}, \quad (13.51)$$

with the Abelian gauge field action S_A in (13.30) and gauge-fixing action S_{gf} in (13.48). Gauge-fixed correlation functions of the gauge field follow from (13.51) as

$$\langle A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \rangle = \frac{1}{Z[0]} \frac{\delta}{\delta J_{\mu_1}(x_1)} \cdots \frac{\delta}{\delta J_{\mu_n}(x_n)} Z[J] \Big|_{J=0}. \quad (13.52)$$

Equation (13.52) is readily extended to the expectation values of functions and functionals of the gauge field, and in particular to that of gauge-invariant operators $\hat{\mathcal{O}}$, to wit

$$\langle \hat{\mathcal{O}}[A] \rangle = \frac{1}{Z[0]} \left[\hat{\mathcal{O}} \left(\frac{\delta}{\delta J} \right) Z[J] \right]_{J=0}, \quad \text{with} \quad A_{\mu_i}(x_i) e^{\int d^d x J_\mu A_\mu} = \frac{\delta}{\delta J_{\mu_i}(x_i)} e^{\int d^d x J_\mu A_\mu}. \quad (13.53)$$

Equation (13.53) is readily extended to interacting Abelian gauge theories, for example QED. Then the classical action also includes the Dirac part (13.13), respective current terms and an integration over the fermions, see (12.45). This remark concludes our discussion of the functional integral approach of Abelian gauge field theories.

13.2.2. Generating Functional for non-Abelian gauge theories

With the detailed discussion of the Abelian case we are almost done with the non-Abelian case either: none of the steps in the derivation of the generating functional (13.51) are specific to the $U(1)$ gauge group. The only difference is the gauge-field dependence of the Faddeev-Popov determinant. We find

$$\Delta_{\mathcal{F}}[A] = \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right|_{\omega=\omega_0} \simeq \left| \det M(\bar{A}) \right|, \quad \text{with} \quad \mathcal{F}[\bar{A}] = C, \quad (13.54)$$

where we have already included the Gaussian integration over C in our analysis. The Faddeev-Popov operator $M = -\delta \mathcal{F} / \delta \omega$ is derived in an expansion about the solution \bar{A}_μ of the gauge fixing condition. This leads us to

$$M^{ab}(x, y) \simeq -g \frac{\delta \mathcal{F}^a \left(\bar{A}(x) + \frac{1}{g} \bar{\mathcal{D}}\omega(x) \right)}{\delta \omega^b(y)} \Big|_{\omega=0} = - \int_z \frac{\delta \mathcal{F}^a(\bar{A}(x))}{\delta \bar{A}_\mu^c(z)} \frac{\partial \bar{D}_\mu^{cd} \omega^d(z)}{\partial \omega^b(y)} = - \int_z \frac{\delta \mathcal{F}^a(\bar{A}(x))}{\delta \bar{A}_\mu^c(z)} \bar{D}_{\mu,z}^{cb} \delta(z-y). \quad (13.55)$$

In the covariant gauge (13.41), this reduces to

$$M^{ab}(A)(x, y) = -\partial_\mu \mathcal{D}_\mu^{ab} \delta^b(x-y). \quad (13.56)$$

The analysis so far used an infinitesimal expansion about the solution \bar{A}_μ of the gauge fixing condition, see (13.54). Now we utilise the gauge fixing δ -function to substitute this solution by the general gauge field, in order to remove any explicit reference to the gauge fixing,

$$\Delta_{\mathcal{F}}[A] \delta[\mathcal{F}(A) - C] = \left| \det M[\bar{A}] \right| \delta[\mathcal{F}(A) - C] = \left| \det M[A] \right| \delta[\mathcal{F}(A) - C], \quad (13.57)$$

for a general gauge fixing condition $\mathcal{F}(A) - C = 0$. Collecting all the different results, we are led to the generating functional of a non-Abelian gauge theory,

Generating functional of a non-Abelian gauge theory

$$Z[J] \simeq \int \mathcal{D}A |\det M(A)| e^{-S[A] + \int d^d x J_\mu^a A_\mu^a}, \quad (13.58)$$

with the gauge-fixed action

$$S[A] = S_A[A] + S_{\text{gf}}[A], \quad S_A[A] = \frac{1}{4} \int_x F_{\mu\nu}^a(A) F_{\mu\nu}^a(A), \quad S_{\text{gf}}[A] = \frac{1}{2\xi} \int_x \mathcal{F}^a(A) \mathcal{F}^a(A), \quad (13.59)$$

see also (13.18) and (13.48). The field strength tensor $F_{\mu\nu}^a$ is defined in (13.20). Finally, the Faddeev-Popov operator M has been defined in (13.55) and has the short hand notation form

$$M = -\frac{\delta \mathcal{F}}{\delta A_\mu} \mathcal{D}_\mu. \quad (13.60)$$

Equation (13.58) is the generating functional of an interacting theory in contradistinction to the Abelian case. It is precisely this difference that is responsible for asymptotic freedom in a non-Abelian gauge theory. Correlation functions are obtained similarly to (13.52) from the generating functional as

$$\langle A_{\mu_1}^{a_1}(x_1) \cdots A_{\mu_n}^{a_n}(x_n) \rangle = \frac{1}{Z[0]} \frac{\delta}{\delta J_{\mu_1}^{a_1}(x_1)} \cdots \frac{\delta}{\delta J_{\mu_n}^{a_n}(x_n)} Z[J] \Big|_{J=0}, \quad (13.61)$$

and the relation for the observables (13.53) carries over identically.

The final result (13.58) leaves us with an expression for the functional integral, that involves a functional determinant. Evidently, this is not directly useful in perturbation theory: while the Faddeev-Popov determinant allows for a perturbative expansion in powers of the coupling g , this form cannot be cast directly into Feynman rules. Moreover, these intricacies carry over to non-perturbative diagrammatic approach such as the DSE as discussed in Chapter 11. Hence, in a final step we convert the Faddeev-Popov determinant into a functional integral over auxiliary fields with a 'classical' action S_{gh} , where the subscript gh stands for *ghost*, as the auxiliary field, the *ghost field* cannot be measured. While this final step can be done for a general gauge fixing condition, we restrict ourselves to the explicit choice of the covariant gauge (13.41): it is the most versatile and used gauge condition and the following derivation can be followed better within an explicit choice. Then we have

$$\mathcal{F}^a(A) = \partial_\mu A_\mu^a, \quad \text{and} \quad S_{\text{gf}}[A] = \frac{1}{2\xi} \int_x (\partial_\mu A_\mu^a)^2, \quad (13.62)$$

with the Faddeev-Popov operator

$$M^{ab}(A) = -\partial_\mu \mathcal{D}_\mu. \quad (13.63)$$

We now assume that the gauge fixing condition $\mathcal{F}(A) = C$ has only one solution, and the Faddeev-Popov operator M in (13.63) is positive semi-definite, $M(A) \geq 0$ is positive. Note that this indeed does not hold for sufficiently smooth gauges including the covariant gauge. It can be shown that the gauge fixing solution has infinitely many solutions, the *Gribov copies*, and while the Faddeev-Popov determinant is positive in a neighbourhood of $A_\mu \equiv 0$ it turns negative for larger amplitudes. This is a highly intricate and challenging problem and is not fully resolved yet. However, as it goes far beyond the scope of the present lecture course and is not relevant for the applications discussed here, we will work with the above assumption. The existence and consequences of Gribov copies will be mentioned here if this generalisation is relevant for direct extensions of the computations done here.

With the assumption $M(A) \geq 0$ we can drop the absolute value in (13.58) and only have the determinant. Now we recall that the fermionic path integral with a bilinear action of ψ and $\bar{\psi}$ such as the Dirac action simply yields the determinant of the respective operator, see (12.21) in Section 12.1.1. In the case of the Dirac action we arrived at the determinant of the Dirac operator, see (12.55). These considerations lead us to

$$|\det M(A)| = \det M(A) \simeq \int \mathcal{D}c \mathcal{D}\bar{c} e^{-S_{\text{gh}}[c, \bar{c}, A]} \quad \text{with} \quad S_{\text{gh}}[c, \bar{c}, A] = \int d^d x \bar{c}^a M^{ab}(A) c^b, \quad (13.64)$$

and in the covariant gauge we have

$$S_{\text{gh}}[c, \bar{c}, A] = - \int d^d x \bar{c}^a(x) [\partial_\mu \mathcal{D}_\mu^{ab} c^b](x). \quad (13.65)$$

The auxiliary fermionic field c, \bar{c} are called the *ghost* and *anti-ghost* respectively. These names reflect the fact that they cannot be measured and are merely a technical tool to rewrite the Faddeev-Popov determinant as a functional integral with a local action.

We remark that the overall sign of the ghost action (13.65) is pure convention, and the most common convention is to define (13.65) with a global plus instead of a global minus. Then the classical ghost propagator G_c^{cl} has a negative dispersion,

$$S_{\text{gh}}[c, \bar{c}, A] = + \int \bar{c} \partial_\mu \mathcal{D}_\mu c \quad \rightarrow \quad G_c^{\text{cl}} = -\frac{1}{p^2} \delta^{ab} \quad (13.66)$$

While the negative or positive dispersion is simply convention, the ghost does not obey the spin-statistics theorem, which again reflects its status as an auxiliary unphysical field.

In summary we are led to the final expression for the generating functional of non-Abelian gauge theories in terms of local gauge-fixed action (including the ghost action),

Generating functional of a non-Abelian gauge theory with ghosts

$$Z[J, \eta, \bar{\eta}] = \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} e^{-S[A, c, \bar{c}] + \int_x (J_\mu^a A_\mu^a + \bar{\eta} c - \bar{c} \eta)}. \quad (13.67)$$

with

$$S[A, c, \bar{c}] = S_A[A] + S_{\text{gf}}[A] + S_{\text{gh}}[A, c, \bar{c}], \quad (13.68)$$

with S_A and S_{gf} provided in (13.59) and the short hand notation form

$$S_{\text{gh}}[c, \bar{c}, A] = - \int_x \bar{c}^a \partial_\mu \mathcal{D}_\mu^{ab} c^b \quad (13.69)$$

of the ghost action (13.65). The Feynman rules in the covariant gauge follow straightforwardly from the action (13.68) and are summarised in Appendix G in Figure G.1.

We close this Section with the discussion of the generating functionals of connected correlation functions, the Schwinger functional $W[J, \eta, \bar{\eta}] \simeq \ln Z[J, \eta, \bar{\eta}]$, and the generating functional of 1PI correlation functions, the effective action $\Gamma[A, c, \bar{c}]$, the quantum analogue of the classical action. It is defined as the Legendre transformation of $W[J, \eta, \bar{\eta}]$, see (11.1) in Section 11.1. In particular the latter is used in most applications of functional approaches. For example it was key to the computation of the one-loop effective potential of a scalar theory, the Coleman-Weinberg potential, and the description of spontaneous symmetry breaking in Section 11.2 in Chapter 11. Moreover, the DSE is almost always used in its form for the effective action, see (11.72) in Section 13.3.1. Moreover, it is also used for the *Quantum Action Principle*, the construction of the effective action in orders of perturbation theory from the gauge symmetry constraints, see Section 13.4.2.

The effective action is constructed in complete analogy to the effective action of the scalar field theory defined in (11.1) in Section 11.1,

Effective action of a non-Abelian gauge theory

$$\Gamma[A, c, \bar{c}] = \int d^d x (J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta) - W[J, \eta, \bar{\eta}], \quad \text{with} \quad W[J, \eta, \bar{\eta}] = \ln Z[J, \eta, \bar{\eta}], \quad (13.70)$$

where we dropped the supremum condition for the currents $J, \eta, \bar{\eta}$ and we have used the short hand notation

$$J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta = J_\mu^a(x) A_\mu^a(x) + \bar{\eta}^a(x) c^a(x) - \bar{c}^a(x) \eta^a(x). \quad (13.71)$$

The Legendre transformation in (13.70) relates the currents with field derivatives of the effective action. In turn, the fields are given by the derivative of the Schwinger functional with respect to the fields. In summary we have similarly to the scalar case described in (11.4) in Section 11.1,

$$J = \frac{\delta \Gamma}{\delta A}, \quad \bar{\eta} = -\frac{\delta \Gamma}{\delta c}, \quad \eta = -\frac{\delta \Gamma}{\delta \bar{c}}, \quad \text{and} \quad A = \frac{\delta W}{\delta J}, \quad c = \frac{\delta W}{\delta \bar{\eta}}, \quad \bar{c} = \frac{\delta W}{\delta \eta}. \quad (13.72)$$

Note that while minus signs cannot be avoided in the relations in (13.72) due to the Grassmann nature of the ghost, anti-ghost and their currents, their location is subject to the definition of the current terms. For example we could have used $\bar{\eta} \cdot c + \bar{c} \cdot \eta$ instead of $\bar{\eta} \cdot c - \bar{c} \cdot \eta$. Then, the last relation in (13.72) for \bar{c} would inherit a minus sign, while that for η would loose its minus sign. While this choice reflects the symplectic nature of the fermionic field and current space, it is simply that, a choice. The invariant feature these minus signs are related to, is the minus sign for fermionic loops in perturbation theory and beyond, which is also present for ghost loops.

13.3. BRST Symmetry & Unitarity

In the previous section we have introduced a functional integral approach for non-Abelian gauge theories that reduces to Gupta-Bleuler in the case of U(1). In the latter case we were able to define a positive-definite Hilbert space via projection onto the positive norm states, Section 5.2. In the present Section we discuss its extension to non-Abelian theories. Put differently, this amounts to defining transverse gluons or rather the physical Hilbert space. In Abelian theories this was helped by the linearity of the gauge transformation. This entails that the orthogonal split into transversal and longitudinal subspaces and the respective projection operators are field independent: an infinitesimal Abelian gauge transformation $A_\mu \rightarrow A_\mu + (1/g)\partial_\mu \omega$ with the infinitesimal parameter ω changes the gauge field by $\partial_\mu \omega$. This shift is orthogonal to the transversal subspace defined by the transversal projection operator

$$\Pi_{\perp \mu\nu} \partial_\nu \omega = 0, \quad \text{with} \quad \Pi_{\perp \mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}, \quad (13.73)$$

see (7.103) in Section 7.3.1 for the momentum space representation. In a non-Abelian gauge theory the infinitesimal shift is gauge-field dependent, see (13.4). The respective projection operator is necessarily gauge-field dependent and is formally given by

$$\Pi_{\perp \mu\nu}(A) D_\nu \omega = 0, \quad \text{with} \quad \Pi_{\perp \mu\nu}(A) = \delta_{\mu\nu} - D_\mu \frac{1}{D^2} D_\nu, \quad (13.74)$$

where the order of operators D_μ and $D^2 = D_\rho^2$ in the projection operator in (13.74) is relevant as $[D_\mu, D^2] = -2igF_{\mu\rho}D_\rho - 2i(D_\rho F_{\mu\rho}) \neq 0$. This simple consideration emphasises the intricacy of the definition of orthogonal projections as required for the definition of the physical Hilbert space. Even more importantly it raises the question of how classical gauge invariance manifest on the quantum level in the present functional integral approach.

13.3.1. Slavnov-Taylor identity

We start the respective considerations with casting classical gauge invariance in its functional form. The gauge invariance of the classical action, $S_A[A^U] = S_A[A]$ reads infinitesimally

$$S_A[A + \frac{1}{g}D\omega] + O(\omega^2) = S_A[A] - \int_y \frac{1}{g}\omega^a(y)D_\mu^{ab} \frac{\delta}{\delta A_\mu^b(y)} S_A[A] = S_A[A], \quad (13.75)$$

for general $\omega(y)$. This leads us to the local relation

$$D_\mu^{ab} \frac{\delta}{\delta A_\mu^b} S_A[A] = 0, \quad (13.76)$$

where we dropped the space-time argument. Hence,

$$\mathcal{G}_A = D \cdot \frac{\delta}{\delta A}, \quad (13.77)$$

is the generator of gauge transformations acting on functionals of the gauge field. In (13.77) we used a condensed notation for emphasising the structure. The dot stands for the contraction of Lorentz and color indices as written out explicitly in (13.76). In the $U(1)$ case, (13.76) reduces to

$$\partial_\mu \frac{\delta}{\delta A_\mu} S_A[A] = 0. \quad (13.78)$$

Performing the derivatives we arrive at

$$\{D_\mu, D_\nu\} F_{\mu\nu} = 0. \quad (13.79)$$

The field strength is the commutator of the covariant derivatives, see (13.19). Hence, (13.79) holds true as the left hand side is the contraction of a symmetric with an antisymmetric tensor. For the Abelian $U(1)$ case, (13.79) reduces to

$$\{\partial_\mu, \partial_\nu\} F_{\mu\nu} = 0, \quad (13.80)$$

which vanishes for the same reason.

Now we consider the generating functional (13.67), aiming for the quantum analogue of (13.79). Typically, symmetry identities are derived from the functional integral by considering symmetry transformations that leave the classical action invariant. If the invariance also holds for the path integral measure, the theory is invariant. In the present case the infinitesimal gauge transformation via (13.77) is an invariance of the path integral measure. Indeed, we could implement it similarly to the derivation of the Dyson-Schwinger equation, see in Equation (11.66). We find that

$$\int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} D \cdot \frac{\delta}{\delta A} \left[e^{-S[A,c,\bar{c}] + \int_x (J_\mu^a A_\mu^a + \bar{\eta}c - \bar{c}\eta)} \right] = 0, \quad (13.81)$$

where we have used the operator identity

$$\frac{\delta}{\delta A_\mu^b} D_\mu^{ab} = D_\mu^{ab} \frac{\delta}{\delta A_\mu^b}, \quad (13.82)$$

that follows with $f^{abb} = 0$. Hence, the integrand in (13.81) constitutes a total derivative with respect to the gauge field and the integral vanishes leading to the identity. For a gauge invariant classical action this would lead to the equivalent of (13.79) for the effective action after a Legendre transform: Assuming a gauge invariant action $S[A, c, \bar{c}]$, (13.81) leads to $\langle D_\mu^{ab} J^b \rangle = 0$, which is (13.79) with $S \rightarrow \Gamma$, if using the relation $J = \delta\Gamma/\delta A$ as derived in 11.1, see (11.4).

However, the generator of gauge transformations for the gauge field also hits the gauge field in the ghost action and the gauge fixing term. While the transformation of the auxiliary ghost fields can be chosen as needed, it seems natural to augment them with a tensor transformation, as they live in the adjoint representation of the gauge group and have been introduced as a complete set of state for the Faddeev-Popov determinant. This leads us to

$$c \rightarrow c^U = U c U^\dagger, \quad \bar{c} \rightarrow \bar{c}^U = U \bar{c} U^\dagger. \quad (13.83)$$

which is nothing but the unitary coordinate rotation that accompanies the gauge transformation of the gauge field. For infinitesimal transformations, (13.83) takes the form

$$c \rightarrow c + i[\omega, c], \quad \bar{c} \rightarrow \bar{c} + i[\omega, \bar{c}]. \quad (13.84)$$

We remark that the transformation of the ghost is suggestive as in the ghost action $-\int_x \bar{c} \partial_\mu D_\mu c$ it is acted on by a covariant derivative and hence should carry the respective representation. In turn, the anti-ghost is acted on by a partial derivative and lives in the trivial representation of the gauge group. It is precisely this mismatch that makes it impossible to accommodate gauge transformations in the gauge fixed action.

The infinitesimal shift (13.84) of the ghost and the anti-ghost can be implemented in terms of functional ghost and anti-ghost derivatives, similarly to (13.76). For that purpose we consider an infinitesimal transformation of the full action with a parameter ω , dropping higher order terms in ω and only keeping the linear term. This leads us to

$$\begin{aligned} S[A, c, \bar{c}] &\rightarrow S\left[A + \frac{1}{g}D\omega, c + i[\omega, c], \bar{c} + i[\omega, \bar{c}]\right] \\ &= S[c, \bar{c}, A] - \left[\int_x \frac{1}{g}\omega \cdot D \cdot \frac{\delta}{\delta A} - i \int_x \left([\omega, c] \cdot \frac{\delta}{\delta c} + [\omega, \bar{c}] \cdot \frac{\delta}{\delta \bar{c}}\right)\right] S[c, \bar{c}, A], \end{aligned} \quad (13.85)$$

where we have used the short hand notation introduced in (13.77). The generator of gauge transformation for both, gauge fields, (13.2), and ghost, anti-ghost, (13.83), can be read off from (13.85), to wit,

$$\mathcal{G}^a = D_\mu^{ab} \frac{\delta}{\delta A_\mu^b} + g f^{abd} \left[c^b \frac{\delta}{\delta c^d} + \bar{c}^b \frac{\delta}{\delta \bar{c}^d} \right]. \quad (13.86)$$

We can easily verify that \mathcal{G}^a indeed generates the transformations (13.4) and (13.84) by computing its action on the fields. We find

$$\left[- \int_x \omega^b(x) \mathcal{G}^b(x) \right] A_\mu(y) = \mathcal{D}_\mu \cdot \omega(y), \quad (13.87)$$

for the gauge field, and

$$\left[- \int_x \omega^b(x) \mathcal{G}^b(x) \right] c(y) = ig [\omega(y), c(y)], \quad \left[- \int_x \omega^b(x) \mathcal{G}^b(x) \right] \bar{c}(y) = ig [\omega(y), \bar{c}(y)], \quad (13.88)$$

for ghost and anti-ghost respectively. In summary, we can represent the infinitesimal gauge transformation in (13.85) in terms of the generator (13.86), contracted with the parameter ω of the gauge transformation,

$$S \left[A + \frac{1}{g} D\omega, c + i[\omega, c], \bar{c} + i[\omega, \bar{c}] \right] - S[A, c, \bar{c}] = -\frac{1}{g} \int d^d x \omega^a(x) \mathcal{G}^a(x) S[A, c, \bar{c}]. \quad (13.89)$$

The desired symmetry identity follows from the observation that also the generators of gauge transformations for the ghost and anti-ghost can be brought into the form of total functional derivatives, similarly to (13.82). With $f^{abb} = 0$ we find

$$g f^{abd} \left[c^b \frac{\delta}{\delta c^d} + \bar{c}^b \frac{\delta}{\delta \bar{c}^d} \right] = -g f^{abd} \left[\frac{\delta}{\delta c^d} c^b + \frac{\delta}{\delta \bar{c}^d} \bar{c}^b \right]. \quad (13.90)$$

This entails that \mathcal{G} is a total derivative in the space of gauge field, ghost and anti-ghost and hence (13.81) can be extended to the generator of gauge transformations of all fields,

$$\int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{G}^a \left[e^{-S[A, c, \bar{c}] + \int_x (J_\mu^a A_\mu^a + \bar{\eta} c - \bar{c} \eta)} \right] = 0. \quad (13.91)$$

We remark that the identity (13.91) comprises the fact, that a reparametrisation of the functional integral $Z[J]$ in terms of gauge-transformed fields does not change the result. Acting with the functional derivatives in (13.91) on the action and the source term leads us to the *Slavnov-Taylor identity* (STI),

Slavnov-Taylor identity

$$\langle \mathcal{G}^a S[A, c, \bar{c}] \rangle = \bar{D}_\mu^{ab} J_\mu^b + g f^{abd} (\bar{\eta}^d C^b - \bar{C}^b \eta^d), \quad (13.92)$$

with $\bar{D}_\mu = D_\mu(\bar{A})$. In (13.92) we have introduced the expectation or mean values of the fields,

$$\bar{A} = \langle A \rangle, \quad C = \langle c \rangle, \quad \bar{C} = \langle \bar{c} \rangle, \quad (13.93)$$

which leads us to

$$\langle D_\mu^{ab} J_\mu^b(x) + g f^{abd} (\bar{\eta}^d c^b - \bar{c}^b \eta^d) \rangle = \bar{D}_\mu^{ab} J_\mu^b + g f^{abd} (\bar{\eta}^d C^b - \bar{C}^b \eta^d). \quad (13.94)$$

Note that the derivation above without the gauge fixing leads to

$$\bar{D}_\mu J_\mu = 0, \quad (13.95)$$

which would constitute covariant current conservation.

We emphasise that the Slavnov-Taylor identity is a functional identity: The right hand side can be rewritten in terms of derivatives of the effective action $\Gamma[\bar{A}, C, \bar{C}]$ of the gauge-fixed theory, and the expectation value $\langle \mathcal{G} S \rangle$ contains as the DSE for scalar theories loop terms in full propagators and classical and full vertices. As there, we can formulate the STI in (13.92) in terms of the effective action $\Gamma[A, c, \bar{c}]$. Note also that the loop terms only arise from the gauge-fixing and ghost term, as the Yang-Mills action is gauge invariant,

$$\mathcal{G}^a S_A[A] = 0, \quad (13.96)$$

which is nothing but (13.76). Hence we get

$$\langle \mathcal{G}^a S \rangle = \langle \mathcal{G}^a S_{\text{gh}} \rangle + \langle \mathcal{G}^a S_{\text{gf}} \rangle. \quad (13.97)$$

which leads us to the following final form of the STI in (13.92) in terms of currents and expectation values,

$$\bar{D}_\mu^{ab} J_\mu^b + g f^{abd} (\bar{\eta}^d C^b - \bar{C}^b \eta^d) = \langle \mathcal{G}^a S_{\text{gh}} \rangle + \langle \mathcal{G}^a S_{\text{gf}} \rangle. \quad (13.98)$$

In a final step we convert the STI (13.98) into one for the effective action. To that end we use the relations between the currents and the field derivatives of the effective action in (13.72). We also redefine in a slight abuse of notation the mean fields $\bar{A}, C, \bar{C} \rightarrow A, c, \bar{c}$, and hence the effective action has the latter arguments, $\Gamma = \Gamma[A, c, \bar{c}]$. This leads us to

Slavnov-Taylor identity for the effective action

$$D_\mu^{ab} \frac{\delta \Gamma}{\delta A_\mu^b} + g f^{abd} \left(c^b \frac{\delta \Gamma}{\delta c^d} + \bar{c}^b \frac{\delta \Gamma}{\delta \bar{c}^d} \right) = \langle \mathcal{G}^a S_{\text{gh}} \rangle + \langle \mathcal{G}^a S_{\text{gf}} \rangle. \quad (13.99)$$

The right hand side can be expressed in terms of one- and two-loop terms with full propagators and vertices. We recall the identity (11.67) as used in (11.72) and write

$$\langle \mathcal{G}^a S_{\text{gh}} \rangle + \langle \mathcal{G}^a S_{\text{gf}} \rangle = (\mathcal{G}^a S_{\text{gh}})[\hat{\phi}] + (\mathcal{G}^a S_{\text{gf}})[\hat{\phi}], \quad (13.100)$$

where the operator argument $\hat{\phi}$ is given by the sum of the propagators of the fields, contracted with field derivatives and the fields ϕ itself.

$$\hat{\phi}_i = G_{\phi_i \phi_j} \cdot \frac{\delta}{\delta \phi_j} + \phi_i, \quad \text{with} \quad \phi = (A_\mu, c, \bar{c}), \quad G_{\phi_i \phi_j}(x, y) = \langle \phi_i(x) \phi_j(y) \rangle_c \quad (13.101)$$

similarly to (11.72). Indeed, (13.99) is nothing but the DSE of the gauge theory, projected on gauge transformations instead of full shifts in the field.

We readily infer from the fully functional form on the right hand side of (13.100), that it contains the classical breaking of gauge invariance $(\mathcal{G}^a S_{\text{gh}})[\phi] + (\mathcal{G}^a S_{\text{gf}})[\phi]$ as well as loop terms. For example, the variation of the gauge fixing term leads to a single one-loop term proportional to

$$f^{abc} \partial_\mu^x \partial_\nu^y G_{AA_{\mu\nu}}^{bc}[\phi](x, y) \Big|_{x=y} = f^{abc} \partial_\mu \partial_\nu \langle A_\mu^b(x) A_\nu^c(y) \rangle[\phi] \Big|_{x=y}. \quad (13.102)$$

Equation (13.102) vanishes identically on the equations of motions, but its field derivatives do not. A similar analysis of the gauge variation of the ghost term reveals that it contains one-loop and two-loop terms as it is given by an expectation value of three fields.

The form (13.99) of the STI makes its content apparent. The left hand side constitutes an infinitesimal gauge transformation of the fields in the effective action with (13.2) and (13.83). If the effective action is gauge invariant, the right hand side would vanish. However, the two loops terms on the right hand side reflect the fact that the path integral has been gauge-fixed. While this was done with an insertion of unity into the formally gauge invariant path integral without the currents, gauge invariance is lost in the presence of the current. Indeed, the latter vanish on the equations of motion of the theory,

$$\frac{\delta \Gamma}{\delta A_\mu^a} = 0, \quad \frac{\delta \Gamma}{\delta c^a} = 0, \quad \frac{\delta \Gamma}{\delta \bar{c}^a} = 0. \quad (13.103)$$

Then, gauge invariance is restored: on the quantum EoM (13.103) the currents vanish and so does the left hand side of (13.99). This implies that the right hand side of (13.99) vanishes.

13.3.2. BRST-transformations

In summary, we had to introduce the gauge fixing for removing the gauge redundancy. The STI (13.92) encodes the information, how the theory reacts to a gauge transformation. Due to the gauge fixing, the current J_μ^a is not covariantly conserved. However, if we accompany the gauge transformation with a related change of the gauge fixing condition \mathcal{F} such that

$$A \rightarrow A^U, \quad \mathcal{F} \rightarrow \mathcal{F}^U, \quad \text{with} \quad \mathcal{F}^U(A^{\text{gf}U}) = 0, \quad \text{for} \quad \mathcal{F}(A^{\text{gf}}) = 0, \quad (13.104)$$

this combined transformation should leave the path integral invariant. Note that such a procedure does not change the FP-operator for linear gauges. Roughly speaking, this idea is at the root of the BRST-symmetry (Becchi, Rouet, Stora '76, Tyutin '75). For preparing the shift of the gauge fixing condition we rewrite our gauge-fixed action with a Hubbard-Stratonovich transformation,

$$S[A, c, \bar{c}, b] = S_A[A] + S_{\text{gh}}[c, \bar{c}, A] + S_{\text{gf}}[A, b], \quad (13.105)$$

with the gauge fixing action

$$S_{\text{gf}}[A, b] = \int d^d x \left[-\frac{\xi}{2} b^a b^a + b^a \partial_\mu A_\mu^a \right]. \quad (13.106)$$

In (13.105) and (13.106) we have introduced the *Nakanishi-Lautrup* field, yet another auxiliary field. Evaluating $S[A, c, \bar{c}, b]$ on the EoM of the Nakanishi-Lautrup field b ,

$$\left. \frac{\delta S[A, c, \bar{c}, b]}{\delta b} \right|_{\bar{b}} = 0, \quad \text{with} \quad \bar{b} = \frac{1}{\xi} \partial_\mu A_\mu, \quad (13.107)$$

leads us back to the gauge-fixed action $S[A, c, \bar{c}]$ in (13.68) with the general covariant gauge-fixing term,

$$S_{\text{gf}}[A] = \frac{1}{2\xi} \int_x (\partial_\mu A_\mu^a)^2. \quad (13.108)$$

On the level of the functional integral representation for Z in (13.67) this transformation can be introduced within a Gaußian integration over b ,

$$\int \mathcal{D}b e^{-S_{\text{gf}}[A, b]} \simeq e^{-S_{\text{gf}}[A]}, \quad (13.109)$$

where we have dropped the field-independent prefactor on the right hand side. This preparation allows us to introduce step by step the BRST transformation that keeps the full gauge-fixed action invariant, thus accompanying the gauge transformation of the gauge field with a transformation of the measure:

For discussing the latter we first concentrate on a gauge transformation

$$A_\mu \rightarrow A_\mu + D_\mu \omega, \quad (13.110)$$

of the gauge fixing term $S_{\text{gf}}[A, b]$. For the sake of convenience we have absorbed the $1/g$ factor in the definition of ω in (13.110). This corresponds to defining the gauge transformations with $U = \exp\{i g \omega\}$. Applying the transformation (13.110) to the action $S_{\text{gf}}[A, b]$ we get

$$S_{\text{gf}}[A, b] \rightarrow S_{\text{gf}}[A, b] + \int_x b^a \partial_\mu \mathcal{D}_\mu^{ab} \omega^b. \quad (13.111)$$

The shifted term is proportional to the FP-operator. It can be absorbed with a change of the ghost action with the combined BRST-transformation δ_ε with

$$\delta_\varepsilon : \quad A \rightarrow A + \varepsilon Dc, \quad \bar{c} \rightarrow \bar{c} + \varepsilon b \quad (13.112)$$

where the transformation of the gauge field is the gauge transformation (13.110) with the parameter ω of the gauge transformation being identified with

$$\omega^a = \varepsilon c^a, \quad (13.113)$$

with a Grassmann-valued parameter ε . We emphasise that ω in (13.113) is not Grassmann-valued and hence the gauge transformation of the gauge field in (13.112) is a shift with a standard function. Likewise, the shift of the anti-ghost in (13.112) is Grassmann-valued. Accordingly, the transformation (13.112) sustains Grassmann and number properties. Note also that (13.113) suggests the definition of the BRST-transformation as its transformation in (13.84) with $\omega = \varepsilon c$ as in (13.113). This would lead us to

$$\delta_\varepsilon : c \rightarrow c + \text{ig } \varepsilon c^2, \quad \text{with} \quad c^2 = c^a t^a c^b t^b = c^a c^b \frac{1}{2} [t^a, t^b]. \quad (13.114)$$

In any case, with the combined transformation (13.112) of gauge field and anti-ghost, the transformation of the gauge fixing action induced by $\delta_\varepsilon A$ is cancelled by that of the ghost action induced by $\delta_\varepsilon \bar{c}$. This already implies that

$$\delta_\varepsilon b = 0. \quad (13.115)$$

It is left to fix the transformation of the ghost such, that the $\delta_\varepsilon A$ -change of the ghost action is cancelled by that induced by $\delta_\varepsilon c$, and the show that the suggestive complementation (13.114) holds true. To that end we apply the combined transformation on the sum of the integrands of gauge fixing and ghost action,

$$b \cdot \partial_\mu A_\mu - \bar{c} \partial_\mu D_\mu c \xrightarrow{\delta_\varepsilon} b \cdot \partial_\mu A_\mu - \bar{c} \partial_\mu D_\mu c - \bar{c} \partial_\mu \delta_\varepsilon (D_\mu c) \stackrel{!}{=} b \cdot \partial_\mu A_\mu - \bar{c} \partial_\mu D_\mu c. \quad (13.116)$$

Equation (13.116) leads us to the constraint

$$-\text{ig} \left[(\varepsilon D_\mu c), c \right] + D_\mu \delta_\varepsilon c = -\text{ig} \varepsilon \left\{ D_\mu c, c \right\} + D_\mu \delta_\varepsilon c = -\text{ig} D_\mu (\varepsilon c^2) + D_\mu \delta_\varepsilon c = 0, \quad (13.117)$$

which suffices to determine the BRST transformation of c as (13.114).

This leaves us with a complete set of transformations that constitute gauge invariance on the level of the gauge-fixed action. As indicated before, the gauge fixing condition defines a coordinate system in the configuration space of the gauge fields, and a gauge transformation rotates the gauge field in this coordinate system. If this rotation is accompanied by a respective rotation of the coordinate system, the underlying gauge symmetry is manifest. Below we summarise the BRST-transformations (13.112), (13.114) and (13.115), making the rôle in this combined transformation of gauge field and coordinate system explicit,

$$\begin{aligned} \text{gauge transformation:} \quad & \delta_\varepsilon A = \varepsilon Dc, \\ \text{coordinate rotation:} \quad & \delta_\varepsilon c = \varepsilon \text{ig} c^2, \\ \text{rotation of the gauge fixing condition:} \quad & \delta_\varepsilon \bar{c} = \varepsilon b, \\ \text{no transformation of the auxiliary field:} \quad & \delta_\varepsilon b = 0, \end{aligned} \quad (13.118)$$

with

$$\delta_\varepsilon S[A, c, \bar{c}, b] = 0 \quad (13.119)$$

We remark that while the derivation of the BRST transformation was done within a covariant gauge, we have not used the explicit gauge fixing condition at all. In fact, that Nakanishi-Lautrup field is the gauge fixing condition in disguise, see (13.107). Hence the rotation of the gauge fixing condition with $\delta_\varepsilon \bar{c}$ takes care of this. Apart from this it takes a passive rôle in the rotation, $\delta_\varepsilon b = 0$.

13.3.3. Hilbert space

Importantly, the BRST-transformations allow us to construct a Hilbert space analogously to the construction of the Hilbert space in QED. There, we have split the Fock space \mathcal{F} in the subspace of the physical (transversal) polarisations with the creation operators $\alpha_{1/2}^+$, that of the zero-norm states related to α_l^+ and negative norm states created by α_-^+ . Polarisation referred to the momentum vector k_μ . Then, we restricted ourselves to the subspace with semi-positive states, and the physical Hilbert space was constructed from equivalence classes of states, whose difference was a zero norm state, see [Section 5.2](#).

The BRST transformations allow us to define the Hilbert space of a non-Abelian gauge theory in complete analogy. For this construction we first consider the action of a BRST transformation on a general state in the Fock space \mathcal{F} of a non-Abelian gauge theory, and define the BRST-operator Q ,

$$\varepsilon Q|\Psi\rangle := \delta_\varepsilon |\Psi\rangle, \quad (13.120)$$

where Q is the ε -independent part of δ_ε . This leads us to

$$QA = Dc, \quad Qc = igc^2, \quad Q\bar{c} = b, \quad Qb = 0. \quad (13.121)$$

Evidently, the BRST-operator Q is Grassmann-valued and increases the number of ghosts in a state by one, see [\(13.118\)](#). Moreover, the BRST-operator generates gauge transformations on functionals of the gauge field, and hence does not change gauge-invariant functionals $O[A] = O[A^U]$ and gauge-invariant states,

$$Q\hat{O}[A] = 0, \quad QS_A[A] = 0. \quad (13.122)$$

We define our physical subspace \mathcal{F}_{phys} as the kernel of the operator Q ,

$$\mathcal{F}_{phys} = \{|\Psi\rangle \in \mathcal{F} \mid Q|\Psi\rangle = 0\} = \text{Ker}Q. \quad (13.123)$$

[Equation \(13.123\)](#) is a linear sub-space of \mathcal{F} , as Q is a linear operator. Moreover, the physical subspace \mathcal{F}_{phys} contains zero-norm states $|\Psi_0\rangle$ with $\langle\Psi_0|\Psi_0\rangle = 0$ as in QED. The existence of these states follows from

$$Q^2 = 0. \quad (13.124)$$

With the property [\(13.124\)](#) we readily can show, that all states $|\Psi_0\rangle = Q|\Psi\rangle$ are zero norm states by computing their scalar product with themselves. We are led to

$$|\Psi_0\rangle = Q|\Psi\rangle, \quad \text{with} \quad \langle\Psi_0|\Psi_0\rangle = \langle\Psi|Q^2|\Psi\rangle = 0, \quad (13.125)$$

It is left to proof [\(13.124\)](#), which is performed by proving this property on the fields. We start with the BRST transformation of the gauge field in [\(13.121\)](#),

$$Q^2A = Q(Dc) = 0. \quad (13.126)$$

Indeed, we have defined $\delta_\varepsilon c$ such that [\(13.126\)](#) holds true, see [\(13.116\)](#) and [\(13.117\)](#). Acting twice with Q on the ghost field using [\(13.121\)](#) produces a third power of the ghost (as Q increases the ghost number by one), but this c^3 -term vanishes due to symmetry reasons, it necessarily carries both anti-symmetry and symmetry,

$$Q^2c = igQc^2 = -g^2c^2c + g^2cc^2 = 0. \quad (13.127)$$

The action of Q^2 on the remaining two fields vanishes trivially. With (13.121) we find

$$Q^2 c = Qb = 0, \quad \text{and} \quad Q^2 b = 0. \quad (13.128)$$

Note that it is the introduction of the Nakanishi-Lautrup field that guarantees the property (13.124). If we integrate it out, the transformation of the anti-ghost has to be evaluated on the EoM for b , (13.107), and we are left with

$$Q\bar{c}|_{b=\bar{b}} = \frac{1}{\xi} \partial_\mu A_\mu, \quad (13.129)$$

and $Q^2 \bar{c} \neq 0$, but it is still satisfied on-shell (on the EoMs). With the property (13.124) (and locality), Q is a derivative, and we can use this property to construct the physical Hilbert space \mathcal{H} ,

$$\mathcal{H} = \mathcal{F}_{\text{phys}} / \sim, \quad \text{with} \quad \sim: |\Psi_1\rangle \sim |\Psi_2\rangle \quad \text{if} \quad Q(|\Psi_1\rangle - |\Psi_2\rangle) = 0. \quad (13.130)$$

This can be rewritten as:

Physical Hilbert space for non-Abelian gauge theories

$$\mathcal{H}_{\text{phys}} = \text{Ker}Q / \text{Im}Q. \quad (13.131)$$

This construction utilises the BRST cohomology, it is similar to the construction of Hilbert spaces within the de Rham cohomology, where the derivative operator Q is simply the plain derivative $Q = d = dx_\mu \frac{\partial}{\partial x_\mu}$. There, the property (13.124) follows readily from $d^2 = dx_\mu \wedge dx_\nu \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial x_\mu} = 0$.

In summary we have arrived at the Hilbert space of a non-Abelian gauge theory, whose construction foregoes and is pivotal for the discussion of unitarity of non-Abelian gauge theories on the quantum level. This discussion is far beyond the aims of this lecture course and defer the interested reader to the literature. In this context it is worthwhile noting that so far we have ignored the fact that the gauge fixing condition may have several solutions. Indeed for the covariant gauge and other sufficiently smooth gauges there are (infinitely many) Gribov copies, and the cohomological construction has to be modified. Strictly speaking its global use rests on the existence of a BRST charge (ghost number) and not only the existence of the generator of BRST transformations. This intricacies lead to the fact that till today the construction of the physical Hilbert space of non-Abelian gauge theories faces difficulties. We rush to add that this does not affect the computation of expectation values of gauge invariant operators, and hence that of S -matrix elements.

13.4. Quantum Master Equation

In this final Section discussing the quantisation of non-Abelian gauge theories we want to discuss the BRST analogue of the functional STI (13.99). We expect that the BRST symmetry underlying the construction of the functional integral leads us to a linear identity such as the functional Ward identity in QED rather than the STI with loop terms as in (13.99).

13.4.1. Derivation of the quantum master equation

For its derivation we again start with the generating functional (13.67), where we have linearised the gauge fixing with the Gaussian integral (13.109) over the Nakanishi-Lautrup field,

$$Z[J, \eta, \bar{\eta}] = \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}b e^{-S[A, c, \bar{c}, b] + \int_x (J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta)}. \quad (13.132)$$

The STI following from the BRST transformations (13.118) with $\delta_\varepsilon = \varepsilon Q$ can be derived similarly to the STI (13.99): we consider the functional integral with a total derivative of the exponent in (13.132) as the integrand by applying δ_ε to it. For the comparison, see (13.91). As the full action is invariant, $\delta_\varepsilon S[A, c, \bar{c}] = 0$, we find

$$\frac{1}{Z[J, \eta, \bar{\eta}]} \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}b \delta_\varepsilon \left[e^{-S[A, c, \bar{c}, b] + \int_x (J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta)} \right] = \int_x \left(J \cdot \langle \varepsilon Q A \rangle + \bar{\eta} \langle \varepsilon Q c \rangle - \langle \varepsilon Q \bar{c} \rangle \cdot \eta \right) = 0. \quad (13.133)$$

Equation (13.133) carries explicitly the BRST invariance of the gauge fixed actions as all terms on the right hand side of the equations stem from the BRST variation of the current term and express the current conservation of the BRST current. Pulling out the Grassmann parameter ε and using (13.121) leads us to

$$\int_x \left(J \cdot \langle Q A \rangle - \bar{\eta} \cdot \langle Q c \rangle - \langle Q \bar{c} \rangle \cdot \eta \right) = \int_x \left(J \cdot \langle D c \rangle - i g \bar{\eta} \cdot \langle c^2 \rangle - \langle b \rangle \cdot \eta \right) = 0. \quad (13.134)$$

The first two terms in (13.134) still contain expectation values with more than one field, and hence contain loops. The last term is proportional to the gauge fixing with the mean field $\bar{A} = \langle A \rangle$, where we have used (13.129). Now we utilise that the terms multiplying the currents are simply BRST variations of the fields, $Q\phi = (QA, Qc, Q\bar{c})$. Accordingly they could be expressed in terms of derivatives with respect to currents of BRST variations, if we would augment the generating functional with these currents. However, potentially this spoils the form of the BRST-identity (13.134), if these additional source terms are not invariant under BRST transformations. It is here, where the derivative property of Q , $Q^2 = 0$ in (13.124) comes to our aid, as it entails that source terms for the BRST variations are invariant. This leads us to the final form of the generating functional for non-Abelian gauge theories including the BRST source terms,

Generating functional for non-Abelian gauge theories with BRST source terms

$$Z[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}] = \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}b e^{-S[A, c, \bar{c}, b] + \int_x (J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta) + \int_x (L_A \cdot QA + L_c \cdot Qc + L_{\bar{c}} \cdot Q\bar{c})}. \quad (13.135)$$

Equation (13.135) also includes a source term for the Nakanishi-Lautrup field b , as the latter is the BRST variation of the anti-ghost, $Q\bar{c} = b$. With $Q^2 = 0$, (13.134) still holds, and we can express the $\langle Q\phi \rangle$ in terms of derivatives of the Schwinger functional

$$W[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}] = \log Z[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}]. \quad (13.136)$$

This leads us to

$$\langle QA \rangle = \frac{\delta W}{\delta L_A}, \quad \langle Qc \rangle = \frac{\delta W}{\delta L_c}, \quad \langle Q\bar{c} \rangle = \frac{\delta W}{\delta L_{\bar{c}}} = \langle b \rangle. \quad (13.137)$$

Inserting (13.137) into (13.134), leads us to the final linear form of the STI,

$$\int_x \left[J \cdot \frac{\delta W}{\delta L_A} - \bar{\eta} \cdot \frac{\delta W}{\delta L_c} - \frac{\delta W}{\delta L_{\bar{c}}} \cdot \eta \right] = 0, \quad (13.138)$$

where we also can use $\delta W/\delta L_{\bar{c}} = \langle b \rangle$ in the last term. For $L_{\bar{c}} = 0$ and using the covariant gauge this yields $\langle b \rangle = 1/\xi \partial_\mu \bar{A}_\mu$ at $L_{\bar{c}} = 0$, see (13.107).

In a last step we convert (13.138) into a functional relation for the effective action and its derivatives, analogously to the STI (13.99). The BRST sources are spectators in the Legendre transformation done in (13.70), and we define

Effective action of a non-Abelian gauge theory with BRST source terms

$$\Gamma[A, c, \bar{c}; L_A, L_c, L_{\bar{c}}] = \int d^d x (J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta) - W[J, \eta, \bar{\eta}, L_A, L_c, L_{\bar{c}}], \quad (13.139)$$

where the Schwinger functional has been defined in (13.136), and we have used the short hand notation (13.71). The relation between the fields A_μ, c, \bar{c} and the currents $J_\mu, \eta, \bar{\eta}$ is not changed in the presence of the BRST source terms and hence we still find (13.72): the sources are given by the first derivative of the effective action with respect to the fields. As the BRST source terms are spectators in the Legendre transformation in (13.139), the derivative of the effective action with respect to a BRST source is minus that of the Schwinger functional, to wit,

$$\frac{\delta \Gamma}{\delta L_A} = -\frac{\delta W}{\delta L_A}, \quad \frac{\delta \Gamma}{\delta L_c} = -\frac{\delta W}{\delta L_c}, \quad \frac{\delta \Gamma}{\delta L_{\bar{c}}} = -\frac{\delta W}{\delta L_{\bar{c}}}. \quad (13.140)$$

Using (13.72) and (13.140) in (13.138), leads us to the quantum master equation (QME),

Quantum master equation

$$\int d^d x \left[\frac{\delta \Gamma}{\delta L_A} \cdot \frac{\delta \Gamma}{\delta A} + \frac{\delta \Gamma}{\delta L_c} \cdot \frac{\delta \Gamma}{\delta c} + \frac{\delta \Gamma}{\delta L_{\bar{c}}} \cdot \frac{\delta \Gamma}{\delta \bar{c}} \right] = 0. \quad (13.141)$$

Equation (13.141) is the final form of the STI for the effective action. It has the desired algebraic form and features no loop terms.

13.4.2. Applications of the quantum master equation

We close this Section and the whole Chapter on the quantisation of non-Abelian gauge theories with a few illustrative examples for the use and content of the quantum master equation (QME) and some remarks.

Classical and quantum BRST transformations

To begin with, we want to illustrate the symmetry content of the QME (13.141) at the example of the classical action. In the presence of the BRST source terms it is given by

$$S[A, c, \bar{c}; L_A, L_c, L_{\bar{c}}] = S[A, c, \bar{c}] - \int_x (L_A \cdot QA + L_c \cdot Qc + L_{\bar{c}} \cdot Q\bar{c}). \quad (13.142)$$

with $S[A, c, \bar{c}]$ given by (13.68). Accordingly, the variation of the classical action with respect to the BRST sources simply provides the BRST variations of the fields,

$$\frac{\delta S}{\delta L_A} = -QA = -Dc, \quad \frac{\delta S}{\delta L_c} = -Qc = -ig c^2, \quad \frac{\delta S}{\delta L_{\bar{c}}} = -Q\bar{c} = -b, \quad (13.143)$$

where we have used (13.121). Hence, on the classical level the quantum master equation (13.141) reduces to

$$\int_x \left[QA \cdot \frac{\delta}{\delta A} + Qc \frac{\delta}{\delta c} + Q\bar{c} \frac{\delta}{\delta \bar{c}} \right] S[A, c, \bar{c}; L_A, L_c, L_{\bar{c}}] = 0, \quad (13.144)$$

which is nothing but $\delta_\varepsilon S = 0$ after multiplication with ε . Not surprisingly, (13.141) reduces to classical BRST invariance on the classical level. The form (13.144) also emphasises the underlying structure of the quantum master equation: it describes the action of the operator of field derivatives, projected on the direction of the quantum BRST variations $\delta\Gamma/\delta L$. The respective operator BRST operator s_γ is given by

$$s_\Gamma := \int_x \left[\frac{\delta\Gamma}{\delta L_A} \cdot \frac{\delta}{\delta A} + \frac{\delta\Gamma}{\delta L_c} \cdot \frac{\delta}{\delta c} + \frac{\delta\Gamma}{\delta L_{\bar{c}}} \frac{\delta}{\delta \bar{c}} \right], \quad (13.145)$$

and the coefficients

$$\frac{\delta\Gamma}{\delta L_A}, \quad \frac{\delta\Gamma}{\delta L_c}, \quad \frac{\delta\Gamma}{\delta L_{\bar{c}}}, \quad (13.146)$$

are the quantum versions of the classical BRST variations (13.143). They acquire n -loop contributions at the n th order of the perturbative expansion. In a non-perturbative functional approach such as the DSE approach the quantum BRST variations (13.146) have a close loop form in full propagators and vertices.

Quantum action principle

We proceed by illustrating the perturbative use of the quantum master equation (13.141) within the *quantum action principle*. This exploits the iterative structure of perturbation theory to constrain the n th perturbative order in the effective action from the $n - 1$ st order. We start on the classical level, that is $n = 0$. As already discussed around (13.144), at this level the quantum master equation simply entails classical BRST invariance, see (13.144),

$$s_S S[A, c, \bar{c}; L_A, L_c, L_{\bar{c}}] = 0, \quad \text{with} \quad \varepsilon s_S = \varepsilon Q = \delta_\varepsilon = \varepsilon \int_x \left[QA \cdot \frac{\delta}{\delta A} + Qc \cdot \frac{\delta}{\delta c} + Q\bar{c} \cdot \frac{\delta}{\delta \bar{c}} \right]. \quad (13.147)$$

At the one-loop level we utilise the classical results and the QME takes the form

$$\frac{\delta S}{\delta L_{\phi_i}} \cdot \frac{\delta\Gamma}{\delta\phi_i} \Big|_{1\text{-loop}} + \frac{\delta\Gamma}{\delta L_{\phi_i}} \Big|_{1\text{-loop}} \cdot \frac{\delta S}{\delta\phi_i} = 0, \quad (13.148)$$

where $\phi = (A_\mu, c, \bar{c})$ and ϕ_i with $i = 1, 2, 3$ are the components. In (13.148) a sum over all components is implied. The derivatives of the classical action are input and (13.148) is the BRST-symmetry constraint for the one-loop order.

This procedure can be iterated and on the n th order we arrive at

$$\frac{\delta S}{\delta L_{\phi_i}} \cdot \frac{\delta\Gamma}{\delta\phi_i} \Big|_{n\text{-loop}} + \frac{\delta\Gamma}{\delta L_{\phi_i}} \Big|_{1\text{-loop}} \cdot \frac{\delta\Gamma}{\delta\phi_i} \Big|_{(n-1)\text{-loop}} + \dots + \frac{\delta\Gamma}{\delta L_{\phi_i}} \Big|_{n\text{-loop}} \cdot \frac{\delta S}{\delta\phi_i} = 0. \quad (13.149)$$

Equation (13.149) entails that the QME at the n th loop order constrains the n th loop order of the effective action with the input of the $n - 1$ th loop order. Importantly, the higher orders do not feed back to the lower ones. This is similar to the renormalisation conditions, where the higher loops do not affect the renormalisation at the lower loop order, as discussed already in Chapter 7.

BRST without Nakanishi-Lautrup field

The Nakanishi-Lautrup field has facilitated the derivation of the BRST transformations. However, it is by no means pivotal to the BRST setup or to the QME (13.141). It can readily be integrated out, as it only occurs quadratically and linearly in the classical action. Indeed, it has been introduced in the generating functional as a Gaussian integral. The integrating out is most conveniently done by solving the equation of motion for b at vanishing b -current $L_{\bar{c}}$, and consequently inserting the solution \bar{b} of the EoM into the effective action and the QME. The EoM yields

$$\left. \frac{\partial \Gamma}{\partial L_{\bar{c}}} \right|_{L_{\bar{c}}=0} = -\bar{b} = -\frac{1}{\xi} \partial_{\mu} A_{\mu}, \quad (13.150)$$

see also (13.107). Inserting (13.150) into the QME (13.141) leads us to

Zinn-Justin equation

$$\int d^d x \left[\frac{\delta \Gamma}{\delta L_A} \cdot \frac{\delta \Gamma}{\delta A} + \frac{\delta \Gamma}{\delta L_c} \cdot \frac{\delta \Gamma}{\delta c} - \frac{1}{\xi} \partial_{\mu} A_{\mu} \cdot \frac{\delta \Gamma}{\delta \bar{c}} \right] = 0. \quad (13.151)$$

In a last step we use the linear occurrence of the anti-ghost in the classical action in order to make the underlying structure of the last term very apparent: the BRST transformation of the anti-ghost was introduced to adjust for the gauge transformation of the gauge condition, that goes hand in hand with the gauge transformation of the field. Due to the linear occurrence of the anti-ghost, its Dyson-Schwinger equation takes a simple form,

$$\frac{1}{Z[J, \eta, \bar{\eta}, L_A, L_c, 0]} \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \frac{\delta}{\delta \bar{c}} \left[e^{-S[A, c, \bar{c}] + \int_x (J \cdot A + \bar{\eta} \cdot c - \bar{c} \cdot \eta) + \int_x (L_A \cdot Q_A + L_c \cdot Q_c)} \right] = \langle \partial_{\mu} D_{\mu} c \rangle - \eta = 0. \quad (13.152)$$

In terms of the effective action, (13.152) reads

$$\partial_{\mu} \frac{\delta \Gamma}{\delta L_{A_{\mu}}} - \frac{\delta \Gamma}{\delta \bar{c}} = 0 \quad (13.153)$$

Inserting (13.153) into (13.151) leads us to

$$\int_x \left[\frac{\delta \Gamma}{\delta L_A} \cdot \frac{\delta \Gamma}{\delta A} + \frac{\delta \Gamma}{\delta L_c} \cdot \frac{\delta \Gamma}{\delta c} - \frac{1}{\xi} (\partial_{\mu} A_{\mu}) \cdot \partial_{\nu} \frac{\delta \Gamma}{\delta L_{A_{\nu}}} \right] = 0. \quad (13.154)$$

In (13.154) it is evident that the variation of the anti-ghost simply amounts to a gauge transformation of the gauge fixing term as it was introduced in the first place. This is even more evident, if we combine the terms proportional to the BRST variation of the gauge field,

$$\int_x \left[\frac{\delta \Gamma}{\delta L_A} \cdot \left(\frac{\delta \Gamma}{\delta A} + \frac{1}{\xi} \partial_{\mu} A_{\mu} \right) + \frac{\delta \Gamma}{\delta L_c} \cdot \frac{\delta \Gamma}{\delta c} \right] = 0. \quad (13.155)$$

It is a matter of the task at hand, whether (13.141), (13.151) or (13.154), (13.155) are used.

STI's for correlation functions & transversality of the gauge field self energy

Evidently, the master equation (13.141) can be used for deriving relations between (parts of) correlation functions. For this purpose we consider (13.141) or one of its variants (13.151) and (13.154), (13.155) as a (zero) functional and evaluate derivatives of this functional at vanishing fields and BRST currents. We define

$$\text{QME}[\phi, L] = \int d^d x \left(\frac{\delta\Gamma}{\delta L_A} \cdot \frac{\delta\Gamma}{\delta A} + \frac{\delta\Gamma}{\delta L_c} \cdot \frac{\delta\Gamma}{\delta c} + \frac{\delta\Gamma}{\delta L_{\bar{c}}} \cdot \frac{\delta\Gamma}{\delta \bar{c}} \right), \quad (13.156)$$

with

$$\phi = (A_\mu^a, c^a, \bar{c}^a, b^a), \quad L = (L_{A_\mu}^a, L_c^a, L_{\bar{c}}^a). \quad (13.157)$$

The QME (13.141) is nothing but

$$\text{QME}[\phi, L] = 0, \quad (13.158)$$

while its variants (13.151) and (13.154), (13.155) are simply given by an evaluation of (13.158) on the EoM for b , (13.150), with $L_{\bar{c}} = 0$,

$$\text{QME} \left[A, c, \bar{c}, b = \frac{1}{\xi} \partial A, L_A, L_c, L_{\bar{c}} = 0 \right] = 0, \quad (13.159)$$

The QME is used frequently for reconstructing full scattering vertices such as the ghost-gluon, three-gluon and four-gluon vertices from the respective STI's. For example it can be shown that the longitudinal parts of three-point functions such as the ghost-gluon vertex or the three-gluon vertex are simply given by sums and differences of two-point functions, while longitudinal parts of the four-gluon vertex are given by sums of products of three- and two-point functions. These are prominent examples of the general structure, that the STI's relate longitudinal projections of higher correlation functions to sums of products of lower correlation functions. Needless to say, the computation of these algebraic identities typically requires far less computational effort than that of the loop identity (13.99), both analytically within perturbation theory as well as numerically within non-perturbative applications. Moreover, these relations unravel physical gauge invariance of the theory as carried by the gauge-fixed correlation functions. In summary, one can learn a lot from the quantum master equation about the physics mechanisms in non-Abelian gauge theories, and in particular it can be used to significantly reduce the computational tasks at hand.

As a key example for such a useful relation we discuss the STI for the 1PI two-point function of the gauge field,

$$\left. \frac{\delta^2 \Gamma}{\delta A_\nu^a(x) \delta A_\mu^b(y)} \right|_{\phi=\bar{\phi}, L=0} = \left[\langle AA \rangle_c^{-1} \right]_{ab}^{\mu\nu}. \quad (13.160)$$

where the subscript c indicates the connected part of the correlation function, and $\bar{\phi} = (0, 0, 0, \bar{b})$. We will derive that the longitudinal part of the two-point function is simply given by its classical (gauge-fixing) part, which obviously reduces the computational costs of computing the longitudinal quantum corrections to zero.

For showing this property we project the functional QME (13.141) or (13.154) on a relation for the two-point function (13.160). To that end we use that the first term in the QME already contains a first derivative with respect to A_μ . Hence a second $A_\mu^a(x)$ -derivative of this term already yields the two-point function (13.160), contracted with $\delta\Gamma/\delta L_A$. The latter term has ghost number one as the BRST operator increases the ghost number by one. This can be already seen from its classical variant $\delta S/\delta L_A = -Dc$, (13.143). Accordingly, we also take a $c^b(y)$ -derivative of the QME and then set all fields and BRST sources to zero. This leads us to

$$\left. \frac{\delta^2 \text{QME}[\phi, L]}{\delta c^b(y) \delta A_\mu^a(x)} \right|_{\phi=\bar{\phi}, L=0} = \int_z \left[\frac{\delta^2 \Gamma}{\delta c^b(y) \delta L_{A_\nu}^d(z)} \frac{\delta^2 \Gamma}{\delta A_\nu^d(z) \delta A_\mu^a(x)} + \frac{\delta^2 \Gamma}{\delta L_{\bar{c}}^d(z) \delta A_\mu^a(x)} \frac{\delta^2 \Gamma}{\delta c^b(y) \delta \bar{c}^d(z)} \right]_{\phi=\bar{\phi}, L=0} = 0. \quad (13.161)$$

The other terms generated by the two field derivatives of the QME vanish at $\phi = L = 0$ as they are products of terms with non-vanishing ghost number. Hence, they involve products of c, \bar{c} 's as well as BRST currents. The second term on the right hand side of (13.161) is readily computed either from (13.141) or directly from (13.154), where part of the EoMs have already been used. If we start with (13.141), we are led to

$$\frac{\delta^2 \Gamma}{\delta L_{\bar{c}}^d(z) \delta A_{\mu}^a(x)} = -\frac{1}{\xi} \frac{\delta \partial_{\nu} A_{\nu}^d(z)}{\delta A_{\mu}^a(x)} = -\frac{1}{\xi} \delta^{ad} \partial_{\mu}^z \delta(z-x), \quad (13.162)$$

where we have performed the $L_{\bar{c}}$ -derivative first and $\phi = \bar{\phi}, L = 0$ is implied. We also relate the ghost–anti-ghost two point function or inverse propagator to the full BRST-transformation of the gauge field,

$$\frac{\delta^2 \Gamma}{\delta c^b(y) \delta \bar{c}^d(z)} = \frac{\delta}{\delta c^b(y)} \partial_{\nu} \frac{\delta \Gamma}{\delta L_{A_{\nu}}^d(z)} = \partial_{\nu}^z \frac{\delta^2 \Gamma}{\delta c^b(y) \delta L_{A_{\nu}}^d(z)}, \quad (13.163)$$

with $b = \bar{b}$, which amounts to re-deriving (13.154). Note that this relation entails, how a quantum gauge transformation transports the gauge field along the gauge fibre: classically the kernel of this transformation is given by the Faddeev-Popov operator $-1/(\partial_{\mu} D_{\mu})$, on the quantum level this kernel is the full ghost propagator $\langle c \bar{c} \rangle_c$. By using (13.162) and (13.163) in (13.161), we arrive at the final form of the STI for the gauge field two-point function,

$$\int_z \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \left(\frac{\delta^2 \Gamma}{\delta A_{\nu}^d(z) \delta A_{\mu}^a(x)} + \frac{1}{\xi} \partial_{\mu}^z \partial_{\nu}^z \delta(z-x) \right) = 0. \quad (13.164)$$

at $\phi = \bar{\phi}, L = 0$. Equation (13.164) entails that the longitudinal gauge field two point function receives no quantum corrections. To see this more clearly, we first use that

$$\begin{aligned} \int_z \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \frac{\delta^2 S[A, c, \bar{c}, \bar{b}, L=0]}{\delta A_{\nu}^d(z) \delta A_{\mu}^a(x)} &= \int_z \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \frac{\delta^2 S_{\text{fg}}[A]}{\delta A_{\nu}^d(z) \delta A_{\mu}^a(x)} \\ &= \int_z \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \left[-\frac{1}{\xi} \delta^{ad} \partial_{\mu}^z \partial_{\nu}^z \delta(z-x) \right], \end{aligned} \quad (13.165)$$

with the solution \bar{b} in (13.150) of the b -EoM, \bar{b} and $S_{\text{fg}}[A]$ in (13.108). With (13.165), the STI (13.164) reduces to

$$\int_z \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \frac{\delta^2 (\Gamma - S)}{\delta A_{\nu}^d(z) \delta A_{\mu}^a(x)} = 0, \quad (13.166)$$

the STI only constrains the quantum part of the effective action. Moreover, the mixed c, L_A -derivative can be written as a total derivative,

$$\left. \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \right|_{\phi=\bar{\phi}, L=0} = \left. \frac{\delta}{\delta c^b(y)} \frac{\delta \Gamma}{\delta L_{A_{\nu}}^d(z)} \right|_{\phi=\bar{\phi}, L=0} = -\frac{\delta}{\delta c^b(y)} \langle D_{\mu}^{de} c^e(z) \rangle_{\phi=\bar{\phi}, L=0}. \quad (13.167)$$

Now we use that the c -derivative of the mixed correlation function $\langle [A, c] \rangle$ vanishes at $\phi = \bar{\phi}, L = 0$, and we arrive at

$$\left. \frac{\delta^2 \Gamma}{\delta L_{A_{\nu}}^d(z) \delta c^b(y)} \right|_{\phi=\bar{\phi}, L=0} = -\delta^{db} \partial_{\mu}^z \delta(z-y). \quad (13.168)$$

Using (13.168) in (13.166), we arrive at the final form of the STI for the gauge field two-point function. As correlation functions are typically computed in momentum space, we also perform a Fourier transform of the result, leading us to

$$p_\mu \left([\Gamma_{AA}^{(2)}(p)]_{\mu\nu}^{ab} - \frac{1}{\xi} p^2 p_\nu \delta^{ab} \right) = 0, \quad \text{with} \quad [\Gamma_{AA}^{(2)}]_{\mu\nu}^{ab}(p, q) = [\Gamma_{AA}^{(2)}]_{\mu\nu}^{ab}(p) (2\pi)^d \delta(p + q). \quad (13.169)$$

In (13.169) we have used the natural extension of the notation (11.6) to general theories,

$$\Gamma_{\phi_{i_1} \dots \phi_{i_n}}^{(n)}(x_1, \dots, x_n) = \frac{\delta^n \Gamma}{\delta \phi_{i_1}(x_1) \dots \delta \phi_{i_n}(x_n)}. \quad (13.170)$$

Commonly, (13.169) is rewritten in terms of the gauge field propagator, to wit,

$$p_\mu \langle A_\mu^a(p) A_\nu^b(q) \rangle_c = \xi \delta^{ab} p_\nu p^2 (2\pi)^d \delta(p + q). \quad (13.171)$$

and the right hand side vanishes for the Landau gauge with $\xi = 0$. We emphasise that in this case the (13.169) is non-trivial while (13.171) is trivial. The reader may convince themselves that (13.171) is also satisfied in the Landau gauge if (13.169) does not hold and $\Gamma_{AA}^{(2)}$ has non-trivial longitudinal contributions.

In summary the STI for the gauge field propagator entails the important information that there are no longitudinal contributions to the gauge field two-point function. This is similar to QED, where it follows from the Ward identity. We emphasise that this information can also be obtained or confirmed within an explicit computation of the longitudinal two-point function. Evidently this is computationally more demanding and hence this example shows very impressively the usefulness of the QME in terms of STIs for correlation functions.

14. QCD

Quantum ChromoDynamics is the theory of the strong interactions. Its interaction is carried by a non-Abelian gauge field, the gluon, with the gauge group $SU(3)$ and the color charge. The gluon couples to the quarks, and strongly interacting matter makes up most of the visible matter in the universe. In the high energy (ultra-violet) regime the couplings tends towards zero with increasing momenta and QCD processes are described with perturbation theory. The phenomenon, that the theory tends towards a free theory at asymptotically high momentum scales is called *asymptotic freedom*.

In turn, at the low energy (infrared) regime at momentum scales $p^2 \lesssim 1\text{GeV}^2$ its coupling grows strong and QCD turns from a theory of weakly interacting quarks and gluons into an (effective) theory of (weakly) interacting hadrons, the bound states of quarks. This dynamical change of degrees of freedom is governed by two peculiar properties, *confinement* and *strong spontaneous chiral symmetry breaking*, often also called dynamical (spontaneous) chiral symmetry breaking. The epitaph *strong* refers to the strong interactions (QCD) and not to the strength of the symmetry breaking or the coupling. Moreover, *dynamical* refers to the fact, that the (pseudo) Goldstone bosons of spontaneous chiral symmetry breaking are the pions built from *up* (u) and *down* (d) quarks, $\pi^+ \sim u\bar{d}$, $\pi^- \sim \bar{u}d$ and $\pi^0 \sim 1/\sqrt{2}(u\bar{u} - d\bar{d})$. Accordingly, in contradistinction to the Higgs mechanism, the Goldstone bosons are *dynamically* emerging low energy degrees of freedom built from *up* and *down* quarks.

These two phenomena, confinement and dynamical chiral symmetry breaking, are also responsible for the rich structure of hadron resonances, and the rich phase structure at finite temperature and density. Both, the physics of hadron resonances and the phase structure of QCD are experimentally tested at running and planned heavy ion facilities/detectors.

These properties of QCD are summarised as follows:

- (a) *Asymptotic freedom*: the strong fine structure constant decays for large momentum scales,

$$\alpha_s(p^2) = \frac{g^2(p^2)}{4\pi} \xrightarrow{p^2 \rightarrow \infty} 0, \quad (14.1)$$

and the theory tends towards the free theory for $p^2 \rightarrow \infty$, for experimental measurements see 14.3.

- (b) *Confinement*: Mathematically, it is defined as the existence of a mass gap in Yang-Mills theory (Millennium problem Clay Mathematics Institute, formulated by A. Jaffe and E- Witten). It is linked to the fact that (in Yang-Mills theory) the quark–anti-quark potential $V_{q\bar{q}}(r)$ grows linearly at large distances r between the quark and the anti-quark,

$$V_{q\bar{q}}(r \rightarrow \infty) \rightarrow \sigma r, \quad (14.2)$$

where σ is the string constant. Equation (14.2) entails that an (asymptotic) $q\bar{q}$ -state has an infinite energy and hence is not part of the Hilbert space. In QCD with dynamical quarks a dynamical $q\bar{q}$ pair is built when the potential energy is large enough. This pair shields the original one and the potential reduces to a color dipole potential.

- (c) *Strong spontaneous chiral symmetry breaking*: At large momenta $1\text{ GeV} \lesssim p \lesssim 100\text{ GeV}$ quarks exhibit a *current* quark mass generated by the Higgs mechanism at the electroweak scale of $\sim 100\text{ GeV}$. These masses can be found in Table 14.1. At smaller momenta $p^2 \lesssim 1\text{ GeV}^2$ strong spontaneous chiral symmetry breaking generates an additional mass gap for the quarks of the order $\Delta m \approx 300 - 400\text{ MeV}$, leading to the quark *constituent* masses. At these scales also confinement takes place and we cannot isolate a single

Generation	first	second	third	Charge
Mass [MeV]	1.5-4	1150-1350	170×10^3	
Quark	u	c	t	$\frac{2}{3}$
Quark	d	s	b	$-\frac{1}{3}$
Mass [MeV]	4-8	80-130	$(4.1-4.4) \times 10^3$	

Table 14.1.: Quark masses and charges. The scale of strong chiral symmetry breaking is $\Delta m \approx 300 - 400$ MeV, as is Λ_{QCD} . This entails that only $2 + 1$ flavours have to be considered for most applications to the phase structure of QCD.

quark as they are only found as *constituents* of hadrons, e.g. the nucleons. The latter (proton $p \sim uud$ and neutron $n \sim udd$) have a mass of ~ 1 GeV, being built from the up and down quarks.

In the present Chapter we discuss the phenomenon of asymptotic freedom and compute the momentum running of the strong fine structure constant $\alpha_s(p)$ in (14.1) at one-loop. We also discuss how the phenomenon of dynamical spontaneous chiral symmetry is mapped to our investigation of the one-loop effective potential of a scalar theory in Section 11.2, including a rough estimate of its size. Finally, confinement is discussed in the next Chapter, Chapter 15.

14.1. Renormalisation of QCD

For the computation of the running coupling of QCD we have to set-up the renormalisation programme first. The color gauge group SU(3) is coupled to the quarks, while the leptons do not experience the strong force, they live in the trivial representation of SU(3). The (Euclidean) action of QCD including gauge fixing and ghost action is a combination of the gauge fixed Yang-Mills action (13.68) with the gauge fixing term 13.3.2 and the Dirac action for the quarks,

$$S[A, q, \bar{q}] = \int_x \frac{1}{2} \text{tr}_f F_{\mu\nu}^2 + \frac{1}{2\xi} \int_x (\partial_\mu A_\mu^a)^2 - \int_x \bar{c}^a \partial_\mu D_\mu^{ab} c^b - \sum_{f=1}^{N_f} \int_x \bar{q}_f (\not{D} + m_f) q_f. \quad (14.3)$$

with the quark field q includes all flavours, $q = (u, d, s, c, b, t)$, and hence $N_f = 6$. We use a diagonal mass matrix m with entries m_f with $m = \text{diag}(m_u, m_d, m_s, m_c, m_b, m_t)$, where we used the running index $f = u, d, s, c, b, t$. The details of the QCD action (14.3) including the field strength $F_{\mu\nu}$ and the covariant derivative have been provided for general SU(N) gauge theories in Section 13.1 see (13.12) to (13.20).

The quarks carry the fundamental representation of SU(3) indicated with capital Latin letters, $q = (q^A)$ with $A = 1, 2, 3$, and the gluon is given by $A_\mu = A_\mu^a t^a$ with $a = 1, \dots, N_c^2 - 1 = 8$ for $N_c = 3$. The generators are given by $t^a = \frac{1}{2} \lambda^a$, where λ^a are the Gell-Mann matrices. In the fundamental representation they read

$$\lambda^a = \begin{pmatrix} \sigma^a & 0 \\ 0 & 0 \end{pmatrix}, \quad a = 1, 2, 3, \quad \text{with} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (14.4a)$$

for the first three generators with the Pauli matrices σ^a . This also indicates an embedding of SU(2) in SU(3). The remaining five generators are given by

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad (14.4b)$$

$$\Pi(p) = - \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4}$$

Figure 14.2.: Gluon vacuum polarisation in QCD

with

$$\Sigma_s(p) = \frac{1}{4N_c} \frac{1}{m} \text{tr} \Sigma(p), \quad \text{and} \quad \Sigma_D(p) = -\frac{1}{4N_c} \frac{1}{p^2} \text{tr} [i\not{p}\Sigma(p)]. \quad (14.10)$$

The trace tr in (14.10) incorporates both the Dirac trace and the gauge group trace. Inserting the one-loop self energy $\Sigma(p)$ (14.6) into (14.10), we are led to

$$\Sigma_s(p) = -\frac{4}{3}g^2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{(p+l)^2 + m^2} \frac{3+\xi}{l^2}, \quad (14.11)$$

for the scalar part of the self energy. Here we have used that $C_2(R) = t^a t^a = C_R \mathbb{1}$ in a given representation R of $\text{SU}(N_c)$ and in the fundamental representation $R = F$ we have

$$C_F = \frac{N_c^2 - 1}{2N_c}, \quad (14.12)$$

for the normalisation (13.16) of the generators. For a compilation of this and other useful relations see e.g. [6]. For $N_c = 3$ we get $C_F = 4/3$, leading to the prefactor in (14.11). We have also used that the Dirac trace of the product of three γ -matrices vanishes and the trace of two γ -matrices γ_μ and γ_ν is proportional to $\delta_{\mu\nu}$,

$$\text{tr} \gamma_\mu (\not{p} + l) \gamma_\nu = 0, \quad \text{tr} \gamma_\mu \gamma_\nu = 4N_c \delta_{\mu\nu}. \quad (14.13)$$

This already leads us to an important conclusion: the scalar part of the self-energy vanishes in the chiral limit for $m = 0$, and the chiral symmetry is maintained in the loop expansion. Moreover, the momentum integral in (14.11) is only logarithmically divergent, one momentum dimension is provided by the explicit mass factor in front of the integral. This emphasises the fact, that the power counting analysis only provides the potentially largest degree of divergence and symmetries can decrease this degree of divergence, even rendering the momentum integrals finite.

We proceed with the Dirac part. With (14.6) and (14.10) we are led to

$$\Sigma_s(p) = -g^2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{(p+l)^2 + m^2} \frac{1}{l^2} \left((1-\xi) \frac{p \cdot l}{p^2} - (1+\xi) - 2 \frac{(p \cdot l)^2}{p^2 l^2} \right), \quad (14.14)$$

In (14.14) we have also used (6.17). As for the scalar part, the Dirac part is not linearly divergent, but carries a logarithmic dimensionless divergence. Moreover, the divergent term is proportional to ξ and vanishes in the Landau gauge.

The vacuum polarisation or self energy of the gluon is depicted in 14.2. The respective power counting analysis reveals quadratic divergences for all four diagrams and hence,

$$[\Pi(p)] = 2. \quad (14.15)$$

However, a quadratically divergent term would be a mass contribution to the gluon with

$$\Pi_{\mu\nu}^{ab}(p) \propto \Delta m_A^2 \delta_{\mu\nu} \delta^{ab}, \quad (14.16)$$

which is the only possible regular tensor structure. It is here, where our STI analysis in the last Chapter [Chapter 13](#) in [Section 13.4.2](#) pays off the first time: Inserting [\(14.16\)](#) into the STI for the two-point function [\(13.169\)](#) readily leads to

$$m_A^2 = 0, \quad (14.17)$$

a mass contribution to the two-point function of the gluon is forbidden by gauge invariance. Since the classical action has no mass term, we conclude $\Delta m_A^2 = 0$ in order to ensure $m_A^2 = 0$. Note that a full analysis has to exclude irregular tensor structures such as

$$\Pi_{\mu\nu}^{ab}(p) \propto \Delta m_A^2 \delta_{\mu\nu} \Pi_{\mu\nu}^\perp(p), \quad \text{with} \quad \Pi_{\mu\nu}^\perp(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \quad (14.18)$$

with the transverse projection operator already defined in [\(7.103\)](#) in Minkowski space. The mass term [\(14.18\)](#) satisfies the STI [\(13.169\)](#) as $p_\mu \Pi_{\mu\nu}^\perp(p) = 0$. However, such a tensor structure is not generated in perturbation theory as the momentum-derivative of loop contributions to $\Gamma^{(2)}(p)$ vanish at $p = 0$.

Any explicit computation within a gauge invariant regularisation scheme such as dimensional regularisation gives this result. Note however, that within a regularisation scheme such as a momentum cutoff Λ with $l^2 \leq \Lambda^2$ the loop contribution produces a mass contribution $\Delta m_A^2 \propto \Lambda^2$. Then, the bare classical action S_0 has to accommodate a counter term

$$-\Delta m_A^2 (A_\mu^a)^2, \quad (14.19)$$

in order to ensure the satisfaction of the STI [\(13.169\)](#) for $\Gamma_{AA}^{(2)}(p) = S_{0,AA}^{(2)}(p) + \Pi(p)$. This brief analysis emphasises once more the importance of the STIs. They offer far more than a formal development that allows to reduce the numerical costs, they are of crucial importance for maintaining the key feature of gauge invariance in a non-Abelian gauge theory on the quantum level. We remark that one indeed finds a transverse gluon mass gap in covariant gauges

$$\lim_{p^2 \rightarrow 0} p^2 \Pi_{\mu\nu}^\perp(p) \langle A_\mu(p) A_\nu(-p) \rangle_c = 0, \quad (14.20)$$

which is related to confinement. Its computation requires a fully non-perturbative treatment of QCD, which is beyond the scope of the present lecture course.

In summary the STI for the gluon two-point function entails transversality of the latter. A particular consequence is the non-renormalisation of the gauge fixing term, or, put differently, the renormalisation of the gauge field $A_{0,\mu} = Z_A^{1/2} A_\mu$ and the gauge fixing parameter $\xi_0 = \xi Z_\xi$. Furthermore, a respective analysis of all one-loop diagrams reveals that only the (parts of) diagrams related to the terms in the classical action are potentially divergent, as is required for a perturbatively renormalisable QFT. From the analysis above we expect that more non-trivial relations can be extracted from the quantum master equation in terms of the quantum action principle. To that end we split the *finite* effective action $\Gamma[\phi, L]$ in [13.4.2](#) or [\(13.139\)](#) in the bare action S_{bare} and the regularised n th order loop contributions $\Gamma_{\text{reg},n}$, to wit,

$$\Gamma = S_{\text{bare}} + \sum_{n=1}^{\infty} \Gamma_{\text{reg},n}. \quad (14.21)$$

The bare action can be expanded in terms of the renormalised action $S[\phi; m, g, \xi]$ and a series of the n -loop counter terms S_n , where n again labels the loop order. This leads us to

$$S_{\text{bare}} = S[\phi; m, g, \xi] + \sum_{n=1}^{\infty} S_n. \quad (14.22)$$

Inserting (14.22) into (14.21), leads us to a *finite* loop expansion of the effective action,

$$\Gamma[\phi; m, g, \xi] = S[\phi; m, g, \xi] + \sum_{n=1}^{\infty} \Gamma_n, \quad \text{with} \quad \Gamma_n = \Gamma_{\text{reg},n} + S_n. \quad (14.23)$$

Inserting (14.23) into the QME leads to the desired relations between renormalisation factors. For the related investigation it is convenient to introduce a short hand notation for the product of derivatives with respect to $\phi = (A, c, \bar{c})$ and $L = L_A, L_c, L_{\bar{c}}$ in the QME (13.141) or (13.155) and (13.159). The QME takes the form

$$\Gamma * \Gamma := \int_x \frac{\delta \Gamma}{\delta \phi_i(x)} \cdot \frac{\delta \Gamma}{\delta L_{\phi_i}(x)} = 0, \quad (14.24)$$

and another common notation is $(\Gamma, \Gamma) = \Gamma * \Gamma$. Now we derive relations for the n -loop renormalisation on the basis of the QME 14.1 and the fact, that in a renormalisable quantum field theory the loop divergences in Γ_n are local and have the form of the terms in the action. These relations follow by induction. Obviously at tree level and one loop we have the required locality of the divergences, which can be absorbed in terms of counter terms in the bare action. We assume now that the effective action has been made finite at $n - 1$ loop level with this procedure. Now we insert (14.23) into the QME, retaining only the n -loop contributions. This leads us to

$$S * (S_n + \Gamma_{\text{reg},n}) + (S_n + \Gamma_{\text{reg},n}) * S = - \sum_{m=1}^{n-1} \Gamma_m * \Gamma_{n-m}. \quad (14.25)$$

Now we use that the right hand side of (14.25) is finite by induction as it only comprises lower loop contributions to the effective action including the counter terms, see (14.23). Hence, projecting (14.25) onto the sum of divergent parts, we are led to

$$S * (\Gamma_{\text{div},n} + S_n) + (\Gamma_{\text{div},n} + S_n) * S = 0 \quad (14.26)$$

Evidently, (14.26) fixes the counter term actions as

$$S_n = -\Gamma_{\text{div},n} + \Delta S_n, \quad \text{with} \quad \Gamma_{\text{reg},n} = \Gamma_{\text{div},n} + \text{finite}, \quad (14.27)$$

where the finite part ΔS can only have the same form as the terms in the classical action. Note that (14.27) also carries the implicit assumption that the divergent parts of the n -loop contribution of the effective action are local terms that are present in the classical action. This is the assumption of perturbative renormalisability.

Moreover, we conclude directly from (14.26), that the sum of the divergent parts, $\Delta S_n = \Gamma_{\text{div},n} + S_n$, carries the classical BRST symmetry. For the Yang-Mills and Dirac action $S_A + S_D$ this is the classical gauge symmetry. This important property entails that the bare action S_{bare} can be written in terms of the BRST-invariant classical action with bare fields and coupling parameters for a gauge-invariant regularisation scheme,

$$S_{\text{bare}} = S[A_{0,\mu}, c_0, \bar{c}_0, q_0, \bar{q}_0; g_0, m_0, \xi_0] = S[Z_A^{1/2} A_\mu, Z_c^{1/2} c, Z_{\bar{c}}^{1/2} \bar{c}, Z_q^{1/2} q, Z_{\bar{q}}^{1/2} \bar{q}; Z_g g, Z_m m, Z_\xi \xi], \quad (14.28)$$

where $\phi_0 = (A_{0,\mu}, c_0, \bar{c}_0)$ are the bare fields and the Z_{ϕ_i} are the wave function renormalisations, relating the bare fields to the renormalised ones. Moreover, g_0, m_0, ξ_0 are the bare (coupling) parameters, and the renormalisation factors Z_g, Z_m, Z_ξ relate them to the renormalised ones. In summary we have

$$\phi_{0,i} = Z_{\phi_i}^{1/2} \phi_i, \quad \text{and} \quad (g_0, m_0, \xi_0) = (Z_g g, Z_m m, Z_\xi \xi). \quad (14.29)$$

We emphasise that the counter terms S_n depend on Z_i with $i \leq n$ as the action also depends on products of the renormalisation factors. This is seen more explicitly, if we write the QCD action in terms of kinetic terms and vertices. This reads schematically

$$S_{\text{bare}} = \frac{1}{2} Z_{\phi_i \phi_{i_1}} \int_{x_1, x_2} S_{\phi_i \phi_{i_1}}^{(2)}(x_2, x_1) \phi_{i_1}(x_1) \phi_i(x_2) + \frac{1}{3!} Z_{\phi_i \phi_{i_2} \phi_{i_1}} \int_{x_1, x_2, x_3} S_{\phi_i \phi_{i_2} \phi_{i_1}}^{(3)}(x, y) \prod_{j=1}^3 \phi_{i_j}(x_j) \\ + \frac{1}{4!} Z_{\phi_{i_4} \dots \phi_{i_1}} \int_{x_1, \dots, x_4} S_{\phi_{i_4} \dots \phi_{i_1}}^{(4)}(x_1, \dots, x_4) \prod_{j=1}^4 \phi_{i_j}(x_j), \quad (14.30)$$

where the factors $Z_{\phi_{i_1} \dots \phi_{i_n}}$ are the renormalisation factors of the two-, three- and four-point functions and are given by product of the wave function renormalisations and the coupling and mass renormalisations. By identifications of the different terms we find for the renormalisation factors $Z_{\phi_i \phi_i}$ of the two-point functions,

$$Z_{AA}^\perp = Z_A, \quad Z_{AA}^\parallel = \frac{Z_A}{Z_\xi}, \quad Z_{c\bar{c}} = Z_c, \quad Z_{q\bar{q}} = Z_q. \quad (14.31)$$

Note that in case of the kinetic term of the quark, $S_{q\bar{q}}^{(2)}$ we had to take into account that it contains two tensor structures, the Dirac tensor structure with $Z_{q\bar{q}}$ and the scalar tensor structure of the mass term with $Z_{s,q\bar{q}}$, see (14.3). Then, $Z_{q\bar{q}}$ stands for the wave function renormalisation Z_q that multiplies both terms due to the rescaling of the quarks. The mass term carries an additional renormalisation factor. For example, for one flavour we find with (14.29),

$$\int_x Z_{q\bar{q}} (\not{D} + Z_m m) q, \quad \text{with} \quad Z_{q\bar{q}} = Z_q, \quad Z_{s,q\bar{q}} = Z_q Z_m. \quad (14.32)$$

For the vertex factors we get

$$Z_{A^3} = Z_A^{3/2} Z_g, \quad Z_{A^4} = Z_A^2 Z_g^2, \quad Z_{c\bar{c}A} = Z_c Z_A^{1/2} Z_g, \quad Z_{q\bar{q}A} = Z_q Z_A^{1/2} Z_g. \quad (14.33)$$

In summary this leads us to the renormalisation scheme for QCD in covariant gauges. For gauge invariant regularisation schemes (14.31) and (14.33) imply

$$\frac{Z_{A^3}}{Z_A} = \frac{Z_{A^4}}{Z_A^3} = \frac{Z_{c\bar{c}A}}{Z_c} = \frac{Z_{q\bar{q}A}}{Z_q} = Z_A^{1/2} Z_g. \quad (14.34)$$

Equation (14.34) can be used to compute the RG-running of the coupling: for this task we have to compute the μ -dependence of Z_g , where μ is the renormalisation group scale. With (14.34) this amounts to computing the renormalisation factors of vertices and that of the respective legs. In general we conclude

$$Z_g = \frac{Z_{\phi_{i_1} \dots \phi_{i_n}}}{Z_{\phi_{i_1}}^{1/2} \dots Z_{\phi_{i_n}}^{1/2}} \quad \text{with} \quad Z_{\phi_{i_1} \dots \phi_{i_n}} = Z_{A^3}, Z_{c\bar{c}A}, Z_{q\bar{q}A}, Z_{A^4}. \quad (14.35)$$

The notation introduced above allows to directly identify the renormalisation factors with the vertices and fields they are linked to. A more common notation is given by

$$Z_3 = Z_A, \quad \tilde{Z}_3 = Z_c, \quad Z_1 = Z_{A^3}, \quad Z_4 = Z_{A^4}, \quad \tilde{Z}_1 = Z_{c\bar{c}A}, \quad Z_2 = Z_q, \quad Z_{1,F} = Z_{q\bar{q}A}, \quad (14.36)$$

which allows to compare the results more directly with the literature.

Finally, we come back to the STI of the gluon two-point function, (13.169), which has already been used to derive (14.16), the absence of a gluon mass term in QCD. However, with (13.169) all longitudinal quantum corrections vanish, not only the $p = 0$ ones: the longitudinal part of the gluon two-point function stays classical, $Z_A^{\parallel} \equiv 1$. With (14.31) this implies

$$Z_{\xi} = Z_A, \quad (14.37)$$

as already discussed before. This concludes our discussion of the consequences of gauge and BRST invariance for the renormalisation programme in QCD, or, more generally, that for non-Abelian gauge theories coupled to fermions in the fundamental representation.

It is left to specify the renormalisation conditions. For the following computation we resort to the minimal subtraction scheme. We consider a general renormalisation factor Z , which we order in powers of the coupling,

$$Z = 1 + \delta Z = 1 + \sum_{i=1}^{\infty} (g^2)^i (\delta Z_{\text{div}}^{(i)} + \delta Z_{\text{finite}}^{(i)}), \quad (14.38)$$

where the finite parts $Z_{\text{finite}}^{(i)}$ arrange for given renormalisation conditions. In a minimal subtraction (MS) scheme they are simply dropped,

$$\text{MS : } Z_{\text{finite}}^{(i)} = 0. \quad (14.39)$$

We emphasise that this condition depends on the given regularisation scheme, as both, $Z_{\text{div}}^{(i)}$ and $Z_{\text{finite}}^{(i)}$ depend on the regularisation procedure as do the $\Gamma_{\text{div},i}^{(n)}$ and the finite parts.

In the explicit computations we use dimensional regularisation in $d = 4 - 2\varepsilon$ dimensions. Then, minimal subtraction amounts to

Minimal subtraction

$$Z = 1 + \sum_{i=1}^{\infty} (g^2)^i \delta Z_{\text{div}}^{(i)}, \quad \text{with} \quad \delta Z_{\text{div}}^{(i)} = \frac{1}{\varepsilon} c_i. \quad (14.40)$$

where the coefficient c_i of the i th loop contribution carries the information about the RG-running of the renormalisation factors. This concludes our discussion of the renormalisation programme in QCD.

14.2. Running Coupling

In this Section we compute the momentum dependence of the strong coupling which leads us to (14.1). The respective momentum behaviour can also be measured experimentally and the up-to-date experimental data is summarised in Figure 14.3. Most of the experimental measurements come from jet or $t\bar{t}$ production processes, and at the core of most of the measurements is the scattering process of quarks via a gluon exchange. The respective diagram that contributes to the cross section is depicted in Figure 14.4. Both quark-gluon vertices and the gluon propagator are full correlation functions as the cross section is compiled out of tree-level diagrams of full (1PI) correlation functions. The transition amplitude $\mathcal{M}_{q\bar{q} \rightarrow q\bar{q}}$ in Figure 14.4 can be written in terms of 1PI correlation functions and we have schematically

$$\mathcal{M}_{q\bar{q} \rightarrow q\bar{q}}(p_1, p_2, p_3, p_4) \simeq \frac{1}{Z_q} \Gamma_{q\bar{q}A}^{(3)}(p_1, p_2, p) \left[\frac{1}{\Gamma(2)} \right]_{AA} (p) \Gamma_{q\bar{q}A}^{(3)}(-p_3, -p_4, -p) \frac{1}{Z_q}, \quad (14.41)$$

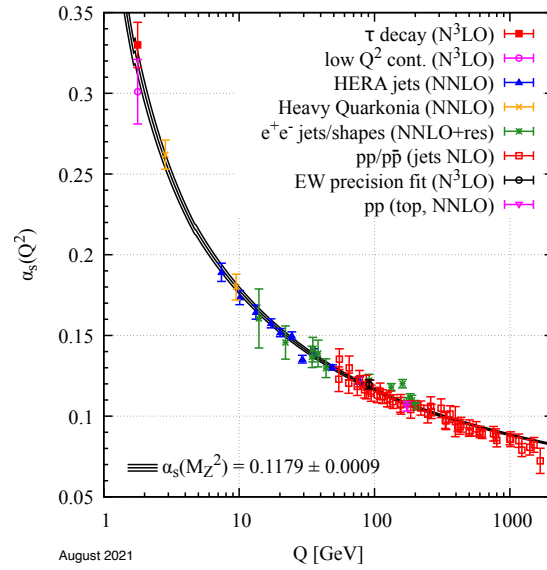


Figure 14.3.: Summary of measurements of α_s as a function of the energy scale Q . The respective degree of QCD perturbation theory used in the extraction of α_s is indicated in brackets (NLO: next-to-leading order; NNLO: next-to-next-to-leading order; NNLO+res.: NNLO matched to a resummed calculation; N3LO: next-to-NNLO). Figure and caption taken from the Review of Particle Physics 2022 [1].

where all momenta are counted ingoing and the gluon momentum is given by $(p_1 + p_2)^2 = p^2 = (p_3 + p_4)^2$. In (14.41) we have also attached the external (on-shell) renormalisation factors $(1/Z_q^{1/2})^4$ from the 'asymptotic' $q\bar{q}$ -states, see (3.204). The process Figure 14.4 can be used to define a running strong coupling with

$$\mathcal{M}_{q\bar{q} \rightarrow q\bar{q}}(p) \simeq 4\pi \frac{\alpha_s(p)}{p^2}, \quad (14.42)$$

where $\alpha_s(p)$ measures the interaction strength of the process in dependence of the exchange momentum. For its computation in perturbation theory we use, that we can map the momentum dependence of the running strong coupling to the RG-scale dependence of the renormalised coupling for asymptotically large momenta:

First we use that (14.41) is a physical process and the renormalisation group equation (7.14) entails that general observables \mathcal{O} do *not* depend on μ . In the present case we either use the dimensionless part of the transition amplitude,

$$\mathcal{O} = p^2 \mathcal{M}_{q\bar{q} \rightarrow q\bar{q}} \simeq \alpha_s(p^2), \quad \text{with} \quad \alpha_s(p^2) = \frac{g(p^2)^2}{4\pi}, \quad (14.43)$$

as an observable or we use the dimensionless part of the respective cross-section. For asymptotically large momenta we can ignore the quark masses and we are led to

$$\mu \frac{d}{d\mu} \mathcal{O} = \left(\mu \frac{\partial}{\partial \mu} + \beta_g g \frac{\partial}{\partial g} \right) \mathcal{O} = 0, \quad \text{with} \quad \beta_g = \mu \frac{d \ln g}{d\mu} = -\mu \frac{d \ln Z_g}{d\mu}. \quad (14.44)$$

Note that the only other (coupling) parameter in the theory is ξ , which cannot occur in observables. The chosen observables are dimensionless and can only depend on dimensionless parameters, that is

$$\mathcal{O}(p) = \mathcal{O} \left(\alpha_{s,\text{ren}}(\mu^2), \frac{p^2}{\mu^2} \right), \quad (14.45)$$

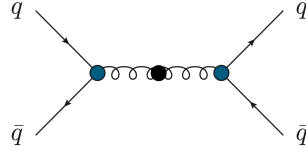


Figure 14.4.: $q\bar{q}$ -scattering with a one-gluon exchange diagram. The quark-gluon vertices and the gluon propagator are full correlation functions.

where the μ -dependence of the renormalised coupling $\alpha_{s,\text{ren}}(\mu^2)$ cancels that of the dimensionless momentum argument $p^2\mu^2$. Now we choose the renormalisation scale to match the momentum scale, $\mu^2 = p^2$, and we arrive at

$$\mathcal{O}\left(\alpha_{s,\text{ren}}(\mu^2), \frac{p^2}{\mu^2}\right) = \mathcal{O}(\alpha_{s,\text{ren}}(p^2), 1). \quad (14.46)$$

This implies

$$\alpha_s(p^2) = \alpha_{s,\text{ren}}(p^2), \quad (14.47)$$

and hence the β -function of the renormalised coupling entails the momentum running of the running coupling: for a negative β -function, $\beta_g < 0$, asymptotic freedom is achieved as

$$p \frac{d\alpha_s(p^2)}{dp} = 2\beta_g(\mu = p) \alpha_s(p^2). \quad (14.48)$$

In summary we have converted the task of computing the momentum-dependence of the running coupling $\alpha_s(p)$ to that of the μ -dependence of $\ln Z_g^2 = 2 \ln Z_g$. The renormalisation factor of the coupling is simply given by the product of that of the vertices and propagators involved in the process [Figure 14.4](#),

$$p^2 \mathcal{M}_{q\bar{q} \rightarrow q\bar{q}}\left(\alpha_{s,\text{ren}}(\mu^2), \frac{p^2}{\mu^2}\right) \propto g^2 \frac{Z_{q\bar{q}A}^2}{Z_A Z_q^2} = g^2 Z_g^2, \quad \text{with} \quad Z_g = \frac{Z_{q\bar{q}A}}{Z_A^{1/2} Z_q}. \quad (14.49)$$

Note that the renormalised strong coupling squared $g(\mu^2)^2$ combines with $1/(4\pi)$ into the argument $\alpha_{s,\text{ren}}(\mu)$ in the dimensionless transition amplitude $p^2 \mathcal{M}_{q\bar{q} \rightarrow q\bar{q}}$ in [\(14.49\)](#), while $Z_g^2(\mu)$ combines with the diagrams to the momentum dependence on p^2/μ^2 of $p^2 \mathcal{M}_{q\bar{q} \rightarrow q\bar{q}}$.

In the following we perform this computation explicitly at one loop by computing the renormalisation factors of the quark-gluon vertex $Z_{q\bar{q}A}$, and that of the quark and gluon propagators Z_q and Z_A respectively. For this computation we use dimensional regularisation mainly for two reasons: Firstly it is a gauge-invariant regularisation scheme, and hence we do not need to introduce gauge-symmetry breaking counter terms such as a mass term for the gluons. Moreover, the STI relations [\(14.34\)](#) holds true.

We rely on the master integrals already introduced in [Chapter 7](#) and listed in [Appendix D](#). For the computation of the one-loop contributions to the quark-gluon vertex and the quark and gluon propagators we encounter loop integrals

$$\mathcal{I}_{\mu\nu} = \int \frac{d^4 l}{(2\pi)^4} \frac{\mathcal{T}_{\mu\nu}}{(l^2)^{n_1} [(l+p)^2]^{n_2} [(l+q)^2]^{n_3}}, \quad \text{with} \quad \mathcal{T}_{\mu\nu} \in \{\delta_{\mu\nu}, p_{\mu_1} p_{\mu_2}, p_{\mu_1} l_{\mu_2}, l_{\mu_1} l_{\mu_2}\}, \quad (14.50)$$

which are all covered directly by the master integrals in [Appendix D](#) after using the Feynman trick [\(D.1\)](#). The master integrals in [Appendix D](#) are computed in $d = 4 - 2\epsilon$ dimensions and have to be multiplied with $\mu^{2\epsilon}$ for keeping their momentum dimension intact, and hence

$$\mathcal{I}_{\mu\nu} = \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} \frac{\mathcal{T}_{\mu\nu}}{(l^2)^{n_1} [(l+p)^2]^{n_2} [(l+q)^2]^{n_3}}. \quad (14.51)$$

$$[\Pi_{3\text{gl}}(p)]_{\mu\nu}^{ab} = - \text{diagram}$$

Figure 14.5.: Tree-gluon diagram $\Pi_{3\text{gl}}(p)$ in Figure 14.2.

With these preparations we start with the computation of the renormalisation factors with the vacuum polarisation of the gluon, depicted in Figure 14.2,

$$\Pi(p) = \Pi_{3\text{gl}}(p) + \Pi_{4\text{gl}}(p) + \Pi_{\text{ghgl}}(p) + \Pi_{\text{qugl}}(p), \quad (14.52)$$

where the different parts on the right hand side stand for the single diagrams in Figure 14.2 including the signs, labelled by the vertices in the diagrams. The computation is done in the Feynman gauge

$$\xi = 1. \quad (14.53)$$

We emphasise that each renormalisation factor in the ratio in (14.49) is ξ -dependent but the ratio is not.

14.2.1. Gluon vacuum polarisation

With the QCD Feynman rules in Appendix G, the first diagram is given by

$$\begin{aligned} [\Pi_{3\text{gl}}(p)]_{\mu\nu}^{ab} &= \frac{1}{2} g^2 \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} f^{acd} \left[\delta_{\mu\rho}(p-l)_\sigma + \delta_{\rho\sigma}(2l+p)_\mu + \delta_{\sigma\mu}(-2p-l)_\rho \right] \\ &\quad \times \frac{\delta_{\rho\rho'}}{l^2} f^{bcd} \left[\delta_{\nu\rho}(-p+l)_{\sigma'} + \delta_{\rho'\sigma'}(-2l-p)_\nu + \delta_{\sigma'\nu}(2p+l)_{\rho'} \right] \frac{\delta_{\sigma\sigma'}}{(l+p)^2}, \end{aligned} \quad (14.54)$$

it is also depicted with all indices in Figure 14.5. The contraction of the group indices and reparametrisations of the integral including the Feynman trick are deferred to Appendix 8.1 with the final result (H.12), which we also include here,

$$\begin{aligned} [\Pi_{3\text{gl}}(p)]_{\mu\nu}^{ab} &= \frac{\delta^{ab}}{2} N_c g^2 \frac{1}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{\Delta^{2-\frac{d}{2}}} \left\{ -\delta_{\mu\nu} p^2 \left[\Gamma\left(2 - \frac{d}{2}\right) \left((1+\alpha)^2 + (2-\alpha)^2 \right) \right. \right. \\ &\quad \left. \left. + \Gamma\left(1 - \frac{d}{2}\right) 3(d-1)\alpha(1-\alpha) \right] + p_\mu p_\nu \Gamma\left(2 - \frac{d}{2}\right) \left[(2-d)(1-2\alpha)^2 + 2(1+\alpha)(2-\alpha) \right] \right\}, \end{aligned} \quad (14.55)$$

with

$$\Delta = \alpha(1-\alpha)p^2. \quad (14.56)$$

We emphasise that the final result (14.55) is not transverse and hence does not satisfy the STI (13.164) of the gluon propagator. However, only for the sum of all diagrams in Yang-Mills theory or in QCD the STI has to hold. Hence, we expect transversality for the one-loop contribution to the Yang-Mills two-point function,

$$p_\mu [\Pi_{\text{YM}}]_{\mu\nu}^{ab} = 0, \quad \text{with} \quad \Pi_{\text{YM}}(p) = \Pi_{3\text{gl}}(p) + \Pi_{4\text{gl}}(p) + \Pi_{\text{ghgl}}(p), \quad (14.57)$$

and for the full QCD result. Consequently also the difference of both one-loop contributions has to be transverse. This entails

$$p_\mu \left[\Pi_{\text{qugl}}(p) \right]_{\mu\nu}^{ab} = 0, \quad (14.58)$$

This short analysis again illustrates the power of the STIs.

We proceed with the second diagram in (14.52) depicted in Figure 14.2, the gluon tadpole $\Pi_{4\text{gl}}$. This diagram leads to a momentum-independent term proportional to $\delta_{\mu\nu}$ which has to vanish according to the STI (13.164), for its momentum space formulation see (13.171). With the Feynman rules in Appendix G we get

$$\left[\Pi_{4\text{gl}}(p) \right]_{\mu\nu}^{ab} = -g^2 N_c \delta^{ab} \delta_{\mu\nu} (d-1) \mu^{2\epsilon} \left[\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \right] = 0, \quad (14.59)$$

in dimensional regularisation. As mentioned before, this originates in the gauge invariance of dimensional regularisation. However, for the present purpose another representation of the tadpole is more convenient. To that end we multiply the integrand with unity, $1 = (p+l)^2/(p+l)^2$, and use the master integrals in Appendix D. This leads us to

$$\left[\Pi_{4\text{gl}}(p) \right]_{\mu\nu}^{ab} = \delta^{ab} \delta_{\mu\nu} p^2 N_c g^2 \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{\Delta^{2-\frac{d}{2}}} (d-1) \left[\Gamma\left(1 - \frac{d}{2}\right) d\alpha(1-\alpha) + \Gamma\left(2 - \frac{d}{2}\right) (1-\alpha)^2 \right], \quad (14.60)$$

with Δ in (14.56). For the pure Yang-Mills or glue contribution we also need the contribution of the ghost loop Π_{ghgl} in (14.52) depicted in Figure 14.2. With the Feynman rules in Appendix G we get

$$\begin{aligned} \left[\Pi_{\text{ghgl}}(p) \right]_{\mu\nu}^{ab} &= -g^2 \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l+p)^2} \frac{1}{l^2} f^{dae} f^{ebd} (l+p)_\mu l_\nu \\ &= \delta^{ab} g^2 N_c \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{\mu^{2\epsilon}}{(k^2 + \Delta)^2} \left[k_\mu k_\nu - \alpha(1-\alpha) p_\mu p_\nu \right], \end{aligned} \quad (14.61)$$

with $l = k - \alpha p$. We also have used the contraction of the structure constants, $f^{dae} f^{ebd} = -N_c \delta^{ab}$ with (H.2) and (H.3). With the master integrals in Appendix D this leads us to the final expression for the ghost loop,

$$\left[\Pi_{\text{ghgl}}(p) \right]_{\mu\nu}^{ab} = \delta^{ab} N_c g^2 \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{\Delta^{2-\frac{d}{2}}} \left[\delta_{\mu\nu} p^2 \Gamma\left(1 - \frac{d}{2}\right) \frac{1}{2} \alpha(1-\alpha) - p_\mu p_\nu \Gamma\left(2 - \frac{d}{2}\right) \alpha(1-\alpha) \right], \quad (14.62)$$

The sum of the results (14.55), (14.60) and (14.69) constitutes the *transverse* one-loop two-point function in Yang-Mills theory, (14.57). Indeed we arrive at the transverse result

$$\left[\Pi_{\text{YM}}(p) \right]_{\mu\nu}^{ab} = -\delta^{ab} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) N_c g^2 \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{\Delta^{2-\frac{d}{2}}} \Gamma\left(2 - \frac{d}{2}\right) \left[\left(1 - \frac{d}{2}\right) (1-2\alpha)^2 + 2 \right]. \quad (14.63)$$

Equation (14.63) is a first key result of this Section. It provides us with the full gluon-two-point function, that can be used to compute the running of the strong coupling in pure Yang-Mills theory. We emphasise in this context that the STI relations between the different couplings allows us to compute this coupling even in Yang-Mills theory from the quark-gluon coupling: we simply have to set $N_f = 0$ for all vertex and two-point function corrections. In the present case of the gluon two-point function this simply amounts to dropping the quark loop.

For the final Yang-Mills results we have to take the limit $\epsilon \rightarrow 0$ with $d = 4 - 2\epsilon \rightarrow 4$. With **(D.6)** this leads us to

Gluon vacuum polarisation in Yang-Mills theory

$$[\Pi_{\text{YM}}(p)]_{\mu\nu}^{ab} = -\delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \frac{g^2 N_c}{16\pi^2} \left[\frac{5}{3} \left(\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2} \right) + \text{finite} \right], \quad (14.64)$$

in the Feynman gauge with $\xi = 1$, see **(14.53)**. Note that the finite terms also include a $-5/3 \ln(4\pi)$ term that stems from the expansion of $1/(4\pi)^{d/2}$ with $d/2 = 2 - \epsilon$. We have already discussed in **Chapter 7**, that singular diagrams in dimensional regularisation always come with the combination

$$\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2}. \quad (14.65)$$

This carries the relation between the regularisation with $1/\epsilon$, the RG-scale $\ln \mu$ and the momentum scaling with $-\ln p^2$. **Equation (14.65)** also opens the possibility to compute the prefactor of the momentum (or μ) running from that of the divergence $1/\epsilon$, which is simpler to compute. We shall use this option later within the computation of the quark-gluon vertex in **Section 14.2.3**.

Note also that the current computation in the Feynman gauge is straightforwardly extended to general ξ , leading to the result

$$[\Pi_{\text{YM}}(p)]_{\mu\nu}^{ab} = -\delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \frac{g^2 N_c}{16\pi^2} \left[\left(\frac{5}{3} + \frac{1}{2}(1 - \xi) \right) \left(\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2} \right) + \text{finite} \right], \quad (14.66)$$

and in the Landau gauge with $\xi = 0$ the prefactor changes from $5/3 \rightarrow 13/6$. It is this gauge which is used for most non-perturbative computations.

The Yang-Mills result **(14.64)** or **(14.66)** is readily upgraded to full QCD. The remaining quark loop contribution Π_{qugl} is computed with the QCD Feynman rules in **Appendix G** as

$$[\Pi_{\text{qugl}}(p)]_{\mu\nu}^{ab} = -(-1)g^2 \mu^{2\epsilon} \text{tr}_f T^a T^b \left[\int \frac{d^d l}{(2\pi)^d} \text{tr}_{\text{Dirac}} \frac{1}{i\cancel{l} + m} \gamma_\nu \frac{1}{i(\cancel{l} + p) + m} \gamma_\mu \right], \quad (14.67)$$

with $T^a = t_f^a$. In **(14.67)** we have restricted ourselves to one flavour, $N_f = 1$. As the quark-gluon vertex is flavour diagonal, the **(14.67)** is generalised straightforwardly to N_f flavours with

$$\Pi_{\text{qugl}}(p) = \sum_{i=1}^{N_f} \Pi_{\text{qugl}}(p, m_i). \quad (14.68)$$

where the second argument on the right hand side indicates the different current quark masses for the different flavours, see **Table 14.1**.

The trace $\text{tr}_f T^a T^b = 1/2 \delta^{ab}$, see **(13.16)**. The momentum integral in the square bracket is nothing but that already encounter in the discussion of the one-loop vacuum polarisation of the photon. The computation is done in detail in between **(7.121)** and **(7.134)** in **Section 7.3.1**. With the result **(7.134)** obtained there, we arrive at the transverse expression for the quark loop,

$$\begin{aligned} [\Pi_{\text{qugl}}(p)]_{\mu\nu}^{ab} &= -\frac{\delta^{ab}}{2} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) g^2 \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{(\Delta + m^2)^{2-\frac{d}{2}}} \frac{4}{d-1} \frac{1}{p^2} \\ &\times \left[(d-2)\Gamma\left(1 - \frac{d}{2}\right)(\Delta + m^2) - (d-2)\Gamma\left(2 - \frac{d}{2}\right)(2\Delta + m^2) + d\Gamma\left(2 - \frac{d}{2}\right)m^2 \right]. \end{aligned} \quad (14.69)$$

Note that the expression seemingly is irregular for $p^2 \rightarrow 0$ due to the $1/p^2$ -term. However, in this limit the second line is proportional to p^2 and the potential irregularity is absent. Finally we take the limit $\epsilon \rightarrow 0$. For $\epsilon \rightarrow 0$ we deduce from (14.69),

$$\begin{aligned} [\Pi_{\text{qugl}}(p)]_{\mu\nu}^{ab} &= \delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \frac{4g^2}{(4\pi)^2} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{(\Delta + m^2)^\epsilon} \alpha(1-\alpha) \Gamma\left(2 - \frac{d}{2}\right) + \text{finite} \\ &= \delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \frac{2}{3} \frac{g^2}{16\pi^2} \left[\left(\frac{1}{\epsilon} - \int_0^1 d\alpha \ln \frac{\Delta + m^2}{\mu^2} \right) + \text{finite} \right]. \end{aligned} \quad (14.70)$$

Equation (14.70) depends on the quark mass m . Finally we aim at the computation of the running coupling at asymptotically large momenta with $m^2/p^2 \rightarrow 0$. This amounts to using $m^2 = 0$, and the logarithm $\ln(\Delta + m^2)/\mu^2$ reduces to $\ln p^2/\mu^2 + \ln \alpha(1-\alpha)$, where we have used (14.56). The α -integral of the latter term can be absorbed into the finite part, while that of the former term simply gives $\ln p^2/\mu^2$.

In summary we get for the one-loop correction of the gluon two-point function Π in QCD for $m = 0$ and N_f flavours,

Gluon vacuum polarisation in QCD

$$[\Pi(p)]_{\mu\nu}^{ab} = \delta^{ab} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \frac{g^2}{16\pi^2} \left[\left(\frac{5}{3} + \frac{1}{2}(1-\xi) \right) N_c - \frac{2}{3} N_f \right] \left(\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2} \right) + \text{finite}, \quad (14.71)$$

which reduces to the Yang-Mills result (14.64) for $N_f = 0$. Equation (14.71) is transverse, as required by the STI (13.171). While it shares this property with the vacuum polarisation Π_{QED} of the photon, its ξ -dependence reflects the fact that in contradistinction to the Π_{QED} , the vacuum polarisation Π in QCD cannot be related directly to the running coupling, and to any observable for that matter.

14.2.2. Quark self-energy

As discussed below (14.49), the other two-point function required for the computation of the running coupling from the quark-gluon scattering, is the quark-self energy. As for the vacuum polarisation of the gluon we perform this computation in the Feynman gauge with $\xi = 1$, and go straight away into the chiral limit $m = 0$. With the Feynman rules in Appendix G we get

$$\Sigma(p) = -g^2 \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} T^a \gamma_\mu \frac{1}{i(l + \not{p})} \gamma_\mu T^a, \quad (14.72)$$

with $T^a = t_f^a$, separately for each flavour. Equation (14.72) is a matrix in the fundamental representation of the gauge group and in Dirac space. The sum over the generators $T^a T^a = (N_c^2 - 1)/N_c \mathbb{1}_f$ has been discussed in Appendix 8.1, see (H.3) and (H.4). We arrive at

$$\Sigma(p) = i\not{p} g^2 \frac{(N_c^2 - 1)}{2N_c} \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha (1 - \alpha)(d - 2) \Gamma\left(2 - \frac{d}{2}\right) \frac{\mu^{2\epsilon}}{\Delta^{2-\frac{d}{2}}}, \quad (14.73)$$

and in the limit $\epsilon \rightarrow 0$ we obtain

$$\Sigma(p) = i\not{p} \frac{g^2}{16\pi^2} \frac{N_c^2 - 1}{2N_c} \left[\left(\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2} \right) + \text{finite} \right]. \quad (14.74)$$

The result is readily extended to general ξ , leading to

Quark self energy in QCD

$$\Sigma(p) = i\not{p} \frac{g^2}{16\pi^2} \frac{N_c^2 - 1}{2N_c} \left[\xi \left(\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2} \right) + \text{finite} \right]. \quad (14.75)$$

Interestingly, in the Landau gauge the singular contribution drops out. This entails that the quark propagator does not require renormalisation at one-loop.

14.2.3. Quark-gluon vertex

The final ingredient in the computation of the running coupling is that of the one-loop correction of the quark-gluon vertex. After the very detailed computations for the propagators we shorten that of the vertex and concentrate on the singular piece alone. The full vertex correction is given by

$$\Gamma_{q\bar{q}A}^{(3)}(p, -q, q - p) \Big|_{1\text{loop}} = V_{\text{qugl}^3}(p, q) + V_{3\text{glqugl}^2}. \quad (14.76)$$

With the Feynman rules in [Appendix G](#) we get for the vertex correction with three quark-gluon vertices,

$$V_{\text{qugl}^3}(p, q) = ig^3 T^b T^a T^b \int \frac{d^d l}{(2\pi)^d} \frac{\gamma_\nu (l + \not{q}) \gamma_\mu (l + \not{p}) \gamma_\nu}{l^2 (l + q)^2 (l + p)^2}, \quad (14.77)$$

with $T^a = t_f^a$. [Equation \(14.77\)](#) is straightforwardly computed using the Feynman trick and the master integrals in [Appendix D](#). For the present analysis we are only interested in the singular piece which we get by evaluating [\(14.77\)](#) at $p = q = 0$, and using the same completion with unity as for the tadpole diagram in the gluon vacuum polarisation. This allows us to extract the coefficient of the logarithmic RG-scale and momentum running from that of the divergence $1/\epsilon$. We proceed by using the symmetry of the angular integration, we get for the numerator $l \gamma_\mu l \rightarrow 1/d \gamma_\rho \gamma_\mu \gamma_\rho l^2$, and hence the integrand in [\(14.77\)](#) reduces to

$$\frac{\gamma_\nu l \gamma_\mu l \gamma_\nu}{l^2 l^2} \simeq \frac{1}{d} \frac{\gamma_\nu \gamma_\rho \gamma_\mu \gamma_\rho \gamma_\nu}{l^2 l^2} = \gamma_\mu \frac{(d - 2)^2}{d} \frac{1}{(l^2)^2}. \quad (14.78)$$

We also use the following relation for the generators in the fundamental representation,

$$T^b T^a T^b = T^b T^b T^a + T^b [T^a, T^b] = C_2(f) T^a + iT^b f^{abc} T^c = \left[C_2(f) - \frac{1}{2} C_2(ad) \right] T^a. \quad (14.79)$$

with (H.3) and (H.4). For the products of Dirac matrices we use the identities (6.17) and further ones following from the Clifford algebra. The remaining momentum integral is treated as the tadpole with a completion of unity, and we arrive at the final result

$$V_{\text{qugl}^3}(0, 0) \simeq igT^a \gamma_\mu \frac{g^2}{16\pi^2} \left(\frac{1}{2} \frac{N_c^2 - 1}{N_c} - \frac{N_c}{2} \right) \left[\Gamma \left(2 - \frac{d}{2} \right) + \text{finite} \right]. \quad (14.80)$$

With a similar analysis we compute the vertex correction with one three-gluon vertex,

$$V_{3\text{glqugl}^2}(0, 0) \simeq igT^a \gamma_\mu \frac{g^2}{16\pi^2} \frac{3}{2} N_c \left[\Gamma \left(2 - \frac{d}{2} \right) + \text{finite} \right] \quad (14.81)$$

In summary we obtain

Quark-gluon vertex

$$\Gamma_{q\bar{q}A}^{(3)}(0, -0, 0) \Big|_{1\text{loop}} = igT^a \gamma_\mu \frac{g^2}{16\pi^2} \left(\frac{1}{2} \frac{N_c^2 - 1}{N_c} + \frac{N_c}{2} \right) \left[\Gamma \left(2 - \frac{d}{2} \right) + \text{finite} \right], \quad (14.82)$$

where the group factor in is the sum of the second Casimir of the adjoint and fundamental representations,

$$\frac{1}{2} \frac{N_c^2 - 1}{N_c} + \frac{N_c}{2} = C_2(f) + C_2(ad). \quad (14.83)$$

For general gauge fixing parameter ξ and taking the limit $\epsilon \rightarrow 0$, the result (14.82) reads,

$$\Gamma_{q\bar{q}A}^{(3)}(0, -0, 0) \Big|_{1\text{loop}} = igT^a \gamma_\mu \frac{g^2}{16\pi^2} \left[\left(1 - \frac{1 - \xi}{4} \right) N_c + \xi \frac{N_c^2 - 1}{2} \right] \left[\frac{1}{\epsilon} + \text{finite} \right]. \quad (14.84)$$

This concludes the discussion of quark- and gluon two-point functions and the quark-gluon vertex.

14.2.4. Running coupling

With the results (14.71), (14.75) and (14.82) we are in the position to compute the RG-scale running of the renormalised coupling and hence the asymptotic momentum running of the running coupling, utilising (14.47) and (14.48). For the computation of the β -function (14.44), we use the relation (14.49), which leads us to

$$\beta_g = -\mu \frac{d \ln Z_g}{d\mu} = -\mu \frac{d \ln Z_{q\bar{q}A}}{d\mu} + \mu \frac{d \ln Z_q}{d\mu} + \frac{1}{2} \mu \frac{d \ln Z_A}{d\mu}. \quad (14.85)$$

Now we put to work the relation (14.65): For a given renormalisation factor $Z = 1 + \delta Z$ we find for the divergent piece, together with the logarithmic RG-scaling,

$$\delta Z \simeq \#_Z \left(\frac{1}{\epsilon} - \ln \frac{p^2}{\mu^2} \right), \quad \text{and} \quad \mu \frac{d \ln Z}{d\mu} = 2 \#_Z = 2\epsilon \delta Z|_{O(1/\epsilon)}. \quad (14.86)$$

Equation (14.86) entails that twice the prefactor of the divergent $1/\epsilon$ term provides the RG-scaling of Z . We also note in passing that at one loop we can rewrite Z_g in terms of the δZ 's,

$$Z_g = 1 + \delta Z_{q\bar{q}A} - \delta Z_q - \frac{1}{2} \delta Z_A, \quad (14.87)$$

which reflects the full additive relation in (14.85). This leads us to

$$\beta(g) = 2 \lim_{\epsilon \rightarrow 0} \left[\epsilon \left(-\delta Z_{q\bar{q}A} + \delta Z_q + \delta Z_A \right) \right]. \quad (14.88)$$

Now we collect the δZ 's from (14.71), (14.75) and (14.82). For the quark and gluon wave function renormalisations we find

$$\delta Z_A \approx \frac{g^2 N_c}{16\pi^2} \left[\left(\frac{5}{3} + \frac{1}{2}(1-\xi) \right) - \frac{2}{3} N_f \right] \frac{1}{\epsilon}, \quad \delta Z_q = -\frac{g^2}{16\pi^2} \frac{N_c^2 - 1}{2N_c} \xi \frac{1}{\epsilon}, \quad (14.89)$$

and for the vertex renormalisation factor we find

$$\delta Z_{q\bar{q}A} \approx -\frac{g^2}{16\pi^2} \left[\left(1 - \frac{1-\xi}{4} \right) N_c + \xi \frac{N_c^2 - 1}{2N_c} \right] \frac{1}{\epsilon}. \quad (14.90)$$

We insert (14.89) and (14.90) into (14.88) and arrive at

$$\beta(g) = \frac{g^2}{16\pi^2} 2 \left[\underbrace{\left(1 - \frac{1-\xi}{4} \right) N_c + \xi \left(\frac{N_c^2 - 1}{2N_c} \right)}_{-\delta Z_{q\bar{q}A}} + \underbrace{\left(-\frac{N_c^2 - 1}{2N_c} \xi \right)}_{\delta Z_q} + \underbrace{\frac{1}{2} \left(\frac{5}{3} + \frac{1}{2}(1-\xi) \right) N_c - \frac{2}{3} N_f}_{\delta Z_A} \right] + O(g^4). \quad (14.91)$$

First we notice that the ξ -dependent terms in (14.91) cancel out, and the result is ξ -independent as it must. In summary we are led to the one-loop β -function of QCD,

One-loop β -function of QCD

$$\beta(g) = -\frac{g^2}{16\pi^2} \left[\frac{11}{3} N_c - \frac{2}{3} N_f \right]. \quad (14.92)$$

Note that the Yang-Mills β -function is obtained for $N_f = 0$, which amounts to dropping the contribution of the quark-loop to the vacuum polarisation. Curiously, the wave function of the quark and the renormalisation factor of the quark-gluon vertex still contribute, and we compute the running Yang-Mills coupling from a scattering process with quarks. The respective approximation to QCD is the quenched approximation, in which all diagrams with quark loops are dropped but mixed diagrams are still present.

Equation (14.92) entails that QCD is asymptotically free for

$$N_f < \frac{11}{2} N_c + O(2\text{loop}), \quad (14.93)$$

which of course includes Yang-Mills theory with $N_f = 0$.

For the following discussion of $\alpha_s(p^2)$ we define the β -function in terms of the n -loop coefficients. For a convenient parametrisation we utilise, that the expansion parameter of perturbation theory is $\alpha_s/(4\pi)$, which is also reflected in the one-loop β -function in (14.92). Moreover, we pull out a global minus sign that signals asymptotic freedom and yield

$$\beta(\alpha_s) = -\frac{\alpha_s}{4\pi} \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi} \right)^n. \quad (14.94)$$

with the one- and two-loop coefficients

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f, \quad \beta_1 = \frac{102}{3} N_c - \frac{38}{3} N_f. \quad (14.95)$$

and further coefficients have been computed up to β_5 . The running coupling $\alpha_s(p)$ is obtained with integrating the β -function (14.94) in a given order of perturbation theory. To that end we use the implicit solution of (14.48),

$$\frac{d\alpha_s}{\alpha_s} \frac{1}{2\beta(\alpha_s)} = \frac{dp}{p}, \quad \longrightarrow \quad \int_{\alpha_s}^{\alpha_s(p^2)} \frac{dx}{x} \frac{1}{\beta_g(x)} = \ln \frac{p^2}{\mu^2}, \quad \text{with } \alpha_s = \alpha_s(\mu^2). \quad (14.96)$$

At one-loop, the implicit solution in (14.96) can be solved, to wit,

One-loop running coupling of QCD

$$\alpha_s(p^2) = \frac{\alpha_s}{1 + \beta_0 \frac{\alpha_s}{4\pi} \ln \frac{p^2}{\mu^2}}, \quad (14.97)$$

which vanishes for $p^2 \rightarrow \infty$ for (14.93). Equation (14.97) has a seeming dependence on μ via that on $\alpha_s = \alpha_s(\mu^2)$ and the explicit μ -dependence. Its μ -independence is clearly seen by using

$$1 = \frac{\beta_0}{4\pi} \alpha_s \ln e^{\frac{4\pi}{\beta_0 \alpha_s}}, \quad (14.98)$$

for unity in the denominator in (14.97). Then the denominator has a global factor $\alpha_s \beta_0 / (4\pi)$ and α_s cancels out with the numerator. We get

$$\alpha_s(p^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln \frac{p^2}{\Lambda_{\text{QCD}}^2}}, \quad \text{with } \Lambda_{\text{QCD}}^2 = \mu^2 e^{-\frac{4\pi}{\beta_0 \alpha_s}} \quad \text{and} \quad \mu \frac{d\Lambda_{\text{QCD}}^2}{d\mu} = 0. \quad (14.99)$$

with the physical scale Λ_{QCD} at one loop. In (14.99), any seeming reference to the RG-scale is removed, and the dependence of the running coupling on the QCD scale Λ_{QCD} is made explicit. In perturbation theory, Λ_{QCD} is related to the location of the infrared Landau pole at $p^2 = \Lambda_{\text{QCD}}^2$, beyond perturbation theory it is related to the dynamically generated mass gap in QCD.

In the remainder of this Section we discuss the phenomenological consequences of the occurrence of the running coupling for scattering processes and the interpretation of experimental accelerator data. To begin with, the strength of QCD scattering processes with a given transfer momentum p^2 is directly proportional to the size of the running coupling at this momentum scale. While the β -function and the solution for the running coupling (14.97) or (14.99) resolves the momentum dependence, its overall size has to be determined with an experimental measurement at some momentum scale $p^2 = \mu^2$, see also its parametrisation in (14.99). Note that then μ should not be understood as the RG-scale but as some reference scale where the running coupling is known. For example, we can fix it at the Z-mass scale, where we have

$$\alpha_s(M_Z^2) \approx 0.118, \quad \text{with } M_Z \approx 91.19 \text{ GeV}, \quad (14.100)$$

see the Review of Particle Physics 2022 [1]. Using the one-loop relation for Λ_{QCD} in (14.99) with $\mu = M_Z$ and $N_f = 5$, we arrive at

$$\Lambda_{\text{QCD}}^{(1\text{loop})} = 88 \text{ MeV}, \quad \text{and} \quad \alpha_s(m_\tau) = 0.273, \quad \text{with } m_\tau \approx 1.77 \text{ GeV}. \quad (14.101)$$

The running coupling in (14.101) was determined with five massless flavours as the top quark decouples below its mass threshold of $m_t \approx 170 \text{ GeV}$. In (14.101) we have also provided the one-loop estimate of the coupling at the tau-scale with $m_\tau = 1.77 \text{ GeV}$, which clearly deviates from the experimental value and the theoretical

prediction in [Figure 14.3](#). Reducing the number of flavours further, the value of the coupling at the tau-scale increases, but the overall slope does not agree. Evidently, the one-loop approximation fails to capture the correct running over large momentum regimes.

The above one-loop estimate can be readily improved in two directions. First of all the successive decoupling of the quarks can be treated more accurately: The quark masses in [Table 14.1](#) lead to mass thresholds that can be accommodated in the RG-equation [\(14.44\)](#) with the inclusion of mass-dependences,

$$\mu \frac{d}{d\mu} O = \left(\mu \frac{\partial}{\partial \mu} + \beta_g g \frac{\partial}{\partial g} - \gamma_m m \partial_m \right) O = 0, \quad \text{with} \quad \beta_g = -\mu \frac{d \ln Z_g}{d\mu}, \quad \gamma_m = \mu \frac{d \ln Z_m}{d\mu}. \quad (14.102)$$

with

$$O(p) = O\left(\alpha_{s,\text{ren}}(\mu^2), \frac{p^2}{\mu^2}, \frac{m^2(\mu^2)}{p^2}\right). \quad (14.103)$$

The definition [\(14.102\)](#) includes the mass thresholds and reduces to the mass-independent ones for $m = 0$.

Secondly, we can go beyond the one-loop approximation used so far. The two-loop β -function can also be integrated, leading to the implicit solution

$$\frac{\alpha_s(p^2)}{4\pi} = \frac{\beta_0}{\beta_1} \frac{1}{1 + W_{-1}(y)}, \quad \text{with} \quad y = -\frac{\beta_0^2}{e\beta_1} \left(\frac{\Lambda_{\text{QCD}}^2}{p^2} \right)^{\frac{\beta_0^2}{\beta_1}}, \quad (14.104)$$

where W_{-1} denotes the lower branch of the real-valued Lambert function, which solves $y = W(y) \exp W(y)$. An approximate solution used often in the literature is given by

$$\alpha_s(p^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \frac{\alpha_s(\mu^2)}{4\pi} \left[1 + \frac{\alpha_s(\mu^2)}{4\pi} \frac{\beta_1}{\beta_0} \right] \ln \frac{p^2}{\mu^2}}. \quad (14.105)$$

The [Equations \(14.104\)](#) and [\(14.105\)](#) together with $N_f = 5$ already reduce the disagreement with the experimental data in [Figure 14.3](#).

The lack of global precision can also be accommodated by choosing a smaller renormalisation scale. For example, if the running coupling is fixed with the experimental values at $p \approx 40 \text{ GeV}$ with

$$\alpha_s(40 \text{ GeV}) \approx 0.140 \text{ GeV}, \quad (14.106)$$

we are led to

$$\alpha_s^{1\text{loop}}(m_\tau) \approx 0.30, \quad \alpha_s^{2\text{loop}}(m_\tau) \approx 0.33. \quad (14.107)$$

Of course, the good accuracy for small momentum values is paid for with a lack of accuracy for larger values. At the Z mass-scale the two-loop coupling with [\(14.106\)](#) yields

$$\alpha_s^{2\text{loop}}(m_\tau) \approx 0.122, \quad (14.108)$$

in comparison to the experimental value [\(14.100\)](#). This entails that even for the large energies at LHC with its successively decreasing coupling we require high loop orders in order to get quantitative precision. We emphasise that this is of paramount importance for the search of new physics beyond the Standard Model (BSM): while QCD sub-processes only constitute (part of) the background, they have to be subtracted accurately in order to get access to potential BSM signals in the data.

In turn, for $p \rightarrow 0$, perturbation theory clearly fails, signalled by the Landau pole. This regime with chiral symmetry breaking and confinement can only be accessed within non-perturbative approaches.

15. Lattice Field Theory

In the present Chapter we introduce a fully non-perturbative approach for solving the functional integral of a quantum field theory, lattice field theory. One of the chief advantages of lattice field theory is the property, that it allows for a numerical full solution of QFTs in terms of the generating functional or its correlation functions. For that purpose the infinite-dimensional statistical integral is approximated by a finite dimensional one on a space-time or spatial lattice.

A simple example is a quantum field theory with a real scalar field $\phi(t, \vec{x})$ in a three-dimensional spatial cube \mathcal{B} with length L in all directions. This cube has the volume $\mathcal{V} = L^3$, and we now approximate the box with a regular rectangular grid of points with a (lattice) distance $a = L/N$ and a (large) natural number N . Then, the field takes real values on the grid points, and the infinite-dimensional path integral turns into a high- but finite-dimensional integral with N^3 computations. Evidently, in the limit $N \rightarrow \infty$ the full path integral is recovered if the limit exists. Moreover, for small enough N , the finite-dimensional integral maybe solved numerically with Monte-Carlo methods, typically applied to high-dimensional integration problems.

This set-up leaves us with the exiting possibility to simply solve interacting quantum field theories numerically within $N \rightarrow \infty$ of the corresponding lattice field theory. An obvious and intriguing advantage of such a formulation is, that it can be formulated for generic coupling strength and does not rely on a small coupling or other constraints, that typically come with systematic expansion schemes. Put differently, lattice field theories are the method of choice for strongly correlated systems, if the $N \rightarrow \infty$ can be taken.

This leaves us with the following tasks:

- Task (i)* Recast a given continuum quantum field theory as the corresponding lattice field theory, while trying to preserve as much of the symmetries of the QFT at hand. Evidently, for the example above with a regular rectangular lattice spatial rotation invariance is broken, and is only recovered in the limit $N \rightarrow \infty$. This statement extends to full space-time symmetries, a fact that e.g. complicates the construction of supersymmetric lattice theories. A further important symmetry that requires attention is the chiral symmetry. This symmetry plays a very important rôle in QCD and beyond, and will be discussed thoroughly, when discussing lattice fermions. Finally, the lattice formulations of gauge theories has great advantages over corresponding (diagrammatic) formulations in the vacuum: lattice gauge theories are explicitly gauge invariant while most continuum approaches are based on gauge fixed formulations.
- Task (ii)* Control of the limit $N \rightarrow \infty$. Here we have to distinguish the thermodynamic limit, $\mathcal{V} \rightarrow \infty$ and the continuum limit, $a \rightarrow 0$. While these limits both require $N \rightarrow \infty$, they are different. In any case they are signalled by the respective scaling behaviour (with either a or \mathcal{V}), and its evaluation is one of the largest systematic error sources in lattice field theories.
- Task (iii)* Complex actions. Many systems with interesting physics, e.g. finite density systems, spin imbalance, mass imbalance, competing orders, require the computation of high dimensional integrals with complex (or at least non-positive) measure factors. The respective numerics may be *NP-hard*.

In this Chapter we provide an introduction to lattice field theories, concentrating on the general structure and not on the computational details. Still, the present introduction allows us to analytically compute the linear confining potential in the strong coupling phase on the lattice. While this is not the physical phase of lattice gauge theory, it is an illuminating computation and result.

15.1. Scalar quantum field theory on the Lattice

Our starting point is the Euclidean generating functional (10.67) for a real scalar field $\phi \in \mathbb{R}$, derived in Section 10.3 as the Wick rotation of the Minkowski path integral. While $\phi \in \mathbb{R}$ described a neutral spin 0 field, a complex scalar field $\phi \in \mathbb{C}$ describes a charged spin 0 field. For the latter field we have the Euclidean path integral

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \exp \left\{ -S[\phi] + \int_x J(x)\phi(x) \right\}, \quad (15.1a)$$

with the path integral measure

$$\mathcal{D}\phi(x) = \prod_x d\phi_x, \quad (15.1b)$$

where $\phi(x)$ is the integration measure of the complex variable $\phi_x = \phi(x)$. The form (15.1b) is already reminiscent of the product form on the space-time lattice Figure 10.1 that was used at an intermediate step of the derivation of the path integral in Chapter 9. The classical action in (15.1a) is given by

$$S[\phi] = \int d^d x \left[\phi^\dagger(x) (-\Delta) \phi(x) + V(\phi^\dagger \phi) \right], \quad \text{with } \Delta = \partial_\mu^2. \quad (15.1c)$$

The potential in (15.1c) is a ϕ^4 -potential with

$$V(\rho) = m_\phi^2 \rho + \frac{\lambda_\phi}{2} \rho^2 \quad \text{where} \quad \phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2), \quad \text{and} \quad \rho = \phi^\dagger \phi = \frac{1}{2} (\phi_1^2 + \phi_2^2). \quad (15.1d)$$

The scalar theory shows a sufficiently rich phase structure and strongly correlated regimes to be of interest in its own, see in particular Section 11.2.1 (3d) and Section 11.2.2 (4d) in Section 11.2. We shall also use it in the present Chapter on lattice field theory to introduce both conceptual particularities on the lattice as well as numerical techniques.

15.1.1. Lattice action of scalar field theories

The derivation of (15.1) in Chapter 9 already carries the discretisation with it, as also mentioned above. See in particular the depiction of the two-dimensional space-time lattice in Figure 10.1 with the temporal lattice distance Δt and the spatial lattice distance ΔL . Instead of taking the limits $\Delta t, \Delta L \rightarrow 0$, we identify both distances with a (hyper)cubic lattice distance a with $\Delta t = \Delta L = a$. A depiction of this lattice in $d = 2$ dimensions is found in Figure 15.1. This lattice is defined by the box $\mathcal{B}_{a,L}$ with periodic boundary conditions.

$$\mathcal{B}_{a,L} = \left\{ x = a n, \quad \text{with } n = n_\mu \hat{e}_\mu, \quad \mu = 1, \dots, d \quad \text{with } n_\mu \in \mathbb{Z} \quad \text{and} \quad |n_\mu| \leq L/a = N \right\}. \quad (15.2)$$

with the Cartesian orthonormal basis

$$(\hat{e}_\mu)_\nu = \delta_{\mu\nu}, \quad \text{with} \quad \hat{e}_\mu \hat{e}_\nu = \delta_{\mu\nu}. \quad (15.3)$$

While for our working horse example we consider a hyper cubic lattice, lattices with different spatial and temporal extend are commonly used, $L = L_i = L_j \neq L_0$ with $i, j = 1, \dots, d-1$. This typically also leads to a different number of points in spatial and temporal directions, with $N_\tau = L_0/a \neq N = N_i = L_i/a$. Note that it

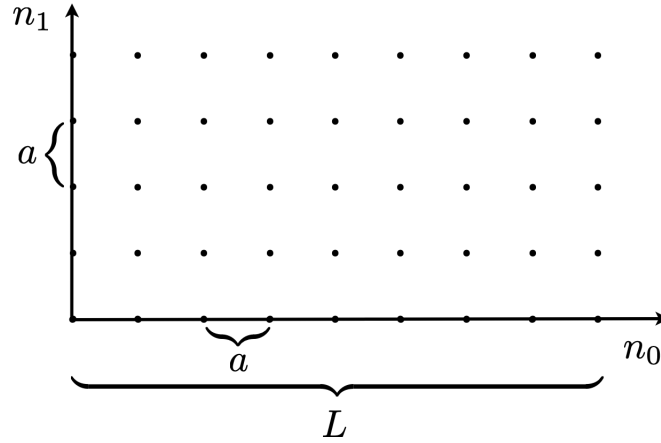


Figure 15.1.: Depiction of a finite 2-dimensional lattice with a lattice size L and a lattice distance a . The space-time vector takes values $x = n_0\hat{e}_0 + n_1\hat{e}_1$, and $L/a = 8$.

is indeed N_τ, N which defines the lattice scales together with the dimensionless parameters of the action. This will be discussed later.

The periodic boundary conditions are either implemented directly on the box by identifying opposite faces or simply considering only fields ϕ with

$$\phi(x + L\hat{\mu}) = \phi(x), \quad \text{for } \mu = 0, 1, \dots, d-1, \quad \text{and } \hat{\mu} = \hat{e}_\mu, \quad (15.4)$$

that live on the *lattice sites* $x = n_\mu a$. With the lattice fields in (15.4), we have to define the lattice version of the classical action (15.1c), as well as the lattice analogue of the path integral measure $\mathcal{D}\phi$ in (15.1a), already indicated in (15.1b), and (10.12) in Chapter 9. To begin with, the space-time integration in the action turns into finite sums,

$$\int_{\mathcal{B}_L} d^d x \rightarrow a^d \sum_{x \in \mathcal{B}_{a,L}}, \quad (15.5)$$

where $\mathcal{B}_L = \mathcal{B}_{0,L} = L^d$, and the sum sums over the $(L/a)^d = N^d$ lattice points, or, more generally, $N_\tau \times N^{d-1}$ lattice points. Quantum field theories on such a lattice are both, ultraviolet finite and infrared finite. The latter property is evident as the theory is defined on a finite volume, and the maximal correlation length on such a lattice is given by the *lattice extend* L . In turn, ultraviolet divergences occur for infinitely small distances, and the minimal distance on the lattice is the *lattice distance* a . Therefore we do not expect any divergences to occur for finite a and L , both are controlled on a finite lattice. However, this also entails that in particular the ultraviolet divergences resurface in the limit $a \rightarrow 0$, which has to be carefully studied, invoking renormalisation group techniques. This will be discussed later.

As already indicated above, it is convenient to express everything in dimensionless quantities in units of the lattice distance. For example, the field can now be written in terms of the dimensionless field $\hat{\phi}$ with

$$\phi(na) = a^{1-d/2} \hat{\phi}_n, \quad \rho(na) = a^{2-d} \hat{\rho}_n, \quad (15.6a)$$

where the prefactor $a^{1-d/2}$ takes care of the canonical dimension of the scalar field ϕ . In particular, in $d = 4$ we have $\phi = 1/a \hat{\phi}$. Moreover, the dimensionless field $\hat{\phi}_n$ can only depend on n and not on the lattice distance a . Similarly we can write the mass m and the coupling λ in (15.1d) in terms of a scaling prefactor in terms of the lattice distance and the dimensionless lattice mass and coupling,

$$m_\phi = \frac{1}{a} \hat{m}_\phi, \quad \lambda_\phi = a^{d-4} \hat{\lambda}_\phi, \quad (15.6b)$$

reflecting the standard canonical momentum dimensions of couplings and fields in QFT. For example, the momentum dimension of the coupling is $4 - d \geq 0$ for $d \leq 4$. This indicates that ϕ^4 -theories are renormalisable for dimensions $d \leq 4$. The upper limit, $d = 4$ is the critical dimension, and there ϕ^4 -theory faces the triviality problem: for all we know it has no UV closure.

With the dimensionless fields, couplings and masses the potential term (15.1d) is readily cast into its lattice form,

$$\int_{\mathcal{B}_L} V(\rho) = a^d \sum_{|n_\mu| \leq N} \left[a^{-d} \hat{m}_\phi^2 \hat{\rho}_n + a^{-d} \frac{\hat{\lambda}_\phi}{2} \hat{\rho}_n^2 \right] = \sum_{|n_\mu| \leq N} \left[\hat{m}_\phi^2 \hat{\rho}_n + \frac{\hat{\lambda}_\phi}{2} \hat{\rho}_n^2 \right]. \quad (15.7)$$

Note, that the lattice distance a has disappeared from (15.7), and the continuum limit $a \rightarrow 0$ has to be taken as the respective limit of the dimensionless mass and coupling \hat{m}_ϕ and $\hat{\lambda}_\phi$. This already indicates the property of lattice field theories, that the continuum limit is achieved within a scaling limit of the dimensionless parameters of the theory. Moreover, the exponential of the potential term factorises in a product of exponentials of the potential $\hat{V}(\hat{\rho}(n)) = \hat{m}_\phi^2 \hat{\rho}(n) + \hat{\lambda}_\phi/2 \hat{\rho}^2(n)$ on the single lattice points,

$$\exp \left\{ - \sum_{|n_\mu| \leq N} \left[\hat{m}_\phi^2 \hat{\rho}(n) + \frac{\hat{\lambda}_\phi}{2} \hat{\rho}^2(n) \right] \right\} = \prod_{|n_\mu| \leq N} e^{-\left[\hat{m}_\phi^2 \hat{\rho}(n) + \frac{\hat{\lambda}_\phi}{2} \hat{\rho}^2(n) \right]} = \prod_{|n_\mu| \leq N} e^{-\hat{V}(\hat{\rho}(n))}. \quad (15.8)$$

Clearly this factorisation takes place for all potentials V without derivative terms. This factorisation mirrors that in the path integral measure (15.1b), see also (10.12). On the lattice and in terms of the dimensionless field $\hat{\phi}_n$, the path integral measure (15.1b) turns into

$$\int \mathcal{D}\phi \rightarrow \prod_{|n_\mu| \leq N} \int_{\mathbb{C}} d\hat{\phi}(n) \simeq \prod_{|n_\mu| \leq N} \int_{\mathbb{R}} d\hat{\phi}_1(n) \int_{\mathbb{R}} d\hat{\phi}_2(n), \quad (15.9)$$

with integrations over the amplitude of the real and imaginary part of the dimensionless scalar field, $\hat{\phi}_1$ and $\hat{\phi}_2$ respectively. This leads to an interesting intermediate result,

$$\int \mathcal{D}\phi e^{-\int_x V(x)} \rightarrow \prod_{|n_\mu| \leq N} \int_{\mathbb{C}} d\hat{\phi}(n) e^{-\hat{V}(\hat{\rho}(n))}, \quad (15.10)$$

in the absence of the kinetic term the path integral factorises in a product of single site models that can be solved independently. We also remark that the above integral is manifestly finite as already argued above. The full lattice theory tends towards (15.10) in the *strong coupling limit*, $\hat{\lambda} \rightarrow \infty$. Seemingly this entails that QFTs with strong *physical* coupling can be solved trivially on the lattice. However, an inspection of (15.6b) leads to the conclusion that a strong but finite coupling λ requires $\hat{\lambda} \rightarrow 0$ for $d < 4$ according to the canonical dimensional running. In the presence of quantum effects this canonical running is augmented by an anomalous part as computed in the last Chapter in QCD, see Section 14.1. This analysis is deferred to Section 15.5.2, for the time being we just keep in mind that the continuum limit obviously requires a scaling limit of the dimensionless lattice parameters.

It is left to put forward the lattice version of the kinetic term, $-\int_x \phi^\dagger \Delta \phi$. This is done with a discretised version of the Laplace operator Δ in (15.1c). To begin with, we define the left, right and symmetric lattice derivatives,

$$\partial_\mu^L \phi(x) = \frac{\phi(x) - \phi(x - \hat{\mu}a)}{a}, \quad \partial_\mu^R \phi(x) = \frac{\phi(x + \hat{\mu}a) - \phi(x)}{a}, \quad \partial_\mu^S \phi(x) = \frac{\phi(x + \hat{\mu}a) - \phi(x - \hat{\mu}a)}{2a}, \quad (15.11a)$$

whose continuum limits with $a \rightarrow 0$ define the standard left, right and symmetric derivatives. In terms of the dimensionless lattice fields these derivatives turn into

$$\partial_\mu^{L,R,S} \phi(x) = a^{-\frac{d}{2}} \hat{\partial}^{L,R,S} \hat{\phi}_n, \quad (15.11b)$$

with the difference operators

$$\hat{\partial}_\mu^L \hat{\phi}_n = \hat{\phi}_n - \hat{\phi}_{n-\hat{\mu}}, \quad \hat{\partial}_\mu^R \hat{\phi}_n = \hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_n, \quad \hat{\partial}_\mu^S \hat{\phi}_n = \frac{1}{2} (\hat{\phi}_{n+\hat{\mu}} - \hat{\phi}_{n-\hat{\mu}}). \quad (15.11c)$$

While all the finite difference operators in (15.11) have the same continuum limit, the convergence of the path integral for the different choices may be quantitatively or even qualitatively different. For example, the symmetric operator $\hat{\partial}^S$ does not single out a direction, which may lead to a quicker convergence. In this context we also remark that the difference operators in (15.11) have a matrix form with $\hat{\partial}_{nm} \hat{\phi}_m = \hat{\partial} \hat{\phi}_n$ with

$$(\partial_\mu^L)_{nm} = \delta_{n,m} - \delta_{n-\hat{\mu},m}, \quad (\partial_\mu^R)_{nm} = \delta_{n+\hat{\mu},m} - \delta_{n,m}, \quad (\partial_\mu^S)_{nm} = \frac{1}{2} (\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}). \quad (15.12)$$

With the matrix representation (15.12) it follows straightforwardly, that the symmetric difference operator $i(\partial_\mu^S)_{nm}$ is a hermitian matrix, while the two other difference operators are not. We close this discussion of lattice derivative operators with the remark that (15.11c) are by no means the only possible choices: Clearly, $\partial^{L,R}$ are the only next neighbour definitions and $\hat{\partial}^S$ already connects next-to-next lattice sites. If we allow for a large and larger amount of lattice sites to be involved, this leads to a large number of possible lattice derivatives. Typically, for standard applications in scalar theories one sticks to the 'canonical' choices, but we shall pick up this discussion again in Section 15.2.1 on lattice fermions.

Finally, with the difference operators (15.11c) we can define lattice Laplacians. As for the derivatives, also the Laplacian is not unique. Here we take the symmetric choice,

$$\hat{\Delta} = \hat{\partial}_\mu^L \hat{\partial}_\mu^R = \hat{\partial}_\mu^R \hat{\partial}_\mu^L, \quad \text{with} \quad \hat{\Delta}_{nm} = \sum_{\mu>0} [\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}]. \quad (15.13)$$

Evidently, this matrix is hermitian, while the choices $(\hat{\partial}_\mu^{L,R})^2$ are not. With the lattice Laplacian (15.13) we arrive at the lattice version of the kinetic term,

$$\int_x \phi^\dagger (-\Delta + m^2) \phi \rightarrow \sum_{n,m} \hat{\phi}_n^\dagger K_{nm} \hat{\phi}_m, \quad (15.14)$$

where we included the mass term from the potential and the restriction $|n_\mu|, |m_\mu| \leq N$ is implied in the sum. The hermitian kinetic operator K_{nm} in (15.14) reads

$$K_{nm} = - \sum_{\mu>0} [\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}] + \hat{m}^2 \delta_{nm} = - \sum_{\mu>0} [\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m}] + (\hat{m}^2 + 2d) \delta_{nm}. \quad (15.15)$$

This leads us to our final expression for the lattice path integral of a complex scalar field in a finite volume $\mathcal{V} = L^d$ with the lattice distance a ,

$$Z[\hat{J}] = \frac{1}{\mathcal{N}} \int \prod_n d\hat{\phi}_n \exp \left\{ -S[\hat{\phi}] + 2 \sum_n \hat{J}_n \hat{\phi}_n \right\}, \quad (15.16)$$

with $2\hat{J}_n\hat{\phi}_n = \hat{J}_{1,n}\hat{\phi}_{1,n} + \hat{J}_{2,n}\hat{\phi}_{2,n}$, the normalisation \mathcal{N} and the lattice action

$$S[\hat{\phi}] = - \sum_{n,m} \hat{\phi}_n^\dagger K_{nm} \hat{\phi}_m + \frac{\hat{\lambda}}{2} \sum_n \hat{\rho}_n^2, \quad \text{with} \quad \hat{\rho}_n = \hat{\phi}_n^\dagger \hat{\phi}_n. \quad (15.17)$$

We have already considered this theory in the absence of a kinetic term, see (15.10). Now we discuss the opposite limit, we drop the interaction term (but keep the mass term). This leads us to the (Gaußian) free theory, which can be integrated analytically. We get from (15.16) for the generating functional of the free theory, $Z_0[J]$,

$$Z_0[J] \simeq \frac{1}{\det K_{nm}} e^{\frac{1}{2} \sum_{n,m} \hat{J}_n K_{nm}^{-1} \hat{J}_m}. \quad (15.18)$$

where both the Gaußian integral over the real and imaginary part of the field lead to a factor $1/\sqrt{\det \bar{K}}$. With derivatives w.r.t. the current we generate the correlation functions of this theory with

$$\frac{\partial^m Z_0[J]}{\partial \hat{J}_{n_1} \cdots \partial \hat{J}_{n_m}} = \langle \hat{\phi}_{n_1} \cdots \hat{\phi}_{n_m} \rangle. \quad (15.19)$$

These correlation functions are either vanishing ($m = 2i + 1$) or are sums of products of the two-point function. The latter is given by

$$\frac{1}{Z_0[J]} \frac{\partial^m Z_0[J]}{\partial \hat{J}_n \partial \hat{J}_m} = \langle \hat{\phi}_n \hat{\phi}_m \rangle = K_{nm}^{-1}, \quad (15.20)$$

the standard result in QM and QFT: the second derivative of (the logarithm of) the generating functional is the propagator. In the current free theory it is the classical propagator $1/K$.

It is very instructive to transform fields and correlation functions to their momentum representation with a Fourier transformation, i.e. the dispersion $K(p)$ and the propagator $1/K(p)$. In momentum space we will see the ultraviolet finiteness explicitly. In order to facilitate the access we use a lattice of infinite extent, $L \rightarrow \infty$, related to the *thermodynamic* limit of the theory. Not that this limit is different from the continuum limit, $a \rightarrow 0$ (augmented with an appropriate rescaling of the couplings), and they should not be confused.

In any case the Fourier transform on a lattice with infinite extent is defined by

$$\hat{\phi}(\hat{p}) = \sum_{n \in \mathbb{Z}^d} \hat{\phi}_n e^{-i\hat{p}_\mu n_\mu}, \quad \hat{\phi}_n = \int_{-\pi}^{\pi} \frac{d^d \hat{p}}{(2\pi)^d} \hat{\phi}(\hat{p}) e^{i\hat{p}_\mu n_\mu}, \quad (15.21)$$

with the dimensionless momentum $\hat{p} = a p$ for the 'standard' momentum p , defined similarly to $x = a n$. The integration over the dimensionless momentum is restricted to the Brillouin zone

$$\mathcal{B}_{\hat{p}} = [-\pi, \pi]^d, \quad (15.22)$$

originally introduced in condensed matter physics, see also Wigner-Seitz cells. A shift of the momentum \hat{p}_μ by $2\pi m_\mu$ with $m_\mu \in \mathbb{Z}$ leads to

$$e^{i(\hat{p}_\mu + 2\pi m_\mu)n_\mu} = e^{i\hat{p}_\mu n_\mu}, \quad (15.23)$$

as $e^{i\hat{m}_\mu n_\mu} = 1$. Accordingly, physical momenta are restricted by $-\pi/a \leq p_\mu \leq \pi/a$, and while we will not do lattice perturbation theory in this lecture course, all momentum loops in a loop expansion or any other diagrammatic expansions are manifestly ultraviolet finite. Note also that this convenient property comes at the

price of violating Euclidean symmetry, instead the full Euclidean group $O(d)$ in d dimensions the lattice only admits rotations by $n\pi$ with $n \in \mathbb{Z}$.

We now perform a Fourier transform of the kinetic operator K_{nm} in (15.15) to $K(\hat{p}, \hat{q})$,

$$K_{nm} = \int_{-\pi}^{\pi} \frac{d^d \hat{p}}{(2\pi)^d} K(\hat{p}) e^{i\hat{p}(n-m)}, \quad \text{with} \quad K(\hat{p}) = 4 \sum_{\mu=0}^{d-1} \sin^2\left(\frac{\hat{p}_\mu}{2}\right) + \hat{m}^2, \quad (15.24)$$

where we have used

$$\delta_{nm} = \int_{-\pi}^{\pi} \frac{d^d \hat{p}}{(2\pi)^d} e^{i\hat{p}(n-m)}. \quad (15.25)$$

Note that the classical dispersion is obtained for $\hat{p} \rightarrow 0$, where $K(\hat{p}) \rightarrow \hat{p}^2 + \hat{m}^2$. The physical dispersion is given by $1/a^2 K(\hat{p}) \rightarrow p^2 + m^2$. This entails that in the continuum limit only the neighbourhood of the zeros of $\sin^2(\hat{p}_\mu/2)$ survives, all other momenta are suppressed by $1/a^2$. This is clearly seen by re-instating all dimensions to the action. Then we have for the kinetic term on the lattice,

$$\int_{-\pi}^{\pi} \frac{d^d \hat{p}}{(2\pi)^d} \hat{\phi}^\dagger(-\hat{p}) K(\hat{p}) \phi(\hat{p}) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d p}{(2\pi)^d} \phi^\dagger(-p) \frac{1}{a^2} K(pa) \phi(p). \quad (15.26)$$

For momenta $\hat{p} = pa$ with finite values of $K(pa)$, the expression in (15.26) diverges for $a \rightarrow 0$, as does the full action. Accordingly, configurations $\phi(p)$ with support for these momenta are suppressed in the continuum limit. In turn, configurations with support in momentum regimes $\hat{p} = pa \propto a$ with $K(pa) \propto a^2$ survive. In the present case there is only one such zero at $\hat{p} = 0$. Note that seemingly another definition for the dispersion with $\sin^2(\hat{p}_\mu)$ would work as well, but with this definition there are also zeros of the dispersion at the corners of the Brillouin zone, leading to additional fields in the continuum limit (doublers). This will be important in the case of fermions. In summary this entails that discretisation details matters.

15.2. Fermions on the Lattice

In the last chapter we have discussed the lattice formulation of scalar field theories. As in standard quantum field theory this includes the showcase example, the ϕ^4 -theory, and also covers many interesting phenomena. However, all matter fields in the Standard Model except the Higgs are fermionic, as are also the fundamental fields in many interesting condensed matter and statistical systems. In this Chapter we discuss the lattice formulation of fermionic theories. We shall see that the numerical implementation of these theories is not as straightforward as that of scalar theories or that of gauge theories treated in the next Chapter. In relativistic theories, the respective problems are related to the spin 1/2 nature of fermions which neither allows for a straightforward implementation of the importance sampling due to the Grassmannian nature of the fermionic fields (and the linear dispersion), nor does it admit the straightforward implementation of one Weyl fermion on the lattice due to the linear Dirac dispersion: the fermionic *doubling problem* covered by the *Nielsen-Ninomiya theorem*. We note in passing that, while these properties obstruct the numerical simulation of fermionic theories, they also carry some very interesting (and cool) mathematics and physics.

We initiate this discussion with a short summary on the properties of fermionic continuum path integrals evaluated in Chapter 12: In Euclidean space-time the fermionic path integral of the free Dirac action in analogy of (15.1),

$$Z[\eta, \bar{\eta}] = \frac{1}{\mathcal{N}} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_\psi[\psi, \bar{\psi}] + \int_x (\bar{\eta}\psi - \bar{\psi}\eta)}. \quad (15.27)$$

with the free Euclidean Dirac action

$$S_\psi[\psi, \bar{\psi}] = \int_x \bar{\psi} (\not{\partial} + m_\psi) \psi, \quad \text{where} \quad \not{\partial} = \gamma_\mu \partial_\mu, \quad (15.28)$$

Grassmann fields $\psi, \bar{\psi}$ with $\psi^2 = \bar{\psi}^2 = 0$, and the Euclidean version of the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \text{and} \quad \{\gamma_\mu, \gamma_5\} = 0 \quad \text{with} \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \quad (15.29)$$

with the 4×4 dimensional Dirac matrices in $d = 4$, and the hermitian $\gamma_5 = \gamma_5^\dagger$. In general dimensions, the Clifford algebra is $2^{\frac{d}{2}} \times 2^{\frac{d}{2}}$ in even dimensions $d = 2l$ with $l \in \mathbb{N}$, and $2^{\frac{d-1}{2}} \times 2^{\frac{d-1}{2}}$ in odd dimensions $d = 2l - 1$. While the Euclidean anti-fermion $\bar{\psi}$ is independent from the fermion ψ as there is no Euclidean Dirac conjugation, one still writes $\bar{\psi} = \psi^\dagger \gamma_0$.

In any case we can expand the fermionic fields in a convenient set of basis functions, for example the eigenfunctions of $\not{\partial}$ with

$$\psi(x) = \sum_n a_n \varphi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \varphi_n^\dagger(x), \quad \text{with} \quad i\not{\partial} \varphi_n = \lambda_n \varphi_n, \quad (15.30)$$

with the Grassmann expansion coefficients $a_n a_m = -a_m a_n$, $\bar{b}_n \bar{b}_m = -\bar{b}_m \bar{b}_n$ and $a_n \bar{b}_m = -\bar{b}_m a_n$. Within the basis (15.30) the Dirac action (15.28) has the simple form

$$S_\psi[\psi, \bar{\psi}] = \sum_n \lambda_n \bar{b}_n a_n. \quad (15.31)$$

and the Grassmann measure in (15.27) reads,

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \simeq \prod_n \int d\bar{b}_n da_n. \quad (15.32)$$

Using the above relations in the path integral, it yields

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_\psi[\psi, \bar{\psi}]} \simeq \int \left[\prod_n d\bar{b}_n da_n \right] e^{-\sum_n \lambda_n \bar{b}_n a_n} = \prod_n \left[\int d\bar{b}_n da_n \lambda_n a_n \bar{b}_n \right] = \prod_n \lambda_n = \det(\not{\partial} + m), \quad (15.33)$$

where we have used the rules for Grassmann integrations,

$$\int da a^n = \delta_{1n}. \quad (15.34)$$

15.2.1. Lattice action of fermionic field theories

We now formulate the lattice version of the path integral (15.27) in analogy of the scalar theory.

We conclude our Lattice excursion with a few remarks on fermions. We put fermions on the lattice similarly to scalars: To begin with, we define our dimensionless lattice fields and parameters as in (15.6). The dimensionless fermionic fields read,

$$(\psi(x), \bar{\psi}(x)) \rightarrow \frac{1}{a^{\frac{d-1}{2}}} (\hat{\psi}_n, \hat{\bar{\psi}}_n), \quad (15.35a)$$

and the lattice derivative and mass,

$$\partial_\mu \psi(x) \rightarrow \frac{1}{a^{\frac{d+1}{2}}} \hat{\partial}_\mu \hat{\psi}_n, \quad m_\psi \rightarrow \frac{1}{a} \hat{m}_\psi, \quad (15.35b)$$

where we have chosen the symmetric lattice derivative, $\hat{\partial}_\mu = \hat{\partial}_\mu^S$, see (15.11c), with

$$\hat{\partial}_\mu \hat{\psi}(n) = \frac{1}{2} (\hat{\psi}_{n+\hat{\mu}} - \hat{\psi}_{n-\hat{\mu}}). \quad (15.36)$$

With the definitions (15.35) the lattice version of the Dirac action is given by,

$$S_\psi [\hat{\psi}, \hat{\bar{\psi}}] = \sum_{\substack{n,m \\ \alpha,\beta}} \hat{\bar{\psi}}_{\alpha,n} K_{\alpha\beta,nm} \hat{\psi}_{\beta,m} \quad (15.37a)$$

with the space-time points n, m and the Dirac indices α, β . The kinetic operator or matrix K reads,

$$K_{\alpha\beta,nm} = \sum_{\mu} (\gamma_{\mu})_{\alpha\beta} \frac{\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}}{2} + \hat{m}_\psi \delta_{mn} \delta_{\alpha\beta}. \quad (15.37b)$$

The generating functional (15.27) turns into,

$$Z[\eta, \bar{\eta}] \simeq \int \prod_{n,\alpha} d\hat{\psi}_{\alpha,n} \prod_{m,\beta} d\hat{\bar{\psi}}_{\beta,m} e^{-S_\psi[\hat{\psi}, \hat{\bar{\psi}}] + \sum_{n,\alpha} (\bar{\eta}_{\alpha,n} \hat{\psi}_{\alpha,n} - \hat{\bar{\psi}}_{\alpha,n} \eta_{\alpha,n})}. \quad (15.38)$$

The Grassmann integrals in (15.38) are easily performed as in (15.33), and yield,

$$Z[\eta, \bar{\eta}] \simeq \det K \exp \left\{ - \sum_{\substack{n,m \\ \alpha,\beta}} \bar{\eta}_{\alpha,n} K_{\alpha\beta,nm}^{-1} \eta_{\beta,m} \right\}, \quad (15.39)$$

with the fermionic two-point function

$$\langle \hat{\psi}_\alpha \hat{\bar{\psi}}_\beta \rangle = K_{\alpha\beta,nm}^{-1}. \quad (15.40)$$

Equation (15.39) and (15.40) are lookalikes of (15.18) and (15.20) respectively. Seemingly the only difference is the different power of the determinant. However, another surprise is buried in the kinetic operator. This is more clearly seen in momentum space. There we have

$$K_{\alpha\beta,nm} = \int_{-\pi}^{\pi} \frac{d^4 \hat{p}}{(2\pi)^4} \tilde{K}_{\alpha\beta}(\hat{p}) e^{i\hat{p}(n-m)} \quad (15.41)$$

with

$$\tilde{K}_{\alpha\beta}(\hat{p}) = i (\gamma_{\mu})_{\alpha\beta} \sin \hat{p}_\mu + \hat{m}_\psi. \quad (15.42)$$

Now we proceed with the same argument as around (15.26): only the zeros of the dispersion (15.42) survive in the continuum limit, as

$$\int_{-\pi}^{\pi} \frac{d^d \hat{p}}{(2\pi)^d} \hat{\bar{\psi}}_\alpha(-\hat{p}) K(\hat{p}) \psi_\beta(\hat{p}) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d p}{(2\pi)^d} \bar{\psi}_\alpha(-p) \frac{1}{a} K(pa) \psi(p). \quad (15.43)$$

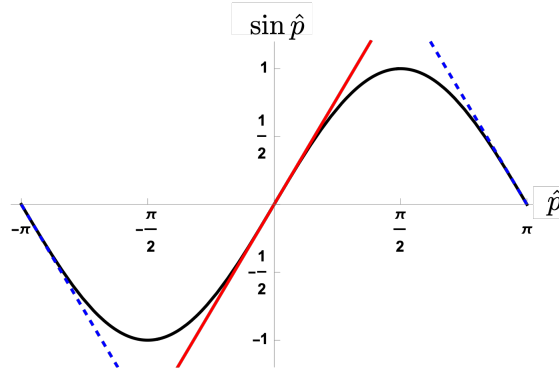


Figure 15.2.: Fermionic lattice dispersion in one dimension

Accordingly, only configurations with support for momenta \hat{p} with $K(pa) \propto a$ lead to a finite action. Let us first look at an oversimplified example, the dispersion in $1 + 0$ dimensions,

$$\tilde{K}(\hat{p}) \sim \sin \hat{p}, \quad (15.44)$$

also depicted in [Figure 15.2](#). Naturally, the dispersion (15.44) vanishes at the middle of the Brillouin zone at $p = 0$ and shows a linear dispersion with a positive slope, $1/a\tilde{K}(\hat{p} \rightarrow 0) \rightarrow 1/a\hat{p} = p$, depicted by the *straight red* tangent line in [Figure 15.2](#). On the boundary points of the Brillouin zone, the dispersion also vanishes with $1/a\tilde{K}(\hat{p}) \rightarrow -p$ depicted by the *dashed blue* tangent line in [Figure 15.2](#). This leaves us with *two* fermions in the continuum limit, one with dispersion p and one with the dispersion $-p$. Note that the latter fermion is distributed at the two boundary points of the lattice $\hat{p} = \pi$ (positive dispersion) and $\hat{p} = -\pi$ (negative dispersion). In summary our attempt of putting one fermion on the lattice ended in having two of them in the continuum limit. This is a baby version of the *fermion doubling problem*.

Let us now try to get rid of the doubler in (15.44), [Figure 15.2](#). Note first that the problem can be avoided by using left or right derivatives instead of the symmetric one (show it). However, these derivatives are not anti-hermitian, which leaves us with complex eigenvalues for the Dirac operator that only turn real in the continuum limit. This is a hefty price to pay. Another possibility consists out of adding higher derivatives terms such as $K(\hat{p}) \rightarrow \sin \hat{p} + r/2 \sin^2 \hat{p}/2$, where the second term is nothing but the lattice Laplacean (15.13) in momentum space, (15.24). Evidently for $\hat{p} \propto a$ this term vanishes linear with the lattice distance in the physical dispersion $1/aK(\hat{p})$, while it leads to a $1/a$ divergence at $\hat{p} \rightarrow \pi, -\pi$. Accordingly, the doubler disappears in the continuum limit.

After this little excursion we come back to the fermionic lattice field theory. As in our baby example we collect all the zeros of the dispersion. For example, in two dimensions we have the four zeros

$$(0, 0), \quad \{(0, \pm\pi)\}, \quad \{(\pm\pi, 0)\}, \quad \{(\pm\pi, \pm\pi)\}. \quad (15.45)$$

More generally we have 2^d zeros in d dimensions: at all points where all p_μ take values $(0, \pm\pi)$,

$$\hat{p}_\mu \in \{0, \pm\pi\} \quad \forall \mu = 1, \dots, d. \quad (15.46)$$

At these points the continuum limit of the propagator is given by

$$\lim_{a \rightarrow 0} \frac{1}{a^3} \int_{-\pi}^{\pi} \frac{d^4 \hat{p}}{(2\pi)^4} \frac{e^{i\hat{p}(n-m)}}{\tilde{K}(\hat{p})} = - \lim_{a \rightarrow 0} \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{i\gamma_\mu \frac{1}{a} \sin ap_\mu + m} = - \sum_j \int_{\varepsilon_1} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{(-1)^{j+1} i\gamma_\mu p_\mu + m} + O(a) \quad (15.47)$$

Note that the sign of the γ -matrices can be flipped by a \hat{p} -dependent similarity transformation $S(\hat{p})$,

$$S(\hat{p})\gamma_\mu S^{-1}(\hat{p}) = (-1)^{j+1}\gamma_\mu. \quad (15.48)$$

In summary we have produced 16 fermions on the lattice. This is the (in) famous doubling problem on the lattice.

We note in passing, that the rather heuristic statement above can be cast in a mathematical form by considering the fermions in momentum space, $\hat{\psi}(\hat{p})$. As in position space, fermions are periodic in momentum space,

$$\hat{\psi}(\hat{p} + \hat{\mu}) = \hat{\psi}(\hat{p} + 2\pi\hat{\mu}), \quad (15.49)$$

which singles out the Brillouin zone in the first place. Therefore, fermions can be seen as maps from the d -dimensional (momentum) torus into the field space, e.g. $\mathbb{T}^4 \rightarrow \mathbb{C}^4$ for a four-dimensional single Dirac fermion. In general even dimensions, the latter has $2^{d/2}$ (complex) components as the Clifford algebra is $2^{d/2}$ -dimensional. Then, left- and right-handed eigenspaces can be defined by momentum dependent projections, and the winding numbers of these projections is related to the total chirality of the theory.

For the purposes of the present lecture course it is sufficient to briefly discuss possible resolutions and concentrate on the simple practical ones, the topic is picked up later, when discussing the continuum limit in more detail, including an renormalisation group (Kadanoff block-spinning) analysis on the lattice.

15.2.2. Wilson fermions

We have already indicated above in the discussion of our baby example, that the introduction of higher derivative terms in the kinetic term gives us the possibility to suppress the doublers in the continuum limit. This is done by augmenting them with masses proportional to $1/a$,

$$K_{\alpha\beta, nm}^{(W)} = K_{\alpha\beta, nm} + \frac{r}{2} \sum_{\substack{n, m \\ a, \beta}} \hat{\psi}_{\alpha, n} \hat{\Delta}_{n, m} \hat{\psi}_{\beta, m}. \quad (15.50)$$

with $K_{\alpha\beta, nm}$ in (15.37b) and the dimensionless Wilson parameter r and the lattice Laplacian (15.13). In Momentum space this kinetic term reads

$$K_{\hat{p}}^{(W)} = i\gamma_\mu \sin \hat{p}_\mu + \hat{m}_\psi + 2r \sum_{\mu=0}^{d-1} \sin^2 \left(\frac{\hat{p}_\mu}{2} \right), \quad (15.51)$$

with the lattice Laplacian in momentum space in (15.24). The Wilson term vanishes for the fermion at $\hat{p} \rightarrow 0$, while it diverges at all the doubler points defined by (15.46),

$$d \frac{r}{2a} \geq \frac{2r}{a} \sum_{\mu=0}^{d-1} \sin^2 \left(\frac{\hat{p}_\mu}{2} \right) \geq \frac{r}{2a}, \quad \text{for } \hat{p} \neq 0 \quad \text{and} \quad \hat{p}_\mu \in \{0, \pm\pi\}. \quad (15.52)$$

Evidently, (15.51) breaks chiral symmetry as it includes the momentum dependent Wilson mass term proportional to the identity in Dirac space. Only in the continuum limit chiral symmetry is restored (in the absence of explicit fermion masses, $m_\psi = 0$). While this is a heavy price to pay for theories with chiral symmetry or that 'close' to chiral symmetry such as present for the light current quark masses of the up and down quark in QCD, it is a simple deformation which is amiable to numerical implementation in simulations.

15.2.3. Staggered fermions

Another possibility is the exploitation of the doublers as physical mode. For example, as we have indicated above, putting one (massless) Weyl fermion on the lattice leads us to 2^d Weyl fermions in the continuum limit

with different chirality. Now, we even can assign different mass gaps to the fermions which allows us to define them as different flavours of the original fermion.

Staggered fermions exploit this possibility. Before we come to some technical details, we would like to give the flavour of the argument (pun intended). The most efficient way of their implementation is a simple counting and distribution of fermionic degrees of freedom. In the following, we restrict ourselves to even dimensions, specifically $d = 2$ and $d = 4$. In even dimensions d , a Dirac fermion has $2^{d/2}$ complex components, that can be distributed over 2^d zeros of the dispersion at the positions (15.46). This leads us to minimally $2^{d/2}$ different Dirac fermions in this formulation, called different *tastes*. For example, in QCD in $d = 4$ dimensions we then would have 4 tastes. It is tempting to identify them with the *flavours* in QCD, so a minimal version of lattice QCD would have four flavours, up, down, strange charm. However, the latter have significantly different masses, and the chiral structure of the theory crucially depends on the mass ordering, typically indicated 2+1+1 in order to single out the two light quarks, u, d with current masses $m_{u,d} \sim 2 - 5$ MeV, the heavier strange quark with a current quark mass of $m_s \sim 10^2$ MeV and the charm quark with a current quark mass of $m_c \sim 10^3$ MeV. However, the fact, that in our QCD example we have four tastes with identical masses, leads us to

$$\exp\left\{\text{tr}_{\text{lattice}} \log K\right\} = \det_{\text{lattice}} K \rightarrow \left(\det_{\text{cont}} [\not{\partial} + m_\psi]\right)^4 = \exp\left\{4 \text{tr}_{\text{cont}} \log (\not{\partial} + m_\psi)\right\}. \quad (15.53)$$

Equation (15.53) suggests to e.g. simply take the square root of (15.53), if considering two-flavour QCD in the isospin-symmetric limit $m_u = m_d$. The latter is typically chosen in many applications as both, the current quark masses m_u, m_d and the mass difference $m_u - m_d$ is rather small in comparison to the mass arising from spontaneous strong chiral symmetry breaking $\sim 300 - 400$ MeV.

Evidently, for one Dirac fermion one has to consider the fourth root of (15.53). Effectively, this is done on the level of the exponent on the left hand side of (15.53)

$$\text{tr}_{\text{lattice}} \log K \rightarrow \frac{1}{N} \text{tr}_{\text{lattice}} \log K \rightarrow \text{tr}_{\text{cont}} \log [\not{\partial} + m_\psi], \quad (15.54)$$

with $N = 16$ in the case of a four-dimensional Dirac fermion. Such a rooting is commonplace, but has its problems. While (15.54) provides the core of the argument, the remaining Dirac fermion is de-localised on the lattice, its component being distributed at the points (15.46).

How one can construct such a fermion technically, is discussed now at the example of a single Dirac fermion in $d = 4$. We recall its action (15.37),

$$S_\psi [\hat{\psi}, \hat{\psi}'] = - \sum_{n,m} \hat{\psi}_n \left[\sum_{\mu=1}^4 \gamma_\mu \frac{\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}}{2} + \hat{m}_\psi \delta_{mn} \right] \hat{\psi}'_m, \quad (15.55)$$

and we make a site-dependent transformation of the fermions in Dirac space such that the Dirac structure 'disappears', and the component fermions $\psi_{\alpha,m}$ run the show. This is done with the *staggered transformations*

$$\hat{\psi}_n = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_0^{n_0} \hat{\psi}'_n, \quad \text{and} \quad \hat{\psi}'_n = \hat{\psi}'_n \gamma_0^{n_0} \gamma_3^{n_3} \gamma_2^{n_2} \gamma_1^{n_1}. \quad (15.56)$$

The staggered transformation rotates the fermions from one lattice site to the next such that the Dirac structure of the next neighbour hopping terms is absorbed into the fields. Moreover, for the site terms we have $\hat{\psi}_n \hat{\psi}'_n = \hat{\psi}'_n \hat{\psi}_n$ owing to $\gamma_\mu^2 = \mathbb{1}$.

Now we consider exemplarily one hopping term in the $\mu = 3$ direction. There we have

$$\hat{\psi}_n \gamma_3 \hat{\psi}'_{n\pm\hat{3}} = \hat{\psi}'_n \gamma_0^{n_0} \gamma_3^{n_3} \gamma_2^{n_2} \gamma_1^{n_1} \gamma_3 \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3\pm 1} \gamma_0^{n_0} \hat{\psi}'_{n\pm\hat{3}} = (-1)^{n_1+n_2} \hat{\psi}'_n \hat{\psi}'_{n\pm\hat{3}}. \quad (15.57)$$

Here, the factor $(-1)^{n_1+n_2}$ counts, how many times we have to have to anti-commute γ_1 and γ_2 with the explicit γ_3 in the middle. Again using $\gamma_\mu^2 = \mathbb{1}$ we arrive at (15.57), and similarly for the other hopping terms.

Inserting all these transformations into (15.55), we are led to,

$$S_\psi[\psi, \bar{\psi}] = \sum_{n,\mu} \left[\frac{1}{2} \eta_{n,\mu} \hat{\psi}'_n (\hat{\psi}'_{n+\hat{\mu}} - \psi'_{\hat{n}-\hat{\mu}}) + \hat{m}_\psi \hat{\psi}'_n \hat{\psi}'_n \right], \quad (15.58)$$

with

$$\eta_{n,1} = 1, \quad \eta_{n,2} = (-1)^{n_1}, \quad \eta_{n,3} = (-1)^{n_1+n_2}, \quad \eta_{n,0} = (-1)^{n_1+n_2+n_3}. \quad (15.59)$$

Importantly, (15.58) is diagonal in Dirac space in the rotated Dirac fields $\psi_{\alpha,n}$, and all components have the same action. Accordingly, the rooting (15.54) simply amounts to dropping the Dirac sum in (15.58), only considering one component, $\hat{\chi}_n = \hat{\psi}_{\alpha,n}$. We arrive at the final result,

$$S_\chi[\chi, \bar{\chi}] = \sum_{n,\mu} \left[\frac{1}{2} \eta_{n,\mu} \hat{\chi}_n (\hat{\chi}_{n+\hat{\mu}} - \chi_{\hat{n}-\hat{\mu}}) + \hat{m}_\psi \hat{\chi}_n \hat{\chi}_n \right], \quad (15.60)$$

With this procedure we have reduced the number of 16 Dirac fermions to 4 Dirac fermions (tastes). In the current lecture course we will rarely use staggered fermions, and if, we will use the rooting prescription indicated above. Hence, for more details as well as a detailed discussion of chiral symmetry and *taste violation*, we refer to the literature, in particular to [7], chapter 4 and [8], chapter 10.1.

15.2.4. Chiral symmetry on the lattice & the fate of the axial anomaly*

However, for its importance as well as its elucidative nature we briefly discuss the fate of (naive) chiral symmetry on the lattice. We emphasise that the derivation below is sketchy and the single step, while straightforward, are a bit tedious. This topic will be picked up later within a more elaborated point of view also involving renormalisation group (RG) arguments.

To begin with, let us assume that we have coupled an external gauge field to the fermion, this will be detailed in the next chapter. Then, the Dirac action reads

$$S_\psi [\hat{\psi}, \hat{\bar{\psi}}] = - \sum_{n,m} \left[\hat{\bar{\psi}}_n \mathcal{D}_{n,m} + \hat{m}_\psi \delta_{mn} \right] \hat{\psi}_m, \quad (15.61)$$

with the naive interacting lattice Dirac operator $\mathcal{D}_{n,m}$ that is proportional to γ_μ and hence it anti-commutes with γ_5 defined in (15.29),

$$\{\mathcal{D}, \gamma_5\} = 0, \quad (15.62)$$

It is precisely the property (15.62), that the Dirac operator of Wilson fermions violates, and indeed one can show, that one either has doublers or one loses (15.62).

In ?? we keep the mass but consider the limit $\hat{m}_\psi \rightarrow 0$. Now we apply local chiral transformations

$$\hat{\psi}_n \rightarrow e^{i\alpha(n)\gamma_5} \hat{\psi}_n, \quad \hat{\bar{\psi}}_n \rightarrow \hat{\bar{\psi}}_n e^{i\alpha(n)\gamma_5}. \quad (15.63)$$

to the action (15.61). Using (15.61) as the action in the fermionic path integral, the theory is still Gaussian (free). Expanding the transformed path integral in powers of α , we are led to the standard (partial) axial current conservation,

$$\hat{\partial}_\mu \langle j_{5,\mu} \rangle = 2\hat{m}_\psi \langle \hat{\bar{\psi}}_n \gamma_5 \hat{\psi}_n \rangle, \quad \text{with} \quad j_{5,\mu} = \hat{\bar{\psi}}_n \gamma_5 \hat{\psi}_n, \quad (15.64)$$

where the derivative is the symmetric lattice derivative. Now we sum over the lattice, and the left hand side vanishes as it is a total derivative. There are no boundary terms due to the periodic boundary conditions. This leads us instantly to

$$\hat{m}_\psi \sum \langle \hat{\bar{\psi}}_n \gamma_5 \hat{\psi}_n \rangle = -\hat{m}_\psi \text{Tr} \gamma_5 \langle \hat{\psi}_n \hat{\bar{\psi}}_n \rangle = 0. \quad (15.65)$$

The expectation value under the trace is nothing but the propagator

$$\langle \hat{\psi}_n \hat{\psi}_m \rangle = \left[\frac{1}{\mathcal{D} + \hat{m}_\psi} \right]_{nm}. \quad (15.66)$$

with the anti-hermitian operator \mathcal{D} with the spectrum

$$\mathcal{D}\hat{\varphi}_j = i\lambda_j\hat{\varphi}_j, \quad \text{with} \quad j \in \mathbb{N}, \quad (15.67)$$

For every $\lambda_j \neq 0$, $\gamma_5\hat{\varphi}_j$ also is an Eigenfunction to the Eigenvalue $-i\lambda_j$. Since $\lambda_j \neq 0$, we have

$$\sum_n \varphi_j(n)\gamma_5\varphi_j(n) = 0. \quad (15.68)$$

This leads us to

$$-\hat{m}_\psi \text{Tr} \gamma_5 \langle \hat{\psi}_n \hat{\psi}_m \rangle = n_+ - n_- = 0, \quad (15.69)$$

where n_+ and n_- is the number Eigenfunctions $\varphi^{(0)}$ of vanishing Eigenvalues with positive and negative chirality respectively.

$$\gamma_5\varphi^{(0)} = \pm\varphi^{(0)}, \quad \text{as} \quad \varphi_{j_1}^{(0)} \{ \gamma_5, \mathcal{D} \} \varphi_{j_2}^{(0)} = 0. \quad (15.70)$$

In conclusion, the total chirality of the fermionic lattice theory is vanishing. Again this reflects the Nielsen-Ninomiya theorem on the lattice. In this formulation is also hints at some serious physics problems, as the (non-vanishing) axial anomaly carries important physics.

15.3. Gauge Fields on the Lattice

With lattice formulations of scalar fields and fermions as introduced in [Section 15.1](#) and [Section 15.2](#) respectively, we close our discussion of lattice formulations of field theories with that of gauge theories. Before introducing lattice gauge theories, we start with a brief reminder of gauge theories in the continuum, see [Chapters 6](#) and [13](#). For the better comparability with lattice gauge theory textbooks we shall use a slightly different notation here than in the these two Chapters and [Chapter 14](#). We start with an Abelian gauge theory with a Dirac fermion,

$$S_\psi[\psi, \bar{\psi}, A_\mu] = \int_x \bar{\psi} (\mathcal{D} + m_\psi) \psi, \quad \text{where} \quad \mathcal{D} = \gamma_\mu D_\mu, \quad D_\mu = \partial_\mu + ieA_\mu. \quad (15.71)$$

with the Abelian gauge field $A_\mu \in \mathbb{R}$, that is in the algebra of the gauge group $U(1)$. Note that the definition of the covariant derivative differs from that in [\(15.71\)](#) with $D_\mu = \partial_\mu - ieA_\mu$ by a relative minus sign. The action [\(15.71\)](#) is invariant under gauge transformations $\Omega(x) = \exp\{i\omega(x)\} \in U(1)$ of the Dirac fermion and the gauge field,

$$\psi(x) \rightarrow \Omega(x)\psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)\Omega^\dagger(x), \quad A_\mu \rightarrow -\frac{i}{e}\Omega(D_\mu\Omega^\dagger) = A_\mu - \partial_\mu\omega. \quad (15.72)$$

The transformation of the gauge field implies that the covariant derivative transforms as a tensor under gauge transformations

$$D_\mu \rightarrow \Omega D_\mu \Omega^\dagger, \quad (15.73)$$

where we use the notation $\Omega \in U(1)$ for gauge transformations, instead of U as in [\(13.2\)](#). The pure gauge field part of the gauge field is given by the standard $U(1)$ action with

$$S_A[A_\mu] = \frac{1}{4} \int_x F_{\mu\nu}^2, \quad \text{with} \quad F_{\mu\nu} = -\frac{i}{e} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (15.74)$$

The relative minus sign in comparison to (13.19) originates in the definition of the covariant derivative.

This set-up is easily generalised to non-Abelian gauge theories, where we restrict ourselves to $SU(N_c)$ theories coupled to fermions ψ^A in the fundamental representation with $A = 1, \dots, N_c$, where N_c is generically called the number of colors, referring to QCD. We have

$$S_\psi[\psi, \bar{\psi}, A_\mu] = \int_x \bar{\psi}^A (\not{D} + m_\psi)^{AB} \psi^B, \quad \text{where} \quad D_\mu^{AB} = \partial_\mu \delta^{AB} + ig A_\mu^c (t^c)^{AB}, \quad (15.75)$$

with $a = 1, \dots, N^2 - 1$, and $A_\mu^a(x) \in \mathbb{R}$.

The action (15.75) is invariant under gauge transformations $\Omega(x) = \exp\{i\omega(x)\} \in SU(N_c)$ with the Lie algebra valued exponent $\omega(x) = \omega^a(x)t^a \in \mathfrak{su}(N_c)$ of the Dirac fermion and the gauge field, similarly to (15.76),

$$\psi(x) \rightarrow \Omega(x)\psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)\Omega^\dagger(x), \quad A_\mu \rightarrow -\frac{i}{g}\Omega(D_\mu\Omega^\dagger), \quad (15.76)$$

the difference being the non-commutativity of the gauge field. The transformation of the gauge field implies that the covariant derivative transforms as a tensor under gauge transformations

$$D_\mu \rightarrow \Omega D_\mu \Omega^\dagger, \quad (15.77)$$

with $D_\mu \Omega^\dagger \psi = (D_\mu \Omega^\dagger)\psi + \Omega^\dagger(D_\mu \psi)$. The pure gauge field action (Yang-Mills action) is given by

$$S_A[A_\mu] = \frac{1}{2} \int_x \text{tr}_f F_{\mu\nu}^2 = \frac{1}{4} \int_x (F_{\mu\nu}^a)^2, \quad (15.78)$$

with the field strength $F_{\mu\nu} = F_{\mu\nu}^a t^a$ with

$$F_{\mu\nu} = -\frac{i}{g} [D_\mu, D_\nu], \quad \text{and} \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c, \quad (15.79)$$

As already mentioned above, QCD has the gauge group $SU(3)$ (three colours), and the fermions are the quarks (u,d,s,c,b,t) or a subset thereof. It is also very common to do simulations in $SU(2)$ theories as these theories share many communalities but $SU(2)$ is far simpler to simulate, in particular at finite densities. There, however, it also differs significantly from QCD with three colours. Moreover, the $SU(2)$ case is also the weak sector of the Standard Model (SM), and with the hyper charge $U(1)$ it adds up to the electroweak sector of the SM.

In the latter case we also have to consider a scalar field, the Higgs field, that carries a representation of the electroweak gauge group. Hence, more generally, we consider the kinetic term of a complex scalar field also carrying the fundamental representation of a non-Abelian gauge group,

$$S_\phi[\phi, A_\mu] = \int_x (\phi^\dagger)^A (D_\mu^2)^{AB} \phi^B, \quad \text{with} \quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2). \quad (15.80)$$

The gauge transformation of the scalar field is given by

$$\phi(x) \rightarrow \Omega(x)\phi, \quad \phi^\dagger(x) \rightarrow \phi^\dagger\Omega^\dagger(x) \quad (15.81)$$

and the gauge invariance of the action (15.80) follows readily.

With the matter actions (15.71), (15.75), (15.80) and the pure gauge theory actions (15.74) and (15.78) we have all ingredients of the Standard Model, except the chiral structure of the electroweak interaction.

15.3.1. Lattice action of gauge field theories

We start our discussion of the action of lattice gauge theories by first working out the lattice analogue of (15.81). The arguments work the same way for the fermionic action.

As in the continuum we introduce the gauge transformation for the complex matter field $\hat{\phi}_n$, living on the sites,

$$\hat{\phi}_n \rightarrow \Omega(n)\hat{\phi}_n, \quad \hat{\phi}_n^\dagger \rightarrow \hat{\phi}_n^\dagger \Omega^\dagger(n), \quad (15.82)$$

where $\Omega(n) \in \text{SU}(N_c)$ or $\Omega(n) \in \text{U}(1)$. The lattice action of the free (complex) scalar theory is given in (15.14) with the dispersion K_{nm} defined in (15.15), which we recall here for the sake of convenience.

$$K_{nm} = - \sum_{\mu>0} [\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m}] + (\hat{m}_\phi^2 + 2d) \delta_{nm}. \quad (15.83)$$

The terms in $\hat{\phi}^\dagger K_{nm} \hat{\phi}$ living on a site (part of K_{nm} proportional to δ_{nm}) are invariant under (15.82), and we have to ensure gauge invariance for e.g.

$$\hat{\phi}_n^\dagger \hat{\phi}_{n+\hat{\mu}}. \quad (15.84)$$

This entails that we have to *transport* the group element $\Omega^\dagger(n + \hat{\mu})$ from the lattice point $n + \hat{\mu}$ to n . To that end we define the *link variable* $U_\mu(n) \in \text{SU}(N_c)$ with

$$U_\mu(n) \rightarrow \Omega(n)U_\mu(n)\Omega^\dagger(n + \hat{\mu}), \quad (15.85)$$

and the parametrisation $U_\mu(n) = e^{i\hat{\theta}_\mu(n)}$ with the Lie algebra field $\theta_\mu(n) \in \text{su}(N_c)$. The link variable is nothing but a parallel transporter along the link between the sites n and $n + \hat{\mu}$. It 'lives' on the link between n and $n + \hat{\mu}$. Accordingly, the term $\hat{\phi}_n^\dagger U_\mu(n) \hat{\phi}_{n+\hat{\mu}}$ is gauge invariant: The link variable $U_\mu(n)$ parallel transports (infinitesimally) the gauge transformation from the lattice site $n + \hat{\mu}$ to the lattice site n . There it is annihilated. Explicitly, this upgrade of (15.84) transforms with

$$\hat{\phi}_n^\dagger U_\mu(n) \hat{\phi}_{n+\hat{\mu}} \rightarrow \hat{\phi}_n^\dagger \Omega^\dagger(n)\Omega(n) U_\mu(n) \Omega^\dagger(n + \hat{\mu})\Omega(n + \hat{\mu}) \hat{\phi}_n. \quad (15.86)$$

Trivially, this also holds for $\hat{\phi}_n^\dagger U_\mu^\dagger(n - \hat{\mu}) \hat{\phi}_{n-\hat{\mu}}$. Then the free action (15.14) turns into,

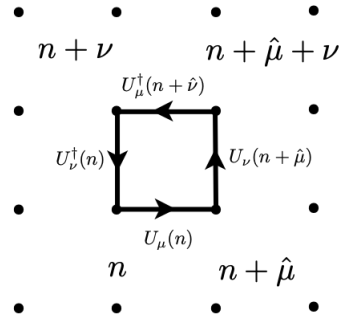
$$S[\hat{\phi}, U] = - \sum_{\substack{n \\ \mu>0}} (\hat{\phi}_n^\dagger U_n^\dagger(n - \hat{\mu}) \hat{\phi}_{n-\hat{\mu}} + \hat{\phi}_n^\dagger U_\mu(n) \hat{\phi}_{n+\hat{\mu}}) + \sum_n \hat{\phi}_n^\dagger \hat{\phi}_n (2d + \hat{m}_\phi^2), \quad (15.87)$$

which reduces to (15.14) for $U_\mu = \mathbb{1}$.

We proceed by showing that the continuum limit of the gauge invariant lattice action is the continuum action (15.80). To that end we write

$$U_\mu(n) = e^{ig_0 a A_\mu(an)} = \mathbb{1} + ig_0 a A_\mu(an) - \frac{1}{2} (ig_0 a A_\mu(an))^2 + O(a^3), \quad (15.88)$$

which defines a gauge field A_μ on the lattice. In the following we only need (15.88) up to the quadratic term in the lattice distance a as displayed in (15.88). This already entails that (15.88) is not the unique definition of the gauge field. Moreover, we shall see that order a^2 terms in the exponent of U_μ vanish in the continuum limit. Accordingly, the gauge field is only defined up to terms $O(a)$, and these terms disappear in the continuum limit.


 Figure 15.3.: Plaquette variable $U_{\mu\nu}(n)$.

Now we expand (15.87) in powers of the lattice spacing a , using (15.88) as well as (15.6). For the sake of simplicity we restrict ourselves to $d = 4$. First, using the expansion in (15.88) we get,

$$S[\hat{\phi}, U] \xrightarrow{a \rightarrow 0} - \sum_{n,m} \hat{\phi}_n^{\dagger} K_{nm} \hat{\phi} - \sum_{\substack{n \\ \mu > 0}} \hat{\phi}_n^{\dagger} [-iag_0 A_{\mu}(na - \hat{\mu}a) \hat{\phi}_{n-\hat{\mu}} + iag_0 A_{\mu}(na) \hat{\phi}_{n+\hat{\mu}}] - \sum_n \hat{\phi}_n^{\dagger} (iag_0 A_{\mu})^2 \hat{\phi}_n, \quad (15.89)$$

where we only keep the terms up to order a^2 . Now we use (15.6) to map our dimensionless scalar lattice fields to the dimensionful ones, $\hat{\phi}_n = a \phi(na)$. Moreover, we use

$$\phi(na \pm a\hat{\mu}) = \phi(na) \pm a\partial_{\mu}\phi(na) + O(a^2), \quad A_{\mu}(na \pm \hat{\mu}a) = A_{\mu}(na) + a\partial_{\mu}A_{\mu}(na) + O(a^2), \quad (15.90)$$

where no sum is implied. Collecting all the terms on the right hand side of (15.89), we arrive at

$$-a^4 \left\{ \sum_{n,m} \phi^{\dagger}(na) \frac{K_{nm}}{a^2} \phi(ma) + ig_0 \sum_{\substack{n \\ \mu > 0}} \phi^{\dagger}(na) [\partial_{\mu}A_{\mu}(na) + A_{\mu}(na)\partial_{\mu}] \phi(na) + \sum_n \hat{\phi}_n^{\dagger} (ig_0 A_{\mu})^2 \hat{\phi}_n + O(a) \right\}. \quad (15.91)$$

We use that in the continuum limit we have $a^4 \sum_n \rightarrow \int_{\mathbb{R}^4}$ and arrive at

$$S[\hat{\phi}, U] \rightarrow \int d^4x \phi^{\dagger}(x) [D_{\mu}^2 + m_{\phi}^2] \phi(x) + O(a). \quad (15.92)$$

In summary, we have obtained the desired result, our gauge invariant lattice action (15.87) reduces to the gauge invariant continuum action (15.80) for $a \rightarrow 0$.

Now we proceed with defining a lattice analogue of the pure gauge action (15.74) or (15.78). The building block of the pure gauge action is the field strength tensor $F_{\mu\nu}$, which in the continuum is the curvature tensor $1/(ig)[D_{\mu}, D_{\nu}]$, see (15.79). The covariant derivative D_{μ} induces an infinitesimal parallel transport in the direction $\hat{\mu}$, and hence the commutator is first transporting in the ν -, then in the μ -direction, and then back in the inverse path. On the lattice, this operation is implemented by the Plaquette variable,

Plaquette variable

$$U_{\mu\nu}(n) = U_{\mu}(n)U_{\nu}(n + \hat{\mu})\hat{U}_{\mu}^{\dagger}(n + \hat{\nu})U_{\nu}^{\dagger}(n). \quad (15.93)$$

As the link variable, the Plaquette variable lives in the gauge group. In analogy to the algebra-valued gauge field defined in the exponent of the link variable in (15.88), we can define an algebra-valued field strength in the exponent of the Plaquette,

$$U_{\mu\nu}(n) = e^{ig_0 a^2 \mathcal{F}_{\mu\nu}(n)} \quad (15.94)$$

with lattice field strength tensor $\mathcal{F}_{\mu\nu}$. In QED we derive from (15.93),

$$\mathcal{F}_{\mu\nu}(n) = \frac{1}{a} \left[\underbrace{(A_\nu(n + \hat{\mu}) - A_\nu(n))}_{\hat{\partial}_\mu^R A_\nu} - \underbrace{(A_\mu(n + \hat{\nu}) - A_\mu(n))}_{\hat{\partial}_\nu^R A_\mu} \right], \quad (15.95)$$

where the left and right derivatives in (15.95) ensure the anti-symmetry of the expression. In non-Abelian gauge theories such as QCD we use the Baker-Campbell-Hausdorff formula for the expansion,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}. \quad (15.96)$$

With (15.96) we arrive at,

$$\mathcal{F}_{\mu\nu}(n) = \frac{1}{a} \left[\hat{\partial}_\mu^R A_\nu - \hat{\partial}_\nu^R A_\mu + ig_0 a [A_\mu, A_\nu](n) + \mathcal{O}(a) \right]. \quad (15.97)$$

Both, (15.95) and (15.97) imply that the Plaquette variable can be expanded in powers of the field strength in the continuum limit, each power going with higher orders of the lattice distance,

$$U_{\mu\nu}(n) \xrightarrow{a \rightarrow 0} 1 + ig_0 a \mathcal{F}_{\mu\nu}(n) - \frac{g_0^2 a^2}{2} \mathcal{F}_{\mu\nu}^2(n) + \mathcal{O}(a^3). \quad (15.98)$$

Summed over μ, ν , the $F_{\mu\nu}^2$ -term in (15.98) is the Yang-Mills action in the continuum, up to prefactors. In turn, the unity and linear term have to be cancelled. Evidently, the linear term is removed by adding the adjoint of the Plaquette to (15.98),

$$U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n) = 2 - g_0^2 a^2 \mathcal{F}_{\mu\nu}^2(n) + \mathcal{F}(a^3). \quad (15.99)$$

Then, the constant term simply can be subtracted and we arrive at a lattice analogue of the pure Yang-Mills action in the continuum,

Wilson action

$$S_W[U_\mu] = \beta \sum_{\mu < \nu} \left(1 - \frac{1}{2N_c} \text{tr}_f (U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right), \quad \text{with} \quad \beta = \frac{2N_c}{g_0^2}. \quad (15.100)$$

Here, β takes the rôle of an expansion parameter getting small for large lattice coupling g_0 , and we shall later derive analytic results in such a *strong coupling expansion* about $\beta = 0$, that is $g_0^2 \rightarrow \infty$. Here we simply check the naïve continuum limit of (15.100), to wit,

$$S_W[U_\mu] \xrightarrow{a \rightarrow 0} S_A[A] = \frac{1}{2} \int d^4x \text{tr}_f F_{\mu\nu}^2, \quad (15.101)$$

where we have used,

$$\sum_{\mu < \nu} \left(1 - \frac{1}{2N_c} \text{tr} (U_{\mu\nu} + U_{\mu\nu}^\dagger) \right) = \sum_{\mu < \nu} \frac{g_0^2 a^2}{2N_c} \text{tr} \mathcal{F}_{\mu\nu}^2 + \mathcal{O}(a^3) = \frac{g_0^2 a^2}{4N_c} \text{tr} \mathcal{F}_{\mu\nu} + \mathcal{O}(a^3). \quad (15.102)$$

This finally leads us to a well-defined generating functional for compact Yang-Mills theory and U(1)-theory on the lattice,

Lattice path integral for non-Abelian gauge theories

$$Z \simeq \int \mathcal{D}U e^{-S_W[U_\mu]}, \quad (15.103)$$

with the Wilson action S_W in (15.100). The path integral measure in (15.103) is the *finite* product of *finite* Haar measures of the link variables,

$$DU = \prod_{\substack{l \\ \uparrow \\ \text{links}}} dU_l. \quad (15.104)$$

\uparrow Haar measure

The Haar measure is the measure on the gauge group, that is invariant under a gauge rotation with a group element,

$$\int dU^V = \int dV U V^\dagger = \int dU = 1. \quad (15.105a)$$

$$\int dU U^{ab} = 0, \quad \int dU U^{ab} U^{cd} = 0, \quad \int dU \underbrace{U^{ab} (U^\dagger)^{bd}}_{\delta_{ad}} = \delta_{ad}, \quad (15.105b)$$

For the gauge group of QCD, $SU(3)$, we also have more specifically,

$$\int dU U^{ab} (U^\dagger)^{cd} = \frac{1}{3} \delta_{ad} \delta_{bc}, \quad \int dU U^{a_1 b_1} U^{a_2 b_2} U^{a_3 b_3} = \frac{1}{3!} \varepsilon_{a_1 a_2 a_3} \varepsilon_{b_1 b_2 b_3}. \quad (15.106)$$

where the $1/3$ reflects the number of colours, $1/N_c$. The relations in (15.106) are confirmed straightforwardly by summing over the indices. For example, summing over b with $b = c$ in the first relation in (15.106) and using (15.105a) leads to δ^{ad} on both sides.

Finally, from the path integral we can derive correlation functions such as,

$$\langle U_{\mu_1}^{a_1 b_1}(n_1) \cdots U_{\mu_m}^{a_m b_m}(n_m) \rangle = \frac{1}{Z} \int DU U_{\mu_1}^{a_1 b_1}(n_1) \cdots U_{\mu_m}^{a_m b_m}(n_m) e^{-S_W[U_\mu]} \quad (15.107)$$

that can be computed either by numerical sampling or, for small lattices or, within appropriate expansion schemes, analytically.

The generating functional (15.103) is the lattice analogue of the generating functional of Yang-Mills theory or pure $U(1)$ theory in the continuum (13.58) in Chapter 13, with

$$Z[J] \simeq \int \mathcal{D}A_\mu e^{-S_A[A] + \int_x J_\mu A_\mu}. \quad (15.108)$$

In (15.108) we have dropped a potential gauge fixing and Faddeev-Popov ghost action that typically have to be implemented in the continuum. Another difference is apparent from the comparison of (15.103) with (15.108): the latter is built on an integration over the gauge field A_μ living in the algebra of the gauge group while the former is built on an integration over the link variable living in the gauge group. Locally, the respective quantum theories are the same, they may differ globally. Indeed, in two-dimensional gauge theories the two quantisations are known to differ.

15.4. The Wilson Loop & the Static Quark Potential

With the Wilson action for Abelian and non-Abelian gauge theories we introduce the strong coupling expansion at the example of the expectation value of the *Wegner-Wilson Loop*, called the *Wilson loop* in the literature. In QCD or rather pure Yang-Mills theory, this observable serves as an order parameter, and below we discuss its physics interpretation and its computation in the limit $\beta \rightarrow 0$.

15.4.1. Wilson loop in QED & QCD

To that end we first consider an electron-positron pair or quark-anti-quark pair, which is created at some initial time, pulled apart, kept at some distance L and then annihilated, see **Figure 15.4** for a depiction of the respective worldline or path C . Here we use the electron-positron ($e^+ - e^-$) pair as an example with particles that can be observed as asymptotic states. In turn, the process of interest with the quark-anti-quark ($q - \bar{q}$) pair does not relate to asymptotic states as quarks are not. It is precisely this property we want to test here.

The physical process can be related to the path integral with the current J_μ of the world line C of the $e^+ e^-$ pair to the photon in the source term $\exp \int_x J_\mu A_\mu$ of the (continuum) path integral (15.108). The precise definition in terms of states will be described below. In any case, the exponent reads

$$\int d^4x J_\mu A_\mu = ie \int_{t_0}^{t_1} dt (A_0(t, \vec{x}) - A_0(t, \vec{y})) + ie \int_{\vec{x}}^{\vec{y}} d\vec{z} (\vec{A}(t_1, \vec{z}) - \vec{A}(t_0, \vec{z})), \quad (15.109)$$

where the difference in the two terms takes into account the parallel horizontal and vertical path segments, see also the charge flow in **Figure 15.4**. The worldline current deduced from (15.109) is given by

$$J_\mu(x) = ie \int_C dz_\mu \delta^{(4)}(z - x), \quad (15.110)$$

and the full source term is given by the *Wilson loop* W_C ,

$$W_C := \exp \int d^4x J_\mu A_\mu = e^{ie \int_C dz_\mu A_\mu(z)}. \quad (15.111)$$

The expectation of the Wilson loop W_C is proportional to the exponential of the free energy $F_{e^+e^-}$ of the $e^+ - e^-$ pair,

$$\langle W_C \rangle \sim e^{-F_{e^+e^-}(C)}, \quad (15.112)$$

and we are interested in its behaviour for large distances L and long times T . As we are working in a Euclidean set-up, the distinction between L and T is an artificial one (for vanishing temperature) and the behaviour in T is the same as that in L .

For (15.112) being a physical observable, the Wilson loop operator (15.111) has to be gauge invariant. To that end we consider the Wilson loop with the gauge-transformed gauge field $A_\mu^\Omega = A_\mu - \partial_\mu \omega$ in (15.72),

$$W_C(A^\Omega) = e^{ie \int_C dz_\mu A_\mu^\Omega(z)} = e^{ie \int_C dz_\mu (A_\mu - \partial_\mu \omega)} = e^{ie \int_C dz_\mu A_\mu}. \quad (15.113)$$

With the same line of arguments one readily proves that an *open Wilson line* with a generic path $C_{x,y}$

$$W_{C_{x,y}}(A) = e^{ie \int_{C_{x,y}} dz_\mu A_\mu(z)} \quad (15.114)$$

transforms similarly to a U(1) link variable on the lattice. Indeed $W_{C_{x,y}}$ is a parallel transport from y to x . As such, its infinitesimal form for $y = x + \epsilon \hat{\mu}$ with $\epsilon \rightarrow 0$ is the continuum version of the the U(1)-link variable (with $a \rightarrow \epsilon$). Under a gauge transformation $\Omega(x)$ the Wilson line (15.114) transforms covariantly,

$$W_{C_{x,y}}(A) = \Omega(x) W_{C_{x,y}}(A) \Omega^\dagger(y), \quad (15.115)$$

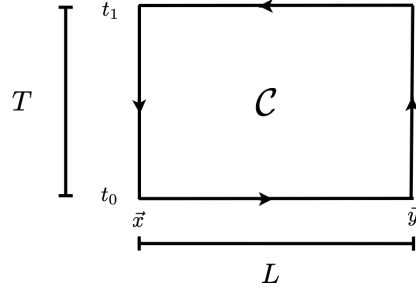


Figure 15.4.: Wordline or path C of a static $e^+ - e^-$ or $q - \bar{q}$ pair created at the time t_0 and pulled apart the distance $L = \|\vec{x} - \vec{y}\|$ and kept at this distance for the time $T = t_1 - t_0$. At the time t_1 it is annihilated. The arrows indicate the electric or colour charge flow.

see (15.85) for comparison. With these definitions we can conclude our derivation of the state or matrix element for the process Figure 15.4. To that end we consider the matrix element of the propagation of an electron (field) from a position y to the position x . It is given by

$$\langle \bar{\psi}(x) W_{C_{x,y}}(A) \psi(y) \rangle, \quad (15.116)$$

where the Wilson line is required for gauge invariance, it parallel transports the gauge group element from y to x . For static quarks the phase carries the full dynamics and for closed worldlines C we are led to the Wilson loop (15.111) in the $U(1)$ theory.

This derivation is now repeated for a non-Abelian gauge group. However, instead of starting with the worldline we rather use the analogue of the matrix element (15.116). The non-Abelian Wilson line has to parallel transport the gauge transformation Ω from y to x which leads us to a product of infinitesimal Wilson lines along the path $C_{x,y}$. This is nothing but a path-ordered exponential similarly to the time-ordered time evolution operator known from the derivation of the path integral. We define

$$U_{C_{x,y}} = \mathcal{P} e^{ig \int_{C_{x,y}} dz_\mu A_\mu(z)}, \quad (15.117)$$

with the path ordering operator \mathcal{P} . For a consecutive order of paths with $C_{x,y}$ being composed out of the path from y to z , $C_{z,y}$ and then from z to x , $C_{x,z}$,

$$\mathcal{P} e^{ig \int_{C_{x,y}} dz_\mu A_\mu(z)} = \mathcal{P} e^{ig \int_{C_{x,z}} dz_\mu A_\mu(z)} \mathcal{P} e^{ig \int_{C_{z,y}} dz_\mu A_\mu(z)}. \quad (15.118)$$

We remark that the definition of the Wilson line (15.118) in terms of the gauge field A_μ makes very explicit its rôle as a parallel transporter. In particular the covariant derivative is given as the parallel transport of the partial derivative,

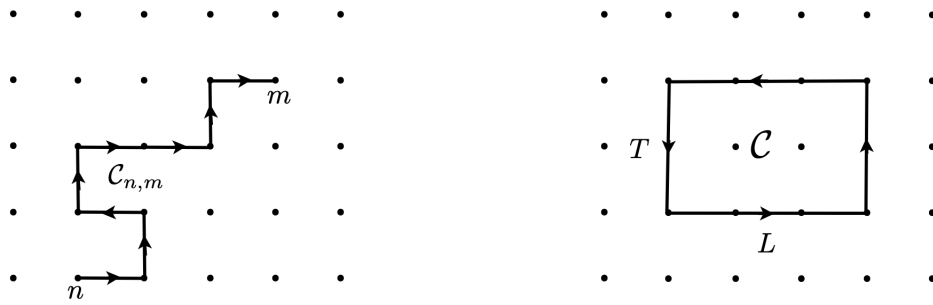
$$U_{C_{y,x}}^\dagger \partial_\mu^x U_{C_{y,x}} = \partial_\mu + ig A_\mu. \quad (15.119)$$

This property can be used to write the Dirac equation in terms of the phase factor $U_{C_{x,y}}$ and the free Dirac operator. The same follows for the respective solution $\psi = U_{C_{x,y}} \psi_{\text{free}}$ of the full Dirac equation.

The closed Wilson loop with a closed Worldline C is given by the trace of the Wilson loop operator U_C ,

$$W_C = \frac{1}{N_c} \text{Tr} U_C, \quad (15.120)$$

the traced Wilson loop in a non-Abelian gauge theory. In summary we conclude that the expectation value of a static $q\bar{q}$ -pair is proportional to that of the traced Wilson loop with


 Figure 15.5.: Wordline or closed paths C and open paths $C_{n,m}$ on the lattice

$$W[L, T] = \langle W_C \rangle = \frac{1}{Z} \int dA W_C(A) e^{-S_A[A]} \quad (15.121)$$

and a similar expression holds for QED. There, we can resort to perturbation theory, and recover the Coulomb potential. This exemplary computation is done in [Appendix I](#), and leads to [\(I.5\)](#),

$$V_{e^+e^-}(L) = -\frac{e^2}{4\pi} \frac{1}{L}, \quad (15.122)$$

as expected. The derivation in [Appendix I](#) also entails, that the contributions come via the resummation of multi-photon exchange diagrams presented in [Figure I.1](#) in [Appendix I](#). The Coulomb potential [\(15.122\)](#) serves as our references result for the computation of the potential of static quarks in the strong coupling expansion in [Section 15.4.2](#).

The whole derivation and the final expression [\(15.121\)](#) is easily translated to the lattice,

Lattice expectation value of the traced Wilson loop

$$W[L, T] = \frac{1}{Z} \int \mathcal{D}U W_C[U] e^{-S_w[U]} \quad (15.123)$$

with the lattice path C depicted in [Figure 15.5](#) and the general Wilson line

$$W_C[U] = \text{tr}_f U_C \quad \text{with} \quad U_{C_{n,m}} = \prod_{l \in C_{n,m}} U_l, \quad (15.124)$$

for a path $C_{n,m}$ also depicted in [Figure 15.5](#). As already discussed above, up to a multiplicative constant, this expectation value is related to the free energy of a quark–anti-quark state at a distance L , hold there for a time T . This leads us to

$$\lim_{T \rightarrow \infty} W[L, T] = F(L) e^{-E(L)T}, \quad (15.125)$$

where, $F(L)$ is the overlap with the ground state.

15.4.2. Static quark potential in the strong coupling expansion

In non-Abelian gauge theories the perturbative computation done in [Appendix I](#) for QED within the Gaussian approximation does not work, as we deal with a strongly-correlated system. If performed it yields a Coloumb potential as in [\(15.122\)](#). The full computation can only be done numerically, and with the current computer resources it is easily done on a laptop for physical lattice sizes. This is beyond the scope of the present lecture course.

As a first step towards the full simulation we perform an analytic computation in the strong coupling expansion with $g_0 \rightarrow \infty$, to wit,

$$\beta = \frac{2N_c}{g_0^2} \rightarrow 0. \quad (15.126)$$

Now we use that the Wilson action [\(15.100\)](#) has a field independent summand which cancels out in the numerator and in the denominator in [\(15.103\)](#). Then the expectation value [\(15.123\)](#) turns into

$$W[L, T] = \frac{\int \mathcal{D}U W_c[U] e^{\beta \sum_P S_P}}{\int \mathcal{D}U e^{\beta \sum_P S_P}} = \langle W_c[U] \rangle \quad (15.127)$$

with the Plaquette action

$$S_P = \frac{1}{2N_c} \text{tr} (U_P + U_P^\dagger) \quad \text{with} \quad U_P = U_{\mu\nu}, \quad (15.128)$$

and \sum_P in [\(15.127\)](#) summing over all plaquettes on the lattice. Now we expand the exponential measure factor $\exp \beta \sum_P S_P$ in powers of the (inverse) coupling β ,

$$e^{\beta \sum_P S_P} = \prod_P e^{\beta S_P} = \prod_P \sum_n \frac{\beta^n}{n!} (S_P)^n. \quad (15.129)$$

Now we use the integration rules of the Haar measure in [\(15.105\)](#) in the denominator of [\(15.127\)](#) as well as the expansion [\(15.129\)](#). This leads us to

$$\int \mathcal{D}U e^{\beta \sum_P S_P} = \int \mathcal{D}U + \mathcal{O}(\beta) = 1 + \mathcal{O}(\beta). \quad (15.130)$$

Hence, the normalisation is nothing but the product of the Haar measure of the link variables, which we have normalised to unity. In the numerator, the respective term of $\mathcal{O}(\beta^0)$ vanished as the integration for the link variables U_l with $l \in C$ vanish: There we have $\int dU_l U_l = 0$, see [\(15.105\)](#).

Accordingly we need at least one U_l^\dagger for a finite result. This can only be achieved by inserting an βU_p^\dagger from the expansion of $e^{\beta \sum_P S_P}$ done in [\(15.129\)](#), see the left figure in [Figure 15.6](#).

However, while this matches the link variable $U_{l_m}^\dagger$, it also creates three 'free' links from U_p^\dagger which have to be matched. This is achieved by augmenting also these lines with the respective βU_p^\dagger 's, yet again generating further 'free' links. The generating of further free links stops for plaquette variables with links being on the contour C . In conclusion, the smallest number of Plaquettes with all links matched is given by the right figure in [Figure 15.6](#). There, the Wilson loop is paired with

$$\overbrace{T/a}^{\hat{T}} \overbrace{L/a}^{\hat{L}} = \frac{A}{a^2} = \hat{A}, \quad (15.131)$$

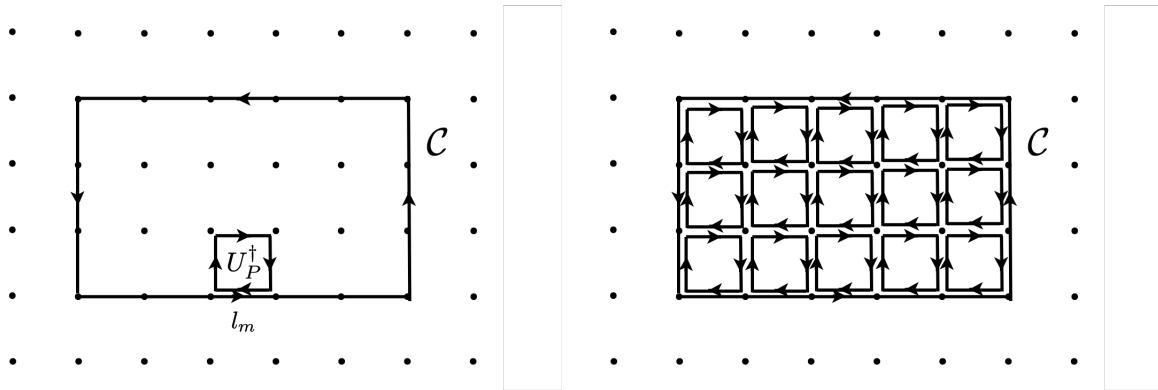


Figure 15.6.: Wilson loop with one adjoint plaquette (left). Wilson loop with plaquettes filling the interior (right).

plaquettes, each of which carries a factor β . We emphasise that the indices on $C_{\mathcal{A}}$ are summed over due to the trace in the Wilson loop. The expression in (15.131) is nothing but the dimensionless area, counting the number of plaquettes inside the path C . This leads us to

$$W[T, L] = \prod_{l \in A_C} \int dU_l U_l^{a_l b_l} (U_l^\dagger)^{c_l d_l} \left(\frac{\beta}{2N_c} \right)^{\hat{A}} + \mathcal{O}(\beta^{\hat{A}+1}), \quad (15.132)$$

where A_C is the area bounded by C . This area contains (including the boundary) $2\hat{A} + \hat{L} + \hat{T}$ links (for each plaquette 2 independent links: $2\hat{A}$, and the remaining half boundary: $\hat{L} + \hat{T}$). At each link we integrate over UU^\dagger with (15.106). Moreover, at each lattice site all group indices of parallel links and adjoint links are the same and summed over,

$$\delta^{a_1 a_2} \delta^{a_2 a_3} \delta^{a_3 a_4} \delta^{a_4 a_1} = N_c. \quad (15.133)$$

As there are $(\hat{L} + 1)(\hat{T} + 1)$ lattice sites within the loop including the boundary, we are led to,

$$\left(\frac{1}{N_c} \right)^{2\hat{A} + \hat{L} + \hat{T}} N_c^{\overbrace{(\hat{L} + 1)(\hat{T} + 1)}^{\# \text{ lattice sites}}} = N_c \left(\frac{1}{N_c} \right)^{\hat{A}}. \quad (15.134)$$

Putting everything together, we arrive at the final result,

Area law for the Wilson loop

$$W[T, L] = N_c \left(\frac{\beta}{2N_c^2} \right)^{\hat{A}} + \mathcal{O}(\beta^{\hat{A}+1}), \quad (15.135)$$

which entails an area law for the Wilson loop. This is the desired result as it entails an growth of the free energy with the area A_C surrounded by the Wilson loop. Evidently, if only the distance L is varied, this entails a linearly rising potential, the two proportionalities go hand in hand. The logarithm of (15.135) is the static $q\bar{q}$ -potential,

Static $q\bar{q}$ -potential

$$\hat{V}(L) = - \lim_{\hat{T} \rightarrow \infty} \frac{1}{\hat{T}} \ln \langle W_C[U] \rangle = \hat{\sigma}(g_0) \hat{L} \quad \text{with} \quad \hat{\sigma} = - \ln \frac{\beta}{2N_c^2}, \quad (15.136)$$

with the string tension $\hat{\sigma}$, measured in lattice units. In summary, lattice Yang-Mills theory shows confinement in the strong coupling limit (15.126). However, we will see shortly that this exciting result does not survive the continuum limit:

To begin with, we have not used in the derivation above that the gauge group is *non*-Abelian. Indeed, it can be straightforwardly carried out in the U(1) case as well without any qualitative changes. In conclusion, also *compact* U(1) theory has a linearly rising static U(1)-potential in the strong coupling limit. Accordingly, *compact* U(1) has a confining phase on the lattice, while it has a Coulomb potential in the continuum (quantised over the algebra). Note however, that also the lattice version has a Coulomb phase with a $1/r$ -potential, and it is this phase which encompasses the continuum limit.

This already casts some doubt on the survival of the $SU(N_c)$ result (15.135) and (15.136) in the continuum. These doubts are solidified if discussing the bare lattice coupling g_0 . In terms of momentum scales, $g_0(a)$ is the coupling of 'classical' Yang-Mills theory, defined at the 'microscopic' scale a , that is related to the momentum scale π/a . In the continuum limit the lattice distance a is approaching zero, and we are probing the coupling $g(a)$ at successively larger momentum scales. Luckily, the running coupling in Yang-Mills theory enjoys asymptotic freedom with the continuum β -function being

$$\beta_g = -\frac{1}{16\pi^2} \frac{11}{3} N_c g_0^3. \quad (15.137)$$

Indeed, this well-known universal result carries over into the present lattice set-up. In any case the sign of the β -function is negative and we have

$$g_0(a) \xrightarrow{\text{cont. limit}} 0. \quad (15.138)$$

Accordingly, the continuum limit is safely governed by (lattice) perturbation theory in the *bare* lattice coupling $g_0(a)$. As it is the only coupling or free parameter in Yang-Mills theory, tuning the continuum limit simply amounts to

$$\beta = \frac{2N_c}{g_0^2} \rightarrow \infty, \quad (15.139)$$

the opposite limit of (15.126), which confirms our suspicion.

15.5. The continuum limit and the renormalisation group

In this Chapter we discuss the continuum limit of lattice field theories. This limit is obtained by keeping the physical scales fixed, while taking the lattice distance to zero, $a \rightarrow 0$. The physical scale are well represented by the correlation length ξ on the lattice that can be extracted from the fundamental two point correlation functions of the theory. For the present introduction we simply consider a real scalar field theory with

$$\lim_{r \rightarrow \infty} \langle \phi(x)\phi(y) \rangle_c \propto e^{-r/\xi}, \quad \text{where} \quad r = \|x - y\|. \quad (15.140)$$

The correlation length is nothing but the inverse mass gap of the theory. Again we consider a simple example, the classical propagator of the three-dimensional scalar field theory with

$$\langle \phi(p)\phi(-p) \rangle_{c,cl} = \frac{1}{p^2 + m_\phi^2}, \quad \longrightarrow \quad \langle \phi(x)\phi(y) \rangle_{c,cl} = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + m_\phi^2} = \frac{1}{8\pi} \frac{1}{r} e^{-m_\phi r}. \quad (15.141)$$

This leads us to $\xi = 1/m_\phi$. More generally, the correlation length is nothing but the distance of the nearest singularity in the complex momentum plane of the correlation function at hand and defines the mass gap of the modes contributing to the correlation function. Evidently this mass gap in general depends on the chosen correlation function as not all of them may overlap with the lowest lying states in the theory. The mass gap of the theory is then given by the smallest mass gap carried by the correlation functions. Again, a simple example is provided by a free scalar theory with two fields, ϕ_1 and ϕ_2 with masses $m_{\phi_1} < m_{\phi_2}$. Clearly, the propagators of the two fields have different correlation lengths $\xi_1 > \xi_2$ and the mass gap of the theory is given by m_{ϕ_1} .

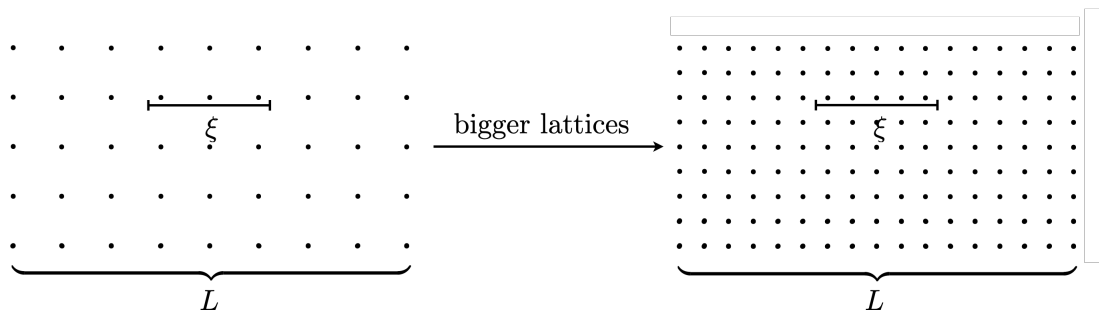


Figure 15.7.: Going from a lattice theory with correlation length ξ , defined on a lattice with lattice spacing a and length L to a theory on a finer lattice with the same correlation length and length, but half the lattice spacing, $a \rightarrow a/2$.

15.5.1. Block-spinning transformations and the RG

So far we have defined lattice field theories by simply introducing an infinite volume lattice with a lattice spacing a as well as defining a discrete version of the classical action, that approaches the continuum action for $a \rightarrow 0$ up to $\mathcal{O}(a)$ -terms. More precisely, the continuum limit is achieved by taking $a \rightarrow 0$, while keeping the physical correlation length ξ fixed.

Let us now go to the more realistic situation in numerical applications: a lattice with a finite extent L in all d directions. Then we deal with a hyper-cubic lattice with $(\hat{L} + 1)^d$ lattice sites (\hat{L}^d independent ones). A step towards the continuum limit in a two-dimensional theory at fixed L is depicted in Figure 15.7 with $a \rightarrow a/2$. This implies $\hat{L} \rightarrow 2\hat{L}$ and hence this procedure increases the number of lattice sites by 2^d .

Typically one does not have the luxury of achieving the continuum limit by simply increasing the number of lattice sites. Instead, one is using the maximal lattice one (or rather the computing resources) numerically can cope with and decreases the lattice spacing, see Figure 15.8. Then, \hat{L} is kept fixed as is ξ . Instead the dimensionless lattice parameters are changed, most clearly this happens for the mass $\hat{m}_\phi = m_\phi a$: at fixed m_ϕ we have $\hat{m}_\phi \rightarrow 2\hat{m}_\phi$. Iterating this procedure i times leads to a lattice spacing $2^i a$ and a lattice extent of $L/2^i$. Evidently, this implies that the correlation length eventually will exceed the lattice extent, $2^i \xi/L > 1$, and the finite volume effects will dominate the physics. Already the cartoon situation in Figure 15.8 makes this abundantly clear.

In summary this asks for a careful, mathematically sound description of such a rescaling of lattice field theories, which should also allow us to facilitate and optimise numerical computations. This is done with the renormalisation group (Stueckelberg, Petermann (1953)), that has been introduced for describing the change of a given theory under general rescalings (and reparametrisations). This will be discussed in the next Chapter, Chapter 16. The discrete version of renormalisation group transformations for lattice theories is the block-spinning transformation (Kadanoff (1966)). At its heart it is Wilson's (1971) renormalisation group, that underlies most modern applications of the renormalisation group.

Such a block-spinning transformation is introduced as a transformation from a finer lattice with lattice distance a to the coarser lattice with lattice distance $2a$. For this transformation we average (*coarse grain*) the field values over square blocks of neighbouring lattice sites, see fig. 15.9. This procedure is called coarse graining and is the *first block-spinning step*. Applied to the finer lattice in Figure 15.7 with lattice distance $a/2$, it brings us back to the original lattice with lattice distance a . Applied to the finer lattice in Figure 15.8, it gives us back the original lattice with half the lattice extent.

Note that this step implies a loss of resolution at fixed correlation length ξ . This is reflected in momentum space by the fact, that the Brillouin zone reduces from $p_\mu \in [-\pi/a, \pi/a]$ to $p_\mu \in [-\pi/(2a), \pi/(2a)]$. This entails that

the quantum fluctuations of the momentum shell

$$|p_\mu| \in [-\pi/(2a), \pi/a], \tag{15.142}$$

are averaged over (integrated out). Put differently, this is an information loss, evident in Figure 15.9, and is mirrored in $\hat{\xi} = \xi/a$, which transforms $\hat{\xi} \rightarrow 1/2 \hat{\xi}$.

However, in a *second block spinning step* we can recover the original situation on the right hand side of fig. 15.9. To that end we rescale ξ by a factor 2 with $\xi \rightarrow 2\xi$. The latter implies that all the dimensionfull parameters in the lattice theory have to be rescaled accordingly, i.e. $m_\phi \rightarrow 1/2 m_\phi$ leading to $\hat{m}_\phi \rightarrow 1/2 \hat{m}_\phi$. Moreover, this implies that L is kept fixed. Then the final lattice field theory has the same correlation length as before as before the two block-spinning steps, and we have,

$$\hat{\xi} \xrightarrow{\text{step 1}} \frac{\hat{\xi}}{2} \xrightarrow{\text{step 2}} \hat{\xi}, \quad \text{and} \quad \hat{m}_\phi \xrightarrow{\text{step 1}} 2\hat{m}_\phi \xrightarrow{\text{step 2}} \hat{m}_\phi, \quad \hat{\lambda} \xrightarrow{\text{step 1\&2}} \hat{\lambda}. \tag{15.143}$$

We emphasise again that on a finite lattice, the combination of the two steps is no identity transformation, as

$$\hat{L} \xrightarrow{\text{step 1\&2}} \frac{\hat{L}}{2}. \tag{15.144}$$

The above example is the simple case of a more coarse grainings and rescalings. For example, in a Ising spin system, the blocking on fundamental plaquettes is not well-defined as the spins can average to zero, while the spin operators on the coarse grained lattice only take values ± 1 . Generally, we introduce a blocking

$$\phi'_{n'} = f_{n'}(\phi_n), \tag{15.145}$$

with a potentially non-linear blocking function $f_{n'}(\phi_n)$. In the example above the n' are the vectors of the center of fundamental plaquettes and $f_{n'}$ simply sums the field over the sites of the plaquette. The coarse grained field ϕ' lives on the coarse-grained lattice. For a given lattice action $S[\phi_n]$ the block-spinning amounts to

$$e^{-S'[\phi'_{n'}]} = \int \prod_n d\phi_n \delta[\phi'_{n'} - f_{n'}(\phi_n)] e^{-S[\phi_n]}, \tag{15.146}$$

which also defines a lattice action on $S'[\phi'_{n'}]$ the coarse-grained lattice. Note that the generating functional Z is unchanged as the Dirac δ -function is removed by the integral over all field values on the sites on the coarse-grained lattice,

$$\int d\phi'_{n'} \delta[\phi'_{n'} - f_{n'}(\phi_n)] = 1, \tag{15.147}$$

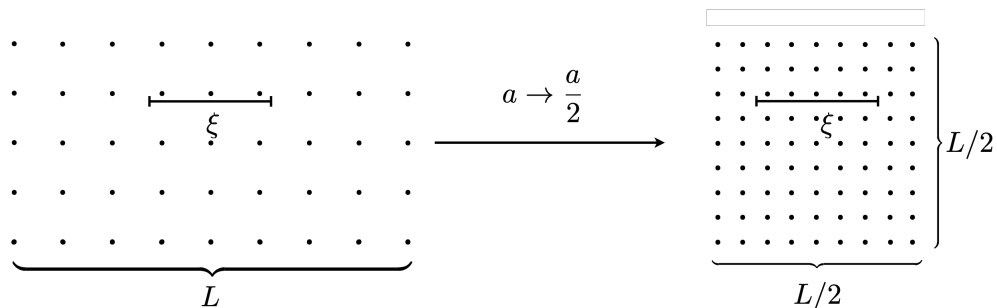


Figure 15.8.: Mapping the lattice theory with correlation length ξ , defined on a lattice with lattice spacing a and length L to a theory on a finer lattice with the same correlation length, but half the lattice spacing and length: $a \rightarrow a/2$ and $L \rightarrow L/2$.

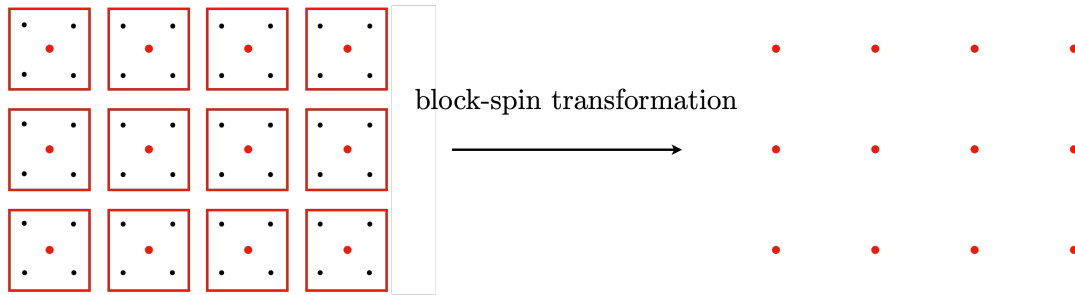


Figure 15.9.: *Block-Spinning* on the lattice: we average (*coarse graining*) the field on the finer lattice (black dots) with lattice distance a on fundamental squares or plaquettes. This defines a field on a lattice with lattice distance $2a$ (red dots).

and hence we conclude

$$Z' = \int \prod_n d\phi'_n e^{-S'[\phi'_n]} = \int \prod_n d\phi'_n \int \prod_n d\phi_n \delta[\phi'_n - f_n(\phi_n)] e^{-S[\phi_n]} = \int \prod_n d\phi_n e^{-S[\phi_n]} = Z. \quad (15.148)$$

Again this emphasises the point that the coarse-grained system has the same expectation values for correlations ϕ'_n , that can also be simulated on the original lattice, but does not resolve smaller distances than $2a$.

In summary, our analysis entails that a lattice action at a given lattice distance a , or rather a/ξ , comprises already quantum effects of momenta $p_\mu > \pi/a$, as these momenta can be integrated out shell-wise, see (15.142), starting at an infinitely fine lattice. Coarser lattices are approached by successive block-spinning or coarse graining within block-spinning RG-steps. The opposite procedure is, strictly speaking only possible on a lattice with infinite extent, therefore called an inverse RG step. Both, RG and inverse RG are used for optimising lattice simulations as well as conceptual investigations.

15.5.2. The Continuum Limit of Lattice Yang-Mills

With the understanding gained in Section 15.5.1 we now come back to the continuum limit of Lattice Yang-Mills theories, already sketchily discussed at the end of Section 15.4.2. In the continuum limit of a lattice theory we keep the physical correlation length

$$\xi = 1/m_{\text{gap}}, \quad (15.149)$$

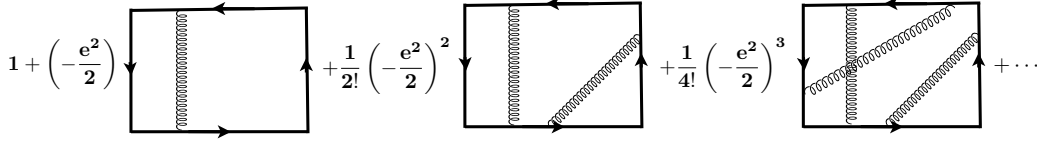
fixed, where m_{gap} is the mass gap of the theory; in a scalar theory the mass gap is simply the pole mass of the scalar field, $m_{\text{gap}} = m_{\phi, \text{pole}}$. Note that the latter is not the mass parameter in the action, but rather defined as the singularity (at $p_0^2 = -m_{\phi, \text{pole}}^2$) of the propagator. In Yang-Mills theory the situation is more complicated as the classical action features no mass scale (the theory has conformal symmetry on the classical level). The quantised theory has a mass gap given by the mass of the lowest glue-ball state.

In any case, (15.149) implies, that the dimensionless mass gap tends to zero on the lattice,

$$\hat{m}_{\text{gap}} = m_{\text{gap}} a \rightarrow 0, \quad (15.150)$$

and hence the correlation length $\hat{\xi} = 1/\hat{m}$ diverges, This is the signature of a 2nd order phase transition. At this point the system has infinite many points inside a physical distance measured in units of the correlation length. As discussed above, in lattice YM we only have the *bare* strong coupling g_0 on the lattice for tuning this limit,

$$\hat{\xi}(g_0) \xrightarrow{g_0 \rightarrow g_*} \infty, \quad (15.151)$$


 Figure 15.10.: Diagrams contributing to the effective potential $V(L)$ up to order g_0^4

where g^* is the fixed point value of the *bare* lattice coupling g_0 .

It is left to extract the dependence of $g_0(a)$ on the lattice spacing a in the continuum limit. To that end we discuss the g_0 and a dependence of a general observable \mathcal{O} in the continuum limit. Its relation to the dimensionless lattice observable $\hat{\mathcal{O}}$ is given by

$$\mathcal{O}(g_0, a) = \left(\frac{1}{a}\right)^{d_{\mathcal{O}}} \hat{\mathcal{O}}(g_0) \quad (15.152)$$

where $d_{\mathcal{O}}$ is the momentum dimension of the observable \mathcal{O} . In the continuum limit we have,

$$\mathcal{O}(g_0 \rightarrow g_*, a \rightarrow 0) = \mathcal{O}_{phys} \quad (15.153)$$

Thus, if we know the functional dependence of \mathcal{O} on g_0 , we know $g_0(a)$ with $\mathcal{O}(g_0, a) = \mathcal{O}_{phys}$. The above argument seemingly implies that $g_0(a)$ depends on the choice of \mathcal{O} . However, the coupling $g_0(a)$ is two-loop *universal*, to that order it does not depend on the observable or the renormalisation procedure. In the continuum limit with $a \rightarrow 0$ we shall see, that also the coupling $g_0 \rightarrow 0$ and hence the continuum limit is governed by the universal running of the coupling.

Let us now take as \mathcal{O} the $q\bar{q}$ -potential V defined in the previous section:

$$V(L, g_0, a) = \frac{1}{a} \hat{V}(\hat{L}, g_0) \quad (15.154)$$

Now, keeping $V(L, g_0, a)$ fixed at its physics value V_{phys} while $a \rightarrow 0$ implies

Renormalisation Group Equation

$$\left(a \frac{\partial}{\partial a} - \beta(g_0) \frac{\partial}{\partial g_0}\right) V(L, g_0, a) = 0 \quad (15.155)$$

with the lattice β -function,

$$\beta(g_0) = -a \frac{\partial g_0}{\partial a}. \quad (15.156)$$

The standard continuum formulation with a cutoff or RG scale Λ is obtained with $\Lambda \sim 1/a$ and hence $\Lambda \partial_{\Lambda} = -a \partial_a$. We now consider the RG-equation up to the forth order in the *bare* lattice coupling g_0^4 . The respective diagrams are depicted in [Figure 15.10](#) and the potential is computed as

$$V(L) = -\frac{g_0^2(a)}{4\pi L} C_2(f) \left[1 + g_0^2(a) \frac{11N_c}{24\pi^2} \ln \hat{L} + \text{const} \right]. \quad (15.157)$$

If we insert [\(15.157\)](#) in [\(15.156\)](#) we get

$$\beta(g_0) = -\frac{g_0^3}{16\pi^2} \frac{11}{3} N_c = \beta_0 g_0^3, \quad (15.158)$$

the well-known continuum result [\(14.92\)](#) in [Section 14.2.4](#). Since $\beta(g_0) < 0$ is smaller (asymptotic freedom), the coupling is driven to zero in the limit $a \rightarrow 0$.

$$a = \frac{1}{\Lambda_L} e^{\frac{1}{2\beta_0 g_0^2}} \quad (15.159)$$

This concludes the discussion of the scaling of lattice Yang-Mills theory in the limit $a \rightarrow 0$. Let us now rewrite the RG-equation (15.155) in terms of physical scales, substituting a with the distance scale L in the heavy quark potential (not to be confused with the extend of the lattice, which is assumed to be infinite here). With (15.157) this leads us to

$$\left(L \frac{\partial}{\partial L} + \beta(g_0) \frac{\partial}{\partial g_0} \right) V(L, g_0, a) = -V(L, g_0, a), \quad (15.160)$$

where the right hand side of (15.160) originates in the $1/L$ factor in (15.157), and constitutes an inhomogeneous term in the RG equation. It simply entails the dimension-scaling of the potential, and hence can be undone if we rescale the potential V with L in order to make it dimensionless. The homogeneous form of the RG has the advantage that we can directly read of the running coupling from it, as discussed below. We define the dimensionless potential \bar{V} with

$$\bar{V}(\hat{L}, g_0) = LV(\hat{L}, g_0), \quad \text{with} \quad \hat{L} = \frac{L}{a}. \quad (15.161)$$

In (15.161) we have used that the dimensionless potential can only depend on dimensionless variables, and hence it only can depend on the ratio of the two length scales L and a . Put differently, the distance scale in the potential has to be measured in units of the lattice distance, the only scale in the system.

Inserting $V = 1/L\bar{V}$ into (15.160) gives us the desired homogeneous RG equation,

$$\left(L \frac{\partial}{\partial L} + \beta(g_0) \frac{\partial}{\partial g_0} \right) \bar{V}(\hat{L}, g_0) = 0. \quad (15.162)$$

Equation (15.162) entails that a change in the physical distance L can be absorbed in a corresponding change of the bare coupling g_0 . It is nothing but the lattice version of the renormalisation group equation for the running coupling of QCD discussed in Section 14.2. Now we perform yet another change of variables, and change from the bare coupling $g_0(a)$ to the running coupling $g(L)$ in analogy to our discussion in QCD, where we went from the renormalised coupling $\alpha_{s,\text{ren}}(\mu)$ to the running coupling $\alpha_s(p)$, see the introduction of Section 14.2. In the present analysis, the rôle of the momentum p is taken by π/L . In this spirit we write

$$\bar{V} = \bar{V}(L, g(L)), \quad \text{with} \quad L \frac{\partial}{\partial L} g(L) = -\beta(\bar{g}(L)), \quad (15.163)$$

in a slight abuse of notation. We emphasise that (15.163) is not simply obtained by substituting the bare coupling in (15.161) by the running coupling. It implies a reshuffling of explicit L and implicit L -dependences as can be seen from the RG equation for the running coupling. The solution of the RG equation of the coupling has been already discussed in Section 14.2.4, see (14.97). In the present case we obtain

$$g^2(L) = \frac{g_0^2}{1 + \beta_0 g_0^2 \ln L^2/a^2}, \quad \text{with} \quad g_0 = g(a). \quad (15.164)$$

Using (15.164) in \bar{V} and V with (15.157) and (15.161) leads to,

$$V(L) = -C_F \frac{\alpha_S(L)}{L}, \quad \text{with} \quad \alpha_s(L) = \frac{\bar{g}^2(L)}{4\pi}. \quad (15.165)$$

The prefactor in (15.165) is the Casimir (14.12) with $C_F = N_c^2 - 1/(2N_c)$, in QCD we have $C_F = 4/3$. Our result (15.165) seemingly depends on a . However, with the relation (15.166), the a -independence is apparent,

Running lattice coupling

$$\alpha_s(L) = \frac{1}{4\pi} \frac{g_0^2}{1 + \beta_0 g_0^2 \ln L^2 \Lambda_L^2 e^{-1/\beta_0 \tilde{a}_0^2}} = \frac{4\pi}{\beta_0} \frac{1}{\ln L^2 \Lambda_L^2}. \quad (15.166)$$

with $\beta_0 = 11/3N_c$, see (14.95), and the lattice version of Λ_{QCD} ,

Lattice Λ_{QCD}

$$\Lambda_L = \frac{1}{a} e^{\frac{1}{2\beta_0 g_0^2}}, \quad \text{with} \quad a \frac{d}{da} \Lambda_L = \left(a \frac{\partial}{\partial a} - \beta(g_0) \frac{\partial}{\partial g_0} \right) \left(\frac{1}{a} e^{\frac{1}{2\beta_0 g_0^2}} \right) = 0. \quad (15.167)$$

Equation (15.166) is the lattice version of (14.99). In the present one-loop treatment the coupling diverges at the infrared Landau pole $L = 1/\Lambda_L$. (14.95) For a better accuracy we can invoke two-loop perturbation theory such as described at the end of Section 14.2.4 below (14.102).

16. Renormalisation Group

Renormalisation group transformations describe general reparametrisations of the theory at hand. Let us first consider Kadanoff's block spinning transformation on the lattice, discussed in [Section 15.5](#). At its heart there is the coarse graining step where one averages over blocks of lattice sites, see [15.9](#). With a coarse graining step we lose resolution and hence we lose access to large momentum modes. Moreover, the lattice action on the coarse-grained lattice is obtained by the same averaging procedure, see [\(15.146\)](#). With the second block spinning step we can rescale the correlation length and other scales such that we return to the original lattice. Still, the lattice action is changed, even though the physics is not. However, within an iteration of this procedure we are led to a fixed point of this transformation, the renormalisation group fixed point. While the RG-steps describe discrete transformations, we have already seen in [Section 15.5](#) that the underlying structure is that of a one-parameter group of scale transformations.

In the present chapter we will exploit both, the group structure of the renormalisation group in terms of a reparametrisation invariance as well as its underlying description of *physical* scale transformation to develop the powerful modern functional renormalisation group framework.

16.1. Wilsonian Renormalisation Group

In [Chapter 15](#) we have regularised the functional integral of a quantum field theory with an explicit ultraviolet cutoff, the inverse lattice spacing. The momenta were limited by $p_\mu^2 \leq \pi^2/a^2$ for all μ and space-time was discrete.

Now we transport this picture to the continuum and explicitly apply a momentum cut-off $\Lambda = 1/a$ to the continuum fields. A respective block-spinning step then can be taken continuously from $\Lambda \rightarrow b\Lambda$. On the level of the generating functional of a real scalar field theory this is implemented by

Generating functional with ultraviolet momentum cutoff

$$Z_\Lambda = \int [D\phi]_\Lambda e^{-S[\phi]}, \quad \text{with} \quad S[\phi] = \int d^d x \left\{ \frac{1}{2} \phi (-\Delta + m^2) \phi + \frac{\lambda}{4!} \phi^4 \right\}, \quad (16.1)$$

with

$$[D\phi]_\Lambda = \prod_{p^2 \leq \Lambda} d\phi(p), \quad \text{or} \quad \phi_\Lambda(p^2 > \Lambda^2) = 0. \quad (16.2)$$

[Equation \(16.1\)](#) has no source term which can be added without problems. In terms of the full functional integral, [\(16.1\)](#) lacks the integration over all modes. This constraint can be easily implemented by adding a quadratic cutoff term to the classical action with

Wilsonian cutoff term

$$S[\phi] \rightarrow S[\phi] + \Delta S_\Lambda[\phi], \quad \text{with} \quad \Delta S_\Lambda[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \phi(-p) R_\Lambda(p^2) \phi(p), \quad (16.3)$$

where the cutoff term ΔS_Λ is introduced to suppress fields that have support for momenta $p^2 > \Lambda^2$. This entails that the *regulator* $R_\Lambda = R_\Lambda^{(\text{UV})}$ is defined as

$$R_\Lambda^{(\text{UV})}(p) = \begin{cases} \infty & p^2 > \Lambda^2 \\ 0 & p^2 \leq \Lambda^2 \end{cases}, \quad \text{with} \quad R_\Lambda^{(\text{UV})}(p) = \Lambda^2 \left(\frac{1}{\theta(\Lambda^2 - p^2)} - 1 \right), \quad (16.4)$$

with the Heaviside θ -function with $\theta(x > 0) = 1$ and $\theta(x < 0) = 0$. **Section 16.1.2** is the starting point for the derivation of functional renormalisation group equations, and in most cases the regulator $R_\Lambda^{(\text{UV})}(p)$ is taken smooth and not with a *sharp* cutoff such as in (16.4). This setting leads to the *Polchinski equation*, a functional renormalisation group equation for the Wilsonian effective action (a variant of the Schwinger functional).

The most fruitful formulation is obtained, if we use an infrared cutoff instead of an ultraviolet one. This is the direct implementation of the coarse graining idea on the lattice: instead of cutting off the ultraviolet modes we average over an area in space time, or, alternatively, we suppress low momentum modes $p^2 \leq k^2$ with an infrared cutoff scale k . This can be done similarly to the ultraviolet cutoff, and the respective sharp cutoff regulator reads

$$R_k^{(\text{IR})}(p) = \begin{cases} 0 & p^2 > k^2 \\ \infty & p^2 < k^2 \end{cases}, \quad \text{with} \quad R_k^{(\text{IR})}(p) = k^2 \left(\frac{1}{\theta(p^2 - k^2)} - 1 \right). \quad (16.5)$$

The respective functional RG equation is the *Wetterich equation*, and its variant for $R_k = k^2$ is the functional Callan-Symanzik equation, first derived by Symanzik. We will come back to these equations and their applications at the end of this Chapter.

16.1.1. Momentum-shell RG

We proceed from (16.1) with an RG step that is emulating the block spinning procedure on the lattice. We aim at representing the generating functional Z for the ultraviolet cutoff $b\Lambda$ in the same form as for Λ . In a first step we split the functional integral measure (16.2) in the part over modes with momenta $p^2 \leq b^2\Lambda^2$ and the rest. Then we integrated out the rest and arrive at a functional integral with the functional integral measure $[D\phi]_{b\Lambda}$. To that end we define the field $\hat{\phi}(p)$, that only has support for momenta p^2 between $b^2\Lambda^2$ and Λ^2 ,

$$\hat{\phi}(p) = \begin{cases} \phi(p) & b\Lambda \leq |p| \leq \Lambda \\ 0 & \text{else} \end{cases}, \quad \text{and} \quad \phi_\Lambda(p) = \phi_{b\Lambda}(p) + \hat{\phi}(p). \quad (16.6)$$

Equation (16.6) allows us to perform the split of the measure, to wit,

$$[D\phi]_\Lambda = [D\phi]_{b\Lambda} D\hat{\phi}. \quad (16.7)$$

With these preparations we write

$$Z_\lambda = \int [D\phi]_{b\Lambda} \int D\hat{\phi} e^{-S[\phi_{b\Lambda} + \hat{\phi}]} = \int [D\phi]_{b\Lambda} e^{-S[\phi_{b\Lambda}]} \int D\hat{\phi} e^{-\int d^d x \left\{ \frac{1}{2} \hat{\phi}(\Delta + m^2) \hat{\phi} + \lambda \left(\frac{1}{6} \phi_{b\Lambda}^3 \hat{\phi} + \frac{1}{4} \phi_{b\Lambda}^2 \hat{\phi}^2 + \frac{1}{6} \phi_{b\Lambda} \hat{\phi}^3 + \frac{1}{4!} \hat{\phi}^4 \right) \right\}}. \quad (16.8)$$

where we have used that

$$\int d^d x \hat{\phi} \Delta \phi_{b\Lambda} = 0 = m^2 \int d^d x \hat{\phi} \phi_{b\Lambda} = 0. \quad (16.9)$$

Equation (16.8) is a functional integral with the measure $[D\phi]_{b\Lambda} \exp\{-S[\phi_{b\Lambda}]\}$, augmented with a further non-trivial measure factor in terms of the $\hat{\phi}$ integral. Note that the latter functional integral is finite as it is infrared and ultraviolet regularised. Integrating over $\hat{\phi}$ leads us to

Generating functional after one Wilsonian RG-step

$$Z_\Lambda = \int [D\phi]_{b\Lambda} e^{-S_{\text{eff}}[\phi]} \quad (16.10)$$

where we now drop the subscript $b\Lambda$ for ϕ , and

$$S_{\text{eff}}[\phi] = S[\phi] - \log \left[\int D\hat{\phi} \exp \left\{ - \int d^d x \left[\frac{1}{2} \hat{\phi}(\Delta + m^2) \hat{\phi} + \lambda \left(\frac{1}{6} \phi^3 \hat{\phi} + \frac{1}{4} \phi^2 \hat{\phi}^2 + \frac{1}{6} \phi \hat{\phi}^3 + \frac{1}{4!} \hat{\phi}^4 \right) \right] \right\} \right]. \quad (16.11)$$

Equation (16.10) has the same form as the original generating functional, but with a different 'classical' action. Indeed, S_{eff} explicitly includes all quantum effects of momentum modes with $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$. Evidently, this RG-step can be repeated and we can store the quantum fluctuations of momentum shells such as $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$ in the *Wilsonian effective action* S_{eff} .

The momentum-shell integration that leads to (16.11) is the continuum version of the first block-spinning step on the lattice. In a second step we want to transform the fields and parameters such, that (16.11) takes again the form (16.1) for $\Lambda, b\Lambda \rightarrow \infty$. This limit is reached if the ultraviolet cutoffs $\Lambda, b\Lambda$ are asymptotically larger than the physics scales such as the mass scale, $m^2/\Lambda^2 \ll 1$. Note that strictly speaking this assumption has to apply to all physical correlation scales in the quantum theory and not to a classical parameter in the action. In this case we can expand the $\hat{\phi}$ -integral in (16.11) about its Gaussian part. The $\hat{\phi}^2$ term reads

$$\int d^d x \hat{\phi}(-\Delta) \hat{\phi} = \int_{b\Lambda \leq p^2 \leq \Lambda^2} \frac{d^d p}{(2\pi)^4} \hat{\phi}(p) p^2 \hat{\phi}(-p), \quad (16.12)$$

with momentum squared values $b\Lambda \leq p^2 \leq \Lambda^2$. This entails that only field with amplitudes $\hat{\phi} \lesssim \frac{1}{\Lambda} \rightarrow 0$ contribute to the $\hat{\phi}$ integral and we arrive at

$$\int D\hat{\phi} e^{-\int d^d x \left\{ \frac{1}{2} \hat{\phi}(-\Delta + m^2 + \frac{1}{2} \phi^2) \hat{\phi} + \frac{\lambda}{6} \phi^3 \hat{\phi} + O(\hat{\phi}^3) \right\}}. \quad (16.13)$$

Equation (16.13) constitutes a Gaussian functional integral with a source term

$$\int \frac{d^d p}{(2\pi)^d} J(-p) \hat{\phi}(p), \quad \text{with} \quad J(x) = -\frac{\lambda}{6} \phi^3(x), \quad (16.14)$$

and $J(p)$ is the Fourier transform of $J(x)$ in a slight abuse of notation with

$$J(p) = \int_{p_1, p_2} \phi(p_1) \phi(p_2) \phi(p - p_1 - p_2). \quad (16.15)$$

The representation of the source term in Fourier space makes apparent, that only (16.14) momenta $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$ are considered in the current as it is coupled to $\hat{\phi}(-p)$. The Gaussian integral (16.13) is readily performed and yields the determinant of the kinetic operator $-\Delta + m^2 + \frac{1}{2}\phi^2$, multiplied by the exponential of

$$\frac{1}{2} \int_{p, q} J(p) G_{b\Lambda, \Lambda}[\phi](p, q) J(q) = \frac{\lambda^2}{72} \int d^d x d^d y \phi^3(x) G_{b\Lambda, \Lambda}[\phi](x, y) \phi^3(y). \quad (16.16)$$

where $G_{b\Lambda, \Lambda}$ is the inverse of $-\Delta + m^2 + \frac{1}{2}\phi^2$ on the momentum shell ($b\Lambda, \Lambda$) as $\hat{\phi}(p) = \hat{\phi}(p) \theta(p^2 - b^2\Lambda^2) \theta(\Lambda^2 - p^2)$. Collecting all the different pieces we arrive at

$$S_{\text{eff}}[\phi] = S[\phi] + \frac{1}{4} \text{tadpole} - \frac{1}{8} \text{fish} + O(\phi^6)$$

Figure 16.1.: Wilsonian effective action after one RG-step from Λ to $b\Lambda$. The double line propagators $G_{b\Lambda,\Lambda}(p)$ indicate the fact that their momentum is restricted to the regime $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$, see (16.21).

Functional momentum-shell integral

$$\int D\hat{\phi} e^{-\int d^d x \{ \frac{1}{2} \hat{\phi} (-\Delta + m^2 + \frac{1}{2} \phi^2) \hat{\phi} + \frac{\lambda}{6} \phi^3 \hat{\phi} + O(\hat{\phi}^3) \}} \simeq \det_{b\Lambda,\Lambda}^{-\frac{1}{2}} \left(-\Delta + m^2 + \frac{\lambda}{2} \phi^2 \right) e^{\frac{1}{2} \int_{p,q} J(p) G_{b\Lambda,\Lambda}[\phi](p,q) J(q) + O(1/\Lambda)}, \quad (16.17)$$

where the subscript $b\Lambda,\Lambda$ indicates that the determinant only runs over momentum values $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$, and we have dropped further terms that are suppressed with powers of $1/\Lambda$. It is left to compute the functional determinant in (16.17). It can be written in terms of a trace, $\det A = e^{\text{Tr} \ln A}$ analogously as in our investigations of the one-loop effective action and potential in Section 11.2. Similarly to there we arrive at

$$\log \det_{b\Lambda,\Lambda}^{-\frac{1}{2}} \left(-\Delta + m^2 + \frac{\lambda}{2} \phi^2 \right) = -\frac{1}{2} \text{Tr}_{b\Lambda,\Lambda} \ln \left(-\Delta + m^2 + \frac{\lambda}{2} \phi^2 \right). \quad (16.18)$$

As in (16.17), the subscript $b\Lambda,\Lambda$ indicates the restriction to momenta in the interval $p \in [b\Lambda, \Lambda]$ with $p = \sqrt{p^2}$. This is best seen if the trace is evaluated in momentum space with

$$\frac{1}{2} \text{Tr}_{b\Lambda,\Lambda} \ln \left(-\Delta + m^2 + \frac{\lambda}{2} \phi^2 \right) = \int_{b^2\Lambda^2 \leq p^2 \leq \Lambda^2} \frac{d^d p}{(2\pi)^d} \ln \left(-\Delta + m^2 + \frac{\lambda}{2} \phi^2 \right)(p, p). \quad (16.19)$$

and hence,

Wilsonian effective action after one momentum-shell integration

$$S_{\text{eff}}[\phi] = S[\phi] + \frac{1}{2} \text{Tr}_{b\Lambda,\Lambda} \ln \left(-\Delta + m^2 + \frac{\lambda}{2} \phi^2 \right) - \frac{\lambda^2}{72} \int d^d x d^d y \phi^3(x) G_{b\Lambda,\Lambda}[\phi](x, y) \phi^3(y) + O\left(\frac{1}{\Lambda}\right). \quad (16.20)$$

Equation (16.20) is the final result for the Wilsonian effective action after a block spinning step at asymptotically large ultraviolet cutoff scales. Evidently, $S_{\text{eff}}[\phi]$ is non-polynomial and we have to discuss the new terms in $S_{\text{eff}}[\phi]$ in comparison to the classical ϕ^4 action. We shall see that in a renormalisable quantum field theory the higher order terms are suppressed with powers of $1/\Lambda$, while the (relevant) terms in the classical action are augmented with a cutoff running with $\log \Lambda$ (marginal terms) and positive powers of Λ (UV-relevant terms). In short, we shall re-derive the perturbative renormalisation programme in a manifestly finite set-up, however, the Wilsonian or functional renormalisation group approach does not stop there. It accommodates general quantum field theories beyond perturbative renormalisation. Its diagrammatic depiction is given in Figure 16.1. The momentum independent tadpole gives a correction to the mass term of the theory, while the fish diagram contributes to the ϕ^4 term. The diagram carries a non-trivial momentum dependence, which is suppressed for $p_i^2/\Lambda^2 \ll 1$, where p_i with $i = 1, \dots, 4$ are the momenta of the fields. We emphasise that the double line propagators $G_{b\Lambda,\Lambda}(p)$ in the diagrams are *all* restricted to the momentum regime $b^2\Lambda^2 \leq p^2 \leq \Lambda^2$, that is

$$G_{b\Lambda,\Lambda}(p) = \frac{1}{p^2 + m^2} \theta(p^2 - b^2\Lambda^2) \theta(\Lambda^2 - p^2). \quad (16.21)$$

Note that the diagrams in 16.1 and hence also (16.20) are not simply given by standard propagators and an integral in loop momenta that is restricted to the momentum shell ($b\Lambda, \Lambda$). Indeed, the latter loop diagram has no shift symmetry in the momentum while the diagrams in Figure 16.1 have, as have all diagrams in (16.20). In summary, the Wilsonian effective action can be expanded as follows

$$S_{\text{eff}}[\phi] = S[\phi] - \frac{\Delta m^2}{2} \int_p \phi(p)\phi(-p) + \frac{1}{4!} \int_{p_1, \dots, p_4} \Delta\lambda(p_1, \dots, p_4) \phi(p_1) \cdots \phi(p_4) + \mathcal{O}(\phi^6). \quad (16.22)$$

with

$$\Delta m^2 = \frac{\lambda}{2} \int \frac{d^d p}{(2\pi)^d} G_{b\Lambda, \Lambda}(p), \quad \Delta\lambda(p_1, \dots, p_4) = 3\lambda^2 \int \frac{d^d p}{(2\pi)^d} G_{b\Lambda, \Lambda}(p) G_{b\Lambda, \Lambda}(p + p_1 + p_2). \quad (16.23)$$

Let us now compute the diagrams or rather the coefficients $\Delta m^2, \Delta\lambda$ defined in (16.23) in an expansion of the external momenta p_i and the mass m measured in the cutoff scale. For the derivation of (16.20) and hence of

$$\frac{m^2}{\Lambda^2} \ll 1, \quad \frac{p_i^2}{\Lambda^2} \ll 1. \quad (16.24)$$

With (16.24) we can expand the propagators in the diagrams about vanishing mass, as the momentum of all propagators is in the regime $b\Lambda^2 \leq p^2 \leq \Lambda^2$,

$$G_{b\Lambda, \Lambda}(p) = \left[\frac{1}{p^2} - \frac{m^2}{p^2} \frac{1}{p^2 + m^2} \right] \theta(p^2 - b^2\Lambda^2) \theta(\Lambda^2 - p^2), \quad (16.25)$$

where the second term is of order $m^2/p^2 \rightarrow 0$. This allows us to compute the two diagrams in Figure 16.1 and further ones in powers and inverse powers of the cutoff scale. We start this investigation with Δm^2 . We use (16.25) and find

$$\Delta m^2 = \frac{\lambda}{2} \int_{b\Lambda}^{\Lambda} \frac{d^d p}{(2\pi)^d} \left[\frac{1}{p^2} - \frac{m^2}{p^2} \frac{1}{p^2 + m^2} \right] = \frac{\lambda}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \Lambda^{d-2} \left[\frac{1 - b^{d-2}}{d-2} - \frac{1 - b^{d-4}}{d-4} \frac{m^2}{\Lambda^2} + \mathcal{O}\left(\frac{m^4}{\Lambda^4}\right) \right], \quad (16.26)$$

where we have used the angular volume Ω_d in (7.34). Note that for $d < 4$ all higher orders vanish in the limit $\Lambda \rightarrow \infty$ and only the first term proportional to Λ^{d-2} survive and diverge for $d > 2$. For $d = 2$ the first term is finite and reads

$$\lim_{\lambda \rightarrow \infty} \Delta m^2 = \frac{\lambda}{4\pi} \ln \frac{1}{b}. \quad (16.27)$$

Note that (16.27) signals a logarithmic divergence as the $\ln 1/b$ term stems from

$$\ln \frac{1}{b} = \ln \Lambda - \ln(b\Lambda). \quad (16.28)$$

For even lower dimensions the mass correction is finite in the limit $\Lambda \rightarrow 0$ and vanishes for $b \rightarrow 0$, that indicates integrating out an infinitesimal momentum shell. Moreover, the mass correction stemming from integrating out the momentum shell between Λ down to $b\Lambda$ is larger than zero as $b < 0$. In turn, if we integrate up to larger cutoff scales, the correction would be negative.

Let us now consider $d = 4$. In four dimensions also the Λ^{d-4}/m^2 -term survives, and we arrive at

$$\lim_{\lambda \rightarrow \infty} \Delta m^2 = \frac{\lambda}{16\pi^2} \left[\Lambda^2 \frac{1 - b^{d-2}}{d-2} + m^2 \ln b \right]. \quad (16.29)$$

We conclude that in $d = 4$ dimensions we have a quadratic divergence proportional to Λ^2 and a logarithmic one, signalled by $\ln(1/b)$. If we consider $d > 4$, also the sub-leading divergence is power like. Moreover, for $d \geq 6$, further divergences successively emerge.

For the time being we concentrate on $d \leq 4$, where the theory is perturbatively renormalisable, see [Section 7.2.1](#). For these dimensions we proceed with the coefficient $\Delta\lambda$ of the ϕ^4 term. Here we only discuss the leading order term as it is straightforward to see that the higher order terms vanish for $\Lambda \rightarrow \infty$. The product of the propagators in the fish diagram reads

$$G_{b\Lambda,\Lambda}(p) G_{b\Lambda,\Lambda}(p + p_1 + p_2) = G_{b\Lambda,\Lambda}(p) G_{b\Lambda,\Lambda}(p) + O\left(\frac{(p_1 + p_2)^2}{\Lambda^2}\right), \quad (16.30)$$

and we find for $\Lambda \rightarrow \infty$,

$$\Delta\lambda(p_1, \dots, p_4) \simeq -3\lambda^2 \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{p^2}\right)^2 = -\frac{3\lambda^2}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \Lambda^{d-4} \frac{(1 - b^{d-4})}{d-4}. \quad (16.31)$$

For $d < 4$ the correction vanishes in the infinite cutoff limit, $\Delta\lambda \rightarrow 0$: the full coupling is given by λ and the theory is super-renormalisable. In turn, for $d = 4$ we find

$$\Delta\lambda(p_1, \dots, p_4) \simeq -\frac{3\lambda^2}{16\pi^2} \ln \frac{1}{b}, \quad (16.32)$$

which signals a logarithmic UV singularity with [\(16.28\)](#). [Equation \(16.32\)](#) reflects the one-loop β -function [Equation \(7.54\)](#) derived in [Section 7.1.4](#), which is obtained via a bd/db derivative,

$$b \frac{d}{db} \Delta\lambda = \frac{3\lambda^2}{16\pi^2}. \quad (16.33)$$

Finally we also consider the source term in [\(16.17\)](#), which is of order ϕ^6 and higher. We first notice that an expansion in $\lambda/2\phi^2$ in the field-dependent propagator $G_{b\Lambda,\lambda}[\phi]$ generates higher order terms in ϕ^2 , connected by propagators,

$$G_{b\Lambda,\lambda}[\phi](x, y) = G_{b\Lambda,\lambda}(x, y) - \int_z G_{b\Lambda,\lambda}(x, z) \frac{\lambda}{2} \phi(z)^2 G_{b\Lambda,\lambda}(z, y) + \dots, \quad G_{b\Lambda,\lambda}(x, y) = G_{b\Lambda,\lambda}[0](x, y). \quad (16.34)$$

Hence, all terms include powers of the momentum-shell propagator $G_{b\Lambda,\lambda}(p, q)$ with $p, q \in [b\Lambda, \Lambda]$. The propagators scale with $1/\Lambda^2$ and subleading terms such as p^2/Λ^4 and vanish in the infinite cutoff limit. Specifically we have

$$-\frac{\lambda^2}{72} \int d^d x d^d y \phi^3(x) G_{b\Lambda,\lambda}(x, y) \phi^3(y) = O\left(\frac{\phi^6}{\Lambda^2}\right). \quad (16.35)$$

Similarly to the above scaling analysis there are further terms in the expansion that carry higher derivative terms or, in momentum space, higher orders of p^2/Λ^2 . Hence, with [\(16.24\)](#) they decay with the respective powers of $1/\Lambda^2$. An example for such a term is

$$\frac{1}{\Lambda^2} \int_x \phi^2 (\partial_\mu \phi)^2, \quad (16.36)$$

where the scaling with $1/\Lambda^2$ follows already from the dimension of the term. In summary our analysis has revealed that the Wilsonian effective action S_{eff} , that results from an integration of the momentum shell $(b\Lambda, \Lambda)$

has the form of the original action $S[\phi]$ with shifted coefficients and further higher order terms in the fields and derivatives, that decay with powers of $1/\Lambda$. Explicitly we have

$$S_{\text{eff}}[\phi] = \int d^d x \left\{ \frac{1}{2} (1 + \Delta Z) \phi(-\Delta) \phi + \frac{1}{2} (m^2 + \Delta m^2) \phi^2 + \frac{1}{4!} (\lambda + \Delta \lambda) \phi^4 \right. \\ \left. + (\alpha + \Delta \alpha) (\partial_\mu \phi)^2 + \frac{\lambda_6 + \Delta \lambda_6}{6!} \phi^6 + \dots \right\}, \quad (16.37)$$

where \dots comprises all higher order terms including the connected terms. The terms on the right hand side of the first line of (16.37) sum up to the classical action $S[\phi]$ with shifted coefficients, and $\phi = \phi_{b\Lambda}$. This part of the effective action has the same form as the action $S[\phi_\Lambda]$ in (16.1). The corrections $\Delta Z, \Delta m^2, \Delta \lambda$ depend on Λ and b . The second line comprises terms that are not present on the classical level and on the classical level we have $\alpha = \lambda_6 = 0$. They are generated from the momentum-shell integration and decay with powers of $1/\Lambda$. There are many more terms even at the same order of $1/\Lambda$, see e.g. (16.36), and we shall illustrate the treatment of the general terms at the example of these two terms.

Equation (16.37) concludes our derivation of the Wilsonian effective action in the path integral. It is left to reparametrise the effective action such, that the functional integral after the Wilsonian RG-step, (16.10), reduces to (16.1). This is achieved with

$$S_{\text{eff}}[\phi] = S[\phi'] + \text{higher order terms}, \quad (16.38)$$

with $\phi(p) = \phi_{b\Lambda}(p)$ with $\phi(p^2 > b^2 \Lambda^2) = 0$, see (16.2). The new field ϕ' after reparametrisation satisfies

$$\phi'(p'^2 > \Lambda^2) = 0, \quad (16.39)$$

which is achieved via the rescalings

$$p' = \frac{p}{b}, \quad x' = xb. \quad (16.40)$$

with

$$\int d^d x' = b^d \int d^d x, \quad \frac{\partial}{\partial x'} = \frac{1}{b} \frac{\partial}{\partial x}. \quad (16.41)$$

Evidently, the path integral measure of the reparametrised field is given by

$$[D\phi']_\Lambda, \quad (16.42)$$

as in (16.1). Moreover, it is apparent from (16.40) that the power of b counts the inverse momentum dimension of the fields and parameters. We now use the coordinate and momentum rescaling in (16.40), the rescaling freedom of the field and a redefinition of the parameters in order to map (16.10) to (16.1),

$$\phi'(x') = [b^{2-d}(1 + \Delta z)]^{\frac{1}{2}} \phi(x), \quad m'^2 = (m^2 + \Delta m^2) \frac{1}{1 + \Delta z} \frac{1}{b^2}, \quad \lambda' = (\lambda + \Delta \lambda) \frac{1}{(1 + \Delta z)^2} b^{d-4}, \\ \alpha' = (\alpha + \Delta \alpha) \frac{1}{(1 + \Delta z)^2} b^d, \quad \lambda'_6 = (\lambda_6 + \Delta \lambda_6) \frac{1}{(1 + \Delta z)^3} b^{2d-6}, \quad \dots \quad (16.43)$$

where the first line in (16.43) summarises the reparametrisations of the field and the parameters in the classical part of the Wilsonian effective action and the second line indicates the rescalings of all the quantum parts that are generated by the momentum-shell integrations.

Now we apply the reparametrisations (16.40), (16.41) and (16.43) to (16.10) with the Wilsonian effective action (16.37). First we remark that the Wilsonian effective action is form-invariant under the reparametrisation and the Λ, b dependence is buried in the shifted coefficients,

Wilsonian effective action after reparametrisation

$$S_{\text{eff}}[\phi'] = \int d^d x' \left\{ \frac{1}{2} \phi'(x') (-\Delta' + m'^2) \phi' + \frac{\lambda'}{4!} \phi'^4 \right\} + \int d^d x' \left\{ \alpha' (\partial'_\mu \phi')^4 + \frac{\lambda'_6}{6!} \phi'^6 + \dots \right\} \quad (16.44)$$

An immediate consequence of the form invariance entailed in (16.44) is, that any further RG-step does not change the form of the effective action. Moreover, the generating functional takes the form

Functional integral after a full Wilsonian RG-step

$$Z_{1,\Lambda} = \int [D\phi']_\Lambda e^{-S_{1,\text{eff}}[\phi']}. \quad (16.45)$$

Equation (16.45) has the same form as (16.1): The integration $[D\phi']_\Lambda$ includes the same momentum modes, but the action has been substituted by $S \rightarrow S_{1,\text{eff}}$. Note that after the momentum shell integration, the Wilsonian effective action $S_{1,\text{eff}}$ includes the quantum effects of the momentum shell $[b\Lambda, \Lambda]$. However, after the rescaling of the momenta with $1/b$ in the second (block-spinning) RG-step in order to obtain (16.45), this momentum shell is rescaled to the shell

$$p \in \left[\Lambda, \frac{\Lambda}{b} \right], \quad \text{after} \quad p \rightarrow \frac{p}{b}, \quad (16.46)$$

which is included in $S_{1,\text{eff}}$. Note that the rescalings and redefinition of the parameters did indeed change the generating functional, as it now includes also the quantum fluctuations in the momentum shell (16.46) in the effective action.

Within a further momentum shell integration starting with (16.45) we again arrive at a similar representation of the generating functional, but the exponent of the path integral has changed yet again to $S_{2,\text{eff}}$. This Wilsonian effective action now includes the quantum effects of the first step, but since we have performed a further rescaling, (16.46) is shifted to $[\Lambda/b, \Lambda/b^2]$, while the second momentum shell integration from $[b\Lambda, \Lambda]$ is shifted to (16.46). Altogether this entails, that $S_{2,\text{eff}}$ includes the quantum fluctuations of the momentum shell

$$p \in \left[\Lambda, \frac{\Lambda}{b^2} \right], \quad (16.47)$$

and after i iterations of this step the Wilsonian effective action $S_{i,\text{eff}}$ includes the quantum fluctuations of the momentum shell

$$p \in \left[\Lambda, \frac{\Lambda}{b^i} \right] \xrightarrow{i \rightarrow \infty} p \in [\Lambda, \infty]. \quad (16.48)$$

The generating functional (16.45) is transformed into

$$Z_{i,\Lambda} = \int [D\phi]_\Lambda e^{-S_{i,\text{eff}}[\phi]}, \quad (16.49)$$

and $Z_{i,\Lambda}$ now also contains quantum fluctuations up to Λ/b^i . Note however, that correlation functions with the functional integral (16.49) only carry momenta with $p \leq \Lambda$ as the momenta above Λ in the shell (16.48) are integrated out and are stored in the non-trivial Wilsonian effective action. In the limit $i \rightarrow \infty$ the Wilsonian effective action runs into an RG fixed point, if the latter exists, as does $Z_{i,\Lambda}$. Assuming the existence of this limit for the theory in hand, we are led to

Generating function with the Wilsonian effective action

$$Z = \int [D\phi]_{\Lambda} e^{-S_{\text{eff}}[\phi]}, \quad \text{with} \quad S_{\text{eff}}[\phi] = S_{i=\infty, \text{eff}}[\phi] \quad \text{and} \quad Z = Z_{\infty, \Lambda}. \quad (16.50)$$

The generating functional Z is Λ -independent, but the correlation functions derived from (16.50) only resolve momenta $p < \Lambda$. We emphasise that the derivation was done within a classical bootstrap procedure, we assumed the existence of the limit and ended up with (16.50). We shall work with this bootstrap in the present section as well as in the next, Section 16.1.2 and discuss the presuppositions for its existence in Section 16.1.3. In the same spirit we remark that the Wilsonian effective action $S_{\text{eff}}[\phi]$ can also formally be written as

Wilsonian effective action

$$S_{\text{eff}}[\phi] \simeq -\ln \int [D\hat{\phi}]_{p^2 \geq \Lambda} e^{-S[\hat{\phi} + \phi]}, \quad \text{where} \quad \phi = \phi_{\Lambda}, \quad \text{with} \quad \phi_{\Lambda}(p^2 > \Lambda^2) = 0. \quad (16.51)$$

where we have used (16.48). Equation (16.50) with (16.48) is the final representation of the generating functional in terms of the Wilson effective action.

16.1.2. Functional flow equations

The fixed point property of the Wilsonian effective action is summarised in a homogeneous RG equation: S_{eff} does not change under a further iteration of the RG-step. Hence it is independent of b ,

$$b \frac{d}{db} S_{\text{eff}} = 0, \quad (16.52)$$

which also applied for Kadanoff block spinning transformations on the lattice, discussed in Chapter 15 in Section 15.5.1. Still, S_{eff} carries the dependence on Λ . The respective renormalisation group equation is readily derived from (16.51). To that end we implement the infrared cutoff in the path integral measure with

$$[D\hat{\phi}]_{p^2 \geq \Lambda} = [D\hat{\phi}] \exp\{-\Delta S_{\Lambda}^{(\text{IR})}[\hat{\phi}]\}, \quad \text{with} \quad \Delta S_{\Lambda}^{(\text{IR})}[\hat{\phi}] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \hat{\phi}(-p) R_{\Lambda}^{(\text{IR})}(p) \hat{\phi}(p), \quad (16.53)$$

where $R_{\Lambda}^{(\text{IR})}(p)$ has been defined in (16.5) with the sharp infrared regulator

$$R_{\Lambda}^{(\text{IR})}(p) = k^2 \left(\frac{1}{\theta(p^2 - \Lambda^2)} - 1 \right). \quad (16.54)$$

Furthermore, we can also introduce smooth regulators where the θ -function is changed for a regularisation of the θ -function. For example we can choose

$$\left(\frac{1}{\theta(p^2 - \Lambda^2)} - 1 \right) \rightarrow \exp\left\{-\frac{p^2}{\Lambda^2}\right\}. \quad (16.55)$$

This has to be augmented with a respective UV-cutoff action 16.1.2 with $R_{\Lambda}^{(\text{UV})}$, leading to

$$[D\phi] = [D\phi] \exp\{-\Delta S_{\Lambda}^{(\text{UV})}[\phi]\} [D\hat{\phi}] \exp\{-\Delta S_{\Lambda}^{(\text{IR})}[\hat{\phi}]\}. \quad (16.56)$$

We defer the general derivation of the RG equation of the Wilsonian effective action and the 1PI effective action for general regulators including smooth ones to Appendix J and proceed here with the sharp cutoff that facilitates some parts of the derivation. The Λ -dependence of the Wilson effective action is described by its Λ -derivative. We use (16.51) with the infrared regularised measure (16.53), and its Λ -derivative reads

Renormalisation group equation for the Wilson effective action

$$\Lambda \frac{d}{d\Lambda} S_{\text{eff}}[\phi] = \left\langle \Lambda \frac{d\Delta S_{\Lambda}^{(\text{IR})}[\hat{\phi}]}{d\Lambda} \right\rangle = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \Lambda \frac{dR_{\Lambda}^{(\text{IR})}(p)}{d\Lambda} \langle \hat{\phi}(p)\hat{\phi}(-p) \rangle[\phi]. \quad (16.57)$$

where the expectation value in (16.57) is given by

$$\langle \hat{\phi}(p)\hat{\phi}(-p) \rangle[\phi] = e^{S_{\text{eff}}[\phi]} \int D\hat{\phi} \hat{\phi}(p)\hat{\phi}(-p) e^{-(S_{\text{int}}[\hat{\phi}+\phi]+S_{\text{kin}}[\phi]+S_{\text{kin},\Lambda}[\hat{\phi}])}, \quad (16.58)$$

with the split of the classical action into the Gaussian part S_{kin} and the interaction part S_{int} . The Gaussian part reads

$$S_{\text{kin}}[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \phi(-p) [p^2 + m^2] \phi(p), \quad S_{\text{kin},\Lambda}[\hat{\phi}] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \hat{\phi}(-p) [p^2 + m^2 + R_{\Lambda}^{(\text{IR})}(p)] \hat{\phi}(p), \quad (16.59)$$

leading to

$$S[\hat{\phi} + \phi] + \Delta S_{\Lambda}^{(\text{IR})}[\hat{\phi}] = S_{\text{int}}[\hat{\phi} + \phi] + S_{\text{kin}}[\phi] + S_{\text{kin},\Lambda}[\hat{\phi}], \quad (16.60)$$

where we used (16.9): $\int_p \phi(p)\hat{\phi}(p) = 0$ for a sharp cutoff.

Equation (16.57) entails the change of the effective action $S_{\text{eff}}[\phi]$ under the integration of an infinitesimal momentum shell around Λ . It is a *functional flow equation* for $S_{\text{eff}}[\phi]$ in terms of the two-point function $\langle \hat{\phi}(p)\hat{\phi}(-p) \rangle[\phi]$ that can be expressed in terms of ϕ -derivatives of $S_{\text{eff}}[\phi]$. It is left to convert the two-point function into this form. For this purpose we use

$$\hat{\phi}(p) \exp\{-S_{\text{kin},\Lambda}[\hat{\phi}]\} = -G_{\Lambda}^{(0)}(p) \frac{\delta}{\delta \hat{\phi}(-p)} \exp\{-S_{\text{kin},\Lambda}[\hat{\phi}]\}, \quad (16.61)$$

with

$$G_{\Lambda}^{(0)}(p) = \frac{1}{p^2 + m^2 + R_{\Lambda}^{(\text{IR})}(p)}. \quad (16.62)$$

Equation (16.63) can be iterated, leading to

$$\hat{\phi}(p)\hat{\phi}(-p) \exp\{-S_{\text{kin},\Lambda}[\hat{\phi}]\} = \left[G_{\Lambda}^{(0)}(p) \frac{\delta}{\delta \hat{\phi}(-p)} G_{\Lambda}^{(0)}(p) \frac{\delta}{\delta \hat{\phi}(p)} + G_{\Lambda}^{(0)}(p) \delta(0) \right] \exp\{-S_{\text{kin},\Lambda}[\hat{\phi}]\}, \quad (16.63)$$

where we shall drop the field-independent term in the square-bracket proportional to $\delta(0)$. This can be achieved cleanly with a point splitting in the momentum variable. Now we use the (16.63) for the representation of $\hat{\phi}(p)\hat{\phi}(-p)$ in terms of $\hat{\phi}$ -derivatives in the functional integral in (16.58). This leads us to

$$\langle \hat{\phi}(p)\hat{\phi}(-p) \rangle[\phi] \simeq G_{\Lambda}^{(0)}(p) G_{\Lambda}^{(0)}(p) e^{S_{\text{eff}}[\phi]} \int D\hat{\phi} e^{-S_{\text{kin},\Lambda}[\hat{\phi}]} \frac{\delta}{\delta \hat{\phi}(p)} \frac{\delta}{\delta \hat{\phi}(-p)} e^{-(S_{\text{int}}[\hat{\phi}+\phi]+S_{\text{kin}}[\phi])}. \quad (16.64)$$

Now we use that the $\hat{\phi}$ -derivatives of $S_{\text{int}}[\hat{\phi} + \phi]$ can be written as ϕ -derivatives and can hence be pulled out of the functional integral. We write

$$e^{S_{\text{eff}}[\phi]} \frac{\delta}{\delta \hat{\phi}(p)} \frac{\delta}{\delta \hat{\phi}(-p)} e^{-(S_{\text{int}}[\hat{\phi}+\phi]+S_{\text{kin}}[\phi])} = e^{S_{\text{int},\Lambda}[\phi]} \frac{\delta}{\delta \phi(p)} \frac{\delta}{\delta \phi(-p)} e^{-S_{\text{int}}[\hat{\phi}+\phi]}, \quad (16.65)$$

where we have defined the interacting part of the Wilsonian effective action with

$$S_{\text{int},\Lambda}[\phi] = S_{\text{eff}}[\phi] - S_{\text{kin}}[\phi]. \quad (16.66)$$

Note that in contradistinction to $S_{\text{int}}[\phi]$, the interacting part $S_{\text{int},\Lambda}[\phi]$ of the Wilsonian effective action also contains quadratic field-dependences. With (16.65) we can readily compute the functional integral in (16.68). Pulling out the ϕ -derivatives we are left with

$$e^{S_{\text{int},\Lambda}[\phi]} \frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(-p)} \int \mathcal{D}\hat{\phi} e^{-S_{\text{int}}[\hat{\phi}+\phi]} = e^{S_{\text{int},\Lambda}[\phi]} \frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(-p)} e^{-S_{\text{int},\Lambda}[\phi]}. \quad (16.67)$$

In summary we arrive at our final expression for the two-point function in (16.68),

$$\langle \hat{\phi}(p)\hat{\phi}(-p) \rangle[\phi] \simeq G_{\Lambda}^{(0)}(p) \left[-S_{\text{int},\Lambda}^{(2)}[\phi] + \left(S_{\text{int},\Lambda}^{(1)}[\phi] \right)^2 \right] (p, -p) G_{\Lambda}^{(0)}(p). \quad (16.68)$$

Inserting (16.68) into the Wilsonian flow equation (16.57), we arrive at our final expression for the functional flow equation for the Wilsonian effective action, the Polchinski equation [9] (1983),

Polchinski equation

$$\Lambda \frac{d}{d\Lambda} S_{\text{int},\Lambda}[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} C_{\Lambda}(p) \left[S_{\text{int},\Lambda}^{(2)}[\phi] - \left(S_{\text{int},\Lambda}^{(1)}[\phi] \right)^2 \right] (p, -p), \quad (16.69)$$

with

$$C(p) = -G_{\Lambda}^{(0)}(p) \Lambda \frac{dR_{\Lambda}^{(\text{IR})}(p)}{d\Lambda} G_{\Lambda}^{(0)}(p) = \Lambda \frac{d}{d\Lambda} G_{\Lambda}^{(0)}(p). \quad (16.70)$$

While we have derived it for the sharp cutoff, it also holds true for smooth regulator functions such as the example in (16.55). This is shown in Appendix J. There we also derive the analogue of (16.69) for the 1PI effective action $\Gamma_k[\phi]$ in the presence of an infrared cutoff term with (16.5) and (16.54) and the infrared cutoff scale k . This is the Wetterich equation [10] (1992),

Wetterich equation

$$k \frac{d}{dk} \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} G_k[\phi](p, -p) k \frac{dR_k^{(\text{IR})}(p)}{dk}. \quad (16.71)$$

with the effective action $\Gamma_k[\phi]$, which is a modification of the Legendre transform $\Gamma_k^{(\text{1PI})}[\phi]$ with

$$\Gamma_k[\phi] = \Gamma_k^{(\text{1PI})}[\phi] - \Delta S_k^{(\text{IR})}[\phi], \quad G_{\Lambda}[\phi] = \frac{1}{\Gamma_k^{(2)}[\phi] + R_k^{(\text{IR})}}, \quad (16.72)$$

see (J.11) and (J.14).

Both functional flow equations as well as that for the partition function, see in Appendix J, are special cases of Wegner's flow equation [11] (1974), which admits general RG-transformations that are also non-linear in the fields. Moreover, the Wetterich equation (16.71) is a generalisation of the functional Calla-Symanzik equation derived by Symanzik in [12] (1970) with the infrared cutoff $R_k(p) = k^2$. This is the first occurrence of a closed exact functional flow equation.

The functional flow equations are 'simple' closed one loop exact equations in full correlation functions. This is most evident for the Wetterich equation, where $G_k[\phi]$ is nothing but the full-field-dependent propagator. However, the same structure holds true for the Polchinski equation due to (16.68). Historically, the Polchinski equation has been derived for the simplification of proofs of perturbative renormalisability: The algebraic closed form of (16.69) provides a simple iterative structure for the estimates of the UV momentum decay of loop diagrams. It is this algebraic structure, which also makes it amiable towards symmetry considerations such as the fate of gauge symmetry in the presence of the momentum cutoff. On the other hand, its form as a functional heat equation has its problems within numerical applications. It easily generates parametric singularities due to the lack of cancellations between $S_{\text{int},\Lambda}^{(2)}$ and $-(S_{\text{int},\Lambda}^{(1)})^2$. It is there, where the Wetterich equation excels as its inverse second derivative structures softens potential singularities in $\Gamma_k^{(2)}$. See also the discussion in Appendix J.

16.1.3. Asymptotic UV scaling of relevant and irrelevant operators & the fixed point action

In the last two Sections, Section 16.1.1 and Section 16.1.2 we have derived the Wilsonian momentum shell RGs and the flow equations for the effective actions, both for the Wilsonian effective action and the 1PI effective action. While the flow equations are certainly well-defined for a given input of an effective action, say the Wilsonian effective action S_{eff} at an input scale Λ , it remains to be shown that for a given theory we can take the iteration order of the momentum shell integration to infinity. Only then is the theory at hand a fundamental theory.

In order to investigate this question, we come back to the expansion of the effective action in terms of local operators with n fields ϕ and m derivatives ∂ that are distributed amongst the n fields. The respective couplings are given by $\lambda_{n,m}$. Examples are given by $\lambda_{4,4}[(\partial_\mu\phi)^2]^2$ or $\lambda_{4,2}\phi^2(\partial_\mu\phi)^2$, see also the terms in (16.37). Evidently, there is more than one operator for a given order n, m and hence several $\lambda_{n,m}^{(i)}$ where i labels the elements in a complete set of operators $\{\mathcal{O}_{n,m}^{(i)}\}$. Schematically we write

$$\lambda_{n,m} \int d^d x \mathcal{O}_{n,m}, \quad \text{with} \quad \mathcal{O}_{n,m} = \partial^n \phi^m, \quad (16.73)$$

as we are only interested in the scaling properties of the local operators under the rescalings with b or $1/b$. Under the rescaling (16.43) of the field and (16.41) of the derivative we infer that the operator $\mathcal{O}_{n,m}$ in scales with

$$\mathcal{O}_{n,m} \rightarrow b^{-d_{n,m}} \mathcal{O}_{n,m} \quad \text{with} \quad d_{n,m} = n \left(\frac{d}{2} - 1 \right) + m, \quad (16.74)$$

where $d/2 - 1$ is the dimension of the field. Accordingly, the coefficient $\lambda_{n,m}$ in 16.1.3 has to scale with

$$\lambda_{n,m} \xrightarrow{b} b^{d_{n,m}-d} \lambda_{n,m}, \quad (16.75)$$

in order to compensate for the scaling of a generic operator $\mathcal{O}_{n,m}$ and the scaling of the space-time integral in 16.1.3. Note that $b > 1$ and hence for $d - d_{n,m} > 0$ the coupling $\lambda_{n,m}$ grows with the rescaling. In turn, for $d - d_{n,m} < 0$ it decays while $d - d_{n,m} = 0$ signals as logarithmic behaviour as discussed around (16.28). Importantly, in the iteration limit $i \rightarrow \infty$ we have to take care of the operators with growing coefficients. These different scaling cases are called

- (a) Relevant: $d - d_{n,m} > 0$.
- (b) Marginal: $d - d_{n,m} = 0$.
- (c) Irrelevant: $d - d_{n,m} < 0$.

Let us now discuss renormalisability: the couplings of relevant and marginal operators grow towards the ultraviolet and have to be rescaled in the RG-procedure of the momentum-shell RG. This entails that the respective terms are part of the ultraviolet effective action and their parameter are physical parameters of the theory. In a perturbatively renormalisable theory these are the parameters of the classical action. However, the setup is more general and *any* fixed point action with a finite number of relevant parameters serves as a UV-closure of a quantum field theory. Note that in general one loses any predictive power if this UV fixed point action has an infinite number of relevant parameters even though there have been considerations that in some cases even this can be accommodated.

This leaves us with the irrelevant terms. A good example is the $\lambda_6 = \lambda_{6,0}$ coupling of the ϕ^4 theory in four dimensions. This is an irrelevant coupling and is generated on one loop by the diagram in ??, where we assume an ultraviolet cutoff for the loop or rather for the propagators. Then, one readily shows that the one-loop correction of a momentum shell integration with $p \in [\Lambda, \Lambda/b]$ scales with

$$\lambda_6^{1\text{loop}} \propto b^2 \frac{\lambda_4^3}{\Lambda^2}, \quad (16.76)$$

with $\lambda_4 = \lambda_{4,0}$. Assuming the scaling of λ_4 with $\log \Lambda$, we arrive at $d_{6,0} = -2$. Let us now consider a classical λ_6 coupling. Then, at one loop we also encounter the diagram in ?? with one six-point coupling and one four-point coupling. This diagram leads to the scaling

$$\lambda_6^{1\text{loop}} \propto \lambda_4 \lambda_6 \log b, \quad (16.77)$$

from the momentum shell integration with $p \in [\Lambda, \Lambda/b]$. Evidently, already at one loop, (16.79) triggers a scaling law $d_{6,0} = 0$ and hence the coupling is relevant. Even worse, it also triggers a logarithmic scaling with the cutoff scale for λ_8 which is signalled by $\ln b$ in the momentum shell integration with $p \in [\Lambda, \Lambda/b]$,

$$\lambda_8^{1\text{loop}} \propto \lambda_6^2 \log b. \quad (16.78)$$

Feeding back this coupling into the momentum shell integration of flow of λ_6 in terms of the tadpole diagram, leads to

$$\lambda_6^{1\text{loop}} \propto \lambda_8 b^2. \quad (16.79)$$

This also triggers higher order couplings λ_n with $n > 8$ to become relevant. Moreover, by iterating this procedure the dimensions $d_{n,m}$ increase with each iteration step. We conclude that the only consistent solution is that λ_6 is absent in the classical action and only is generated by quantum fluctuations. Then, it scales with b^2 and the marginal scaling with $\log b$ in (16.79) reduces to b^2 , which is sustained in each iteration step.

We conclude that the classical or UV fixed point action only can contain a finite number of relevant operators. Moreover, UV-irrelevant operators cannot be present in the classical or UV fixed point action. If these conditions are satisfied, we have

UV fixed point action

$$\lim_{\Lambda \rightarrow \infty} S_{\text{eff}\Lambda} \rightarrow S_{\text{bare}}^{\text{FP}}[\phi] \quad (16.80)$$

We note that above we have only sketched the proof of these statements. Mathematically sound proofs even of perturbative renormalisability require a lot of work, even if using the modern functional renormalisation group approach put forward here.

16.2. Fixed Points

In the last Section we have learned, that the UV effective action is a fixed point of the RG group in the sense of . Evidently, if the limit exists, flow of the effective action of the theory tends towards zero. Not that this does not entail, that the fields or the parameters do not scale, as they have an explicit momentum dimension. However, their dimensionless versions obtained by scaling them with appropriate powers of Λ should loose their Λ -dependence. This consideration already leads us to the general definition of a fixed point:

Fixed points of a theory are points in theory space defined by the values of all couplings in the Wilson effective action, where a scale transformation does not change the physics.

This definition of a fixed point entails that dimensionless ratios of observables/coupling parameters do not change. If we consider a scale in the RG-scale Λ , the latter statement implies that the dimensionless version of every coupling parameter of the theory invariant under a scale transformation. Here, dimensionless means that one is dividing/multiplying the given coupling with the appropriate power of the RG-scale Λ as already eluded to above. Before we cast this statement into an invariance equation, we have to carefully evaluate its presupposition. Naïvely one would simply take the parameters $\lambda_{n,m}$ in the expansion of the Wilson effective action in local operators. However, the $\lambda_{n,m}$ in the ϕ^4 -theory used as an example here, are simply vertex dressings. For example, the dimensionless 'coupling' $\lambda_4 = \lambda_4, 0$ in the four-dimensional ϕ^4 theory is of the type of the quark-gluon vertex dressing $Z_{q\bar{q}A}g$ in QCD, see the discussion in [Sections 14.1](#) and [14.2](#). It is proportional to $Z_\phi^2 \bar{\lambda}_4$, where $\bar{\lambda}_4$ is the RG-invariant coupling. This leads us to the following definition of the dimensionless couplings

UV fixed point action

$$\bar{\lambda}_{n,m} = \frac{\lambda_{n,m} \Lambda^{d_{n,m}-d}}{Z_\phi^{n/2}}, \quad (16.81)$$

where the numerator is the dimensionless vertex dressing and the denominator takes care of the anomalous scaling of the field. The wave function renormalisation or rather the full wave function of the field is not a coupling in the above sense as it can be absorbed in a (linear) redefinition of the field.

Then, a fixed point of the quantum field theory at hand is defined by vanishing momentum scale dependences for all couplings with

$$\Lambda \frac{d}{d\Lambda} \bar{\lambda}_{n,m} = 0, \quad \forall n, m. \quad (16.82)$$

The relations (16.82) can be conveniently formulated in terms of the β -functions of the theory: all β -functions vanish on the fixed point. Its properties and in particular the question of relevant, marginal and irrelevant directions or couplings can be determined in an expansion about the fixed point. To that end we define the β -functions of a given theory with

$$\beta_i(\bar{\lambda}) = \Lambda \frac{d}{d\Lambda} \bar{\lambda}_i, \quad \text{with} \quad \bar{\lambda} = (\lambda_1, \lambda_2, \dots). \quad (16.83)$$

In (16.83) and from now on, we label all couplings with $i \in \mathbb{N}$ instead of using the multi-index n, m . Within this notation, the Fixed Point condition (16.82) reads

Fixed point equation

$$\beta_i(\bar{\lambda}_*) = 0, \quad \text{with} \quad \beta = (\beta_1, \beta_2, \dots). \quad (16.84)$$

We illustrate the content of the fixed point conditions within a first example of the ϕ^4 -theory in four dimensions. In this theory we have two relevant coupling parameters,

$$\bar{\lambda} = \bar{\lambda}_4, \quad \bar{m}^2 = \frac{m^2}{\Lambda^2}, \quad (16.85)$$

and the wave function renormalisation. The Wilsonian β -functions in leading order can be extracted from our momentum shell integration: We first discuss the running of the four-point function. Within one Wilsonian RG-step the momentum shell integration signalled a logarithmic singularity of the four-point function, see (16.32). As discussed there, this is nothing but the one-loop β -function computed in (7.53) in Section 7.1.4. We conclude from that the Wilsonian β -function is given by

$$\beta_\lambda = \Lambda \frac{d}{d\Lambda} \bar{\lambda} = \frac{3\bar{\lambda}^2}{16\pi^2} + O(\bar{\lambda}^3), \quad (16.86)$$

where we have used that the wave function renormalisation is trivial at one loop. Note however, that (16.86) only provides the leading term. There are further contributions proportional to $\bar{\lambda}_6$ and in a general Wilsonian RG-step we also have to consider the wave function renormalisation.

The respective β -function for \bar{m}^2 can be read off from (16.29). However, since m^2 is dimensionful, its Wilsonian flow is dominated by the dimensional running. In combination we arrive at

Wilsonian β -functions in the ϕ^4 theory

$$\beta_\lambda = \Lambda \frac{d}{d\Lambda} \bar{\lambda} = \frac{3\bar{\lambda}^2}{16\pi^2} + O(\bar{\lambda}^3), \quad \beta_m = \Lambda \frac{d}{d\Lambda} \bar{m}^2 = -2\bar{m}^2 + O(\bar{\lambda}). \quad (16.87)$$

In (16.87) we only consider the leading term. As we have already seen in the derivation, it is nothing but the one-loop β -functions of the theory at hand. Moreover, on the right hand side the full Λ -dependent coupling parameters appear, so (16.87) already contains a one-loop RG-improvement. The fixed point condition reads

Fixed point condition in the four-dimensional ϕ^4 theory

$$\beta_m = 0, \quad \beta_\lambda = 0, \quad \beta_{\lambda_6} = 0, \quad \dots, \quad (16.88)$$

where we ordered the β -functions or rather the parameters in terms of their UV relevance. Equation (16.89) has a trivial solution,

Gaussian fixed point in the four-dimensional ϕ^4 theory

$$\bar{m}^2 = 0, \quad \bar{\lambda} = 0, \quad \bar{\lambda}_6 = 0, \quad \bar{\lambda}_i = 0, \quad \forall i. \quad (16.89)$$

Equation (16.89) is called a Gaussian fixed point as it describes a free theory with a Gaussian path integral measure. With the vanishing fixed point coupling the expansion in orders of the coupling, that allowed us to derive the fixed point so easily, is self-consistent. Moreover, perturbation theory is nothing but an expansion about the Gaussian fixed point of a given theory. Therefore the stability of this fixed point or rather its neighbourhood within this expansion decides about the existence of perturbation theory. We visualise this situation within a

a plot of the vector field of the β -functions around the Gaussian fixed point in the $\bar{m}^2, \bar{\lambda}, \bar{\lambda}_6$ plane. The arrows point into the direction of $-\beta$ and indicate the flow towards the infrared. This also requires the β -function of $\bar{\lambda}_6$, which reads

$$\beta_{\bar{\lambda}_6} = \Lambda \frac{d}{d\Lambda} \bar{\lambda}_6 = 2\bar{\lambda}_6 + \mathcal{O}(\bar{\lambda}^3, \bar{\lambda}\bar{\lambda}_6, \bar{\lambda}_8). \quad (16.90)$$

As for the β -function of the mass, it is the dimensional running, that dominates the flow in the vicinity of the Gaussian fixed point.

Part III.

Appendix

A. Coherent states

Here we provide some more details for the computations relevant for the discussion of the coherent state $|\alpha\rangle$ in [Section 2.3.3](#). We start with the ansatz [\(2.130\)](#) in [Section 2.3](#)

$$|\alpha\rangle = \frac{1}{\mathcal{N}(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{\infty} \left(\int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}_i}}} \alpha(\mathbf{p}_i) \right) |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle, \quad (\text{A.1})$$

with

$$a(\mathbf{p}) |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle = \sum_{i=1}^n (2\pi)^3 \sqrt{2\omega_{\mathbf{p}_i}} |\mathbf{p}_1 \cdots \mathbf{p}_{i-1} \mathbf{p}_{i+1} \cdots \mathbf{p}_n\rangle \delta(\mathbf{p} - \mathbf{p}_i). \quad (\text{A.2})$$

From [\(A.2\)](#) and with normal ordering [\(2.117\)](#) it follows that

$$\begin{aligned} & \frac{1}{n!} a(\mathbf{p}) \left(\int \frac{d^3 p'}{(2\pi)^3} \alpha(\mathbf{p}') a^\dagger(\mathbf{p}') \right)^n |0\rangle \\ &= \frac{1}{n!} n \alpha(\mathbf{p}) \left(\int \frac{d^3 p'}{(2\pi)^3} \alpha(\mathbf{p}') a^\dagger(\mathbf{p}') \right)^{n-1} |0\rangle \\ &= \alpha(\mathbf{p}) \frac{1}{(n-1)!} \left(\int \frac{d^3 p'}{(2\pi)^3} \alpha(\mathbf{p}') a^\dagger(\mathbf{p}') \right)^{n-1} |0\rangle \end{aligned} \quad (\text{A.3})$$

and similarly

$$\begin{aligned} & \langle 0| \frac{1}{n!} \left(\int \frac{d^3 p'}{(2\pi)^3} \alpha^*(\mathbf{p}') a(\mathbf{p}') \right)^n a^\dagger(\mathbf{p}) \\ &= \langle 0| \frac{1}{(n-1)!} \left(\int \frac{d^3 p'}{(2\pi)^3} \alpha^*(\mathbf{p}') a(\mathbf{p}') \right)^{n-1} \alpha^*(\mathbf{p}). \end{aligned} \quad (\text{A.4})$$

Now we can calculate

$$\begin{aligned} \langle \alpha|\alpha\rangle &= \frac{1}{\mathcal{N}^2(\alpha)} \sum_{n=0}^{\infty} \left(\frac{1}{n!} \right)^2 \int \prod_{i=1}^{\infty} \left(\frac{d^3 p_i}{(2\pi)^3} \frac{d^3 p'_i}{(2\pi)^3} \alpha^*(\mathbf{p}_i) \alpha(\mathbf{p}'_i) \right) \cdots \\ & \quad \cdots \cdot \langle 0| a(\mathbf{p}_1) \cdots a(\mathbf{p}_n) a^\dagger(\mathbf{p}'_n) \cdots a^\dagger(\mathbf{p}'_1) |0\rangle \\ & \quad \cdot \end{aligned} \quad (\text{A.5})$$

With

$$\begin{aligned}
 & \langle 0|a(\mathbf{p}_1) \cdots a(\mathbf{p}_n) a^\dagger(\mathbf{p}'_n) \cdots a^\dagger(\mathbf{p}'_1)|0\rangle \\
 &= \langle 0|a(\mathbf{p}_1) \cdots \left([a(\mathbf{p}_n), a^\dagger(\mathbf{p}'_n)] + a^\dagger(\mathbf{p}'_n) a(\mathbf{p}_n) \right) \cdots a^\dagger(\mathbf{p}'_1)|0\rangle \\
 &= (2\pi)^3 \delta(\mathbf{p}_n - \mathbf{p}'_n) \langle 0|a(\mathbf{p}_1) \cdots a(\mathbf{p}_{n-1}) a^\dagger(\mathbf{p}'_{n-1}) \cdots a^\dagger(\mathbf{p}'_1)|0\rangle + \dots \\
 &\quad \dots + \langle 0|a(\mathbf{p}_1) \cdots a(\mathbf{p}_{n-1}) a^\dagger(\mathbf{p}'_n) a(\mathbf{p}_n) a^\dagger(\mathbf{p}'_{n-1}) \cdots a^\dagger(\mathbf{p}'_1)|0\rangle \\
 &\quad \vdots \quad (\text{continue normal ordering}) \\
 &= (2\pi)^3 \sum_{i=1}^n \langle 0|a(\mathbf{p}_1) \cdots a(\mathbf{p}_{n-1}) a^\dagger(\mathbf{p}'_n) \cdots \widehat{a^\dagger(\mathbf{p}'_i)} \cdots a^\dagger(\mathbf{p}'_1)|0\rangle \cdot \delta(\mathbf{p}_n - \mathbf{p}'_i), \tag{A.6}
 \end{aligned}$$

where

$$\widehat{a^\dagger(\mathbf{p}'_i)} = 1. \tag{A.7}$$

Finally we get

$$\begin{aligned}
 \langle \alpha|\alpha\rangle &= \frac{1}{\mathcal{N}^2(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(n-1)!} \int \frac{d^3 p_n}{(2\pi)^3} \alpha^*(\mathbf{p}_n) \alpha(\mathbf{p}_n) \cdots \\
 &\quad \cdots \cdot \int \prod_{i=1}^{n-1} \left(\frac{d^3 p_i}{(2\pi)^3} \frac{d^3 p'_i}{(2\pi)^3} \alpha^*(\mathbf{p}_i) \alpha(\mathbf{p}'_i) \right) \langle 0|a(\mathbf{p}_1) \cdots a(\mathbf{p}_{n-1}) a^\dagger(\mathbf{p}'_{n-1}) \cdots a^\dagger(\mathbf{p}'_1)|0\rangle \\
 &\quad \vdots \quad (\text{continue normal ordering}) \\
 &= \frac{1}{\mathcal{N}^2(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int \frac{d^3 p}{(2\pi)^3} \alpha^*(\mathbf{p}) \alpha(\mathbf{p}) \right)^n \\
 &= \frac{1}{\mathcal{N}^2(\alpha)} \exp \left(\int \frac{d^3 p}{(2\pi)^3} |\alpha(\mathbf{p})|^2 \right) \tag{A.8}
 \end{aligned}$$

and it follows that

$$\mathcal{N}(\alpha) = \exp \left(\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} |\alpha(\mathbf{p})|^2 \right). \tag{A.9}$$

B. Normalisation, orthogonality and completeness of $u_s(p), v_s(p)$

In this Appendix we provide the technical details for the computation of the normalisation and orthogonality relations (4.104) as well as the completeness relations (4.103).

The derivation of (4.104) reads,

$$\begin{aligned}
 \bar{u}_r(p) u_s(p) &= u_r^\dagger(p^0) \frac{(\not{p} + m) \gamma^0 (\not{p} + m)}{p^0 + m} u_s(p^0) \\
 \gamma^0 \gamma^\dagger \gamma^0 &= \gamma \quad \rightarrow \quad = u_r^\dagger(p^0) \gamma^0 \frac{(\not{p} + m)(\not{p} + m)}{p^0 + m} u_s(p^0) \\
 u_r^\dagger(p^0) &= u_r(p^0) \quad \rightarrow \quad = u_r(p^0) \gamma^0 \frac{p^2 + m^2 + 2p^0 m \gamma^0}{p^0 + m} u_s(p^0) \\
 u_r(p^0) \gamma^0 \gamma^i u_s(p^0) &= 0 \\
 \gamma^0 u_s(p^0) &= u_s(p^0) \quad \rightarrow \quad = 2u_r(p^0) \gamma^0 \frac{m(p^0 + m)}{p^0 + m} u_s(p^0) \\
 &= 2m\delta_{rs}. \tag{B.1}
 \end{aligned}$$

The analogous steps hold true for the normalisation identity for $\bar{v}_r v_s$. The orthogonality relation in (4.104) follows from $(\not{p} - m)(\not{p} + m) = 0$.

Finally, the completeness relations (4.104) are proven true by showing that they hold within for the complete basis $u_s(p), v_s(p)$ with

$$\begin{aligned}
 \sum_s u_s(p) \bar{u}_s(p) u_r(p) &= \sum_s u_s(p) 2m\delta_{rs} \\
 &= 2m u_r(p) = \frac{2m(\not{p} + m)}{\sqrt{p^0 + m}} u_r(p^0) = \frac{(\not{p} + m)^2}{\sqrt{p^0 + m}} u_r(p^0) \\
 &= (\not{p} + m) u_r(p), \tag{B.2}
 \end{aligned}$$

and

$$\sum_s u_s(p) \bar{u}_s(p) v_r(p) = 0 = (\not{p} + m) v_r(p). \tag{B.3}$$

The relation for $\sum_s v_s(p) \bar{v}_s(p)$ is shown similarly.

C. Properties of Grassmann numbers

In this Appendix we briefly discuss the basic properties of Grassmann numbers. They are anti-commuting numbers: two Grassmann numbers θ and η obey,

$$\theta\eta = -\eta\theta. \quad (\text{C.1})$$

which readily implies

$$\theta^2 = \eta^2 = 0. \quad (\text{C.2})$$

This implements their anti-commuting nature. Grassmann numbers can be multiplied and added to complex numbers. Let us now consider a linear function

$$A + B\theta + C\eta. \quad (\text{C.3})$$

The multiplication of two such functions leads to

$$\begin{aligned} (A_1 + B_1\theta + C_1\eta)(A_2 + B_2\theta + C_2\eta) \\ = A_1A_2 + (B_1A_2 + A_1B_2)\theta + (C_1A_2 + A_1C_2)\eta + (B_1C_2 - C_1B_2)\theta\eta. \end{aligned} \quad (\text{C.4})$$

where the quadratic terms proportional to θ^2 and η^2 are absent due to (C.2). We also conclude from (C.4), that (C.3) is the general function in a Grassmann variable,

$$f(\theta) = A + B\theta. \quad (\text{C.5})$$

Derivatives with respect to Grassmann variables are defined via

$$\frac{d}{d\theta}\theta = 1, \quad \frac{d}{d\theta}1 = 0, \quad \longleftrightarrow \quad \frac{d}{d\theta}f(\theta) = B. \quad (\text{C.6})$$

The integration over Grassmann variables obeys the two rules

$$\int d\theta 1 = 0, \quad \int d\theta \theta = 1, \quad (\text{C.7})$$

where the first rule originates in the differentiation rule and the absence of boundary terms,

$$\int d\theta 1 = \int d\theta \frac{\partial}{\partial\theta}\theta = 0, \quad (\text{C.8})$$

while the second rule is simply the normalisation of the integral. Hence, for a general function $f(\theta)$ defined in (C.5) it follows

$$\int d\theta f(\theta) = \int d\theta(A + B\theta) = \int d\theta((A + B\eta) = B. \quad (\text{C.9})$$

We note that the Grassmann integral is identical to the Grassmann differentiation.

D. Dimensional regularisation

In this Appendix we collect some master integrals in dimensional regularisation as well as the Feynman parametrisation of loop integrals. Loop integrals typically consist out of a product of propagators with different momenta and potentially polynomials of contracted momenta or open momenta in the numerator. The product of propagators with different momenta can be brought into the form of a power of one propagator with the Feynman trick,

$$\prod_{i=1}^n \frac{1}{A_i} = (n-1)! \left[\prod_{i=1}^n \int_0^1 d\alpha_i \right] \frac{\delta\left(1 - \sum_{i=1}^n \alpha_i\right)}{\left(\sum_{i=1}^n \alpha_i A_i\right)^n}, \quad (\text{D.1})$$

see also (7.68) in Chapter 7.

In dimensional regularisation, integrals with a power of the scalar classical propagator take the form

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \frac{1}{(m^2)^{n - \frac{d}{2}}}. \quad (\text{D.2})$$

All the other integrals listed below can be readily derived from (D.2). The first variant is given by

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2 + 2q \cdot p)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \frac{1}{(m^2 - p^2)^{n - \frac{d}{2}}}, \quad (\text{D.3})$$

which can be reduced to (D.2) by a shift of the loop momentum.

Applying a p_μ -derivative to leads us to

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu}{(q^2 + m^2 + 2q \cdot p)^n} = -\frac{\Gamma\left(n - \frac{d}{2}\right)}{(4\pi)^{d/2} \Gamma(n)} \frac{p^\mu}{(m^2 - p^2)^{n - \frac{d}{2}}}. \quad (\text{D.4})$$

With a further p_ν -derivative we obtain

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 + m^2 + 2q \cdot p)^n} = \frac{1}{(4\pi)^{d/2} \Gamma(n)} \left[p^\mu p^\nu \frac{\Gamma\left(n - \frac{d}{2}\right)}{(m^2 - p^2)^{n - \frac{d}{2}}} + \frac{1}{2} \delta^{\mu\nu} \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{(m^2 - p^2)^{n - \frac{d}{2} - 1}} \right], \quad (\text{D.5})$$

which can be iterated further.

The explicit results are obtained in the limit $\epsilon \rightarrow 0$, where we use

$$x\Gamma(x) = \Gamma(x+1), \quad \text{and} \quad \Gamma(-1 + \epsilon) = -\frac{1}{\epsilon} + \gamma - 1 + O(\epsilon). \quad (\text{D.6})$$

E. Functional Derivatives

As indicated in the beginning of [Section 9.3.2](#), functional derivatives are obtained in the $d \rightarrow \infty$ limit of partial derivatives in d -dimensional spaces. Partial derivatives are defined by the rules

$$\frac{\partial q_i}{\partial q_k} = \delta_{ik}, \quad (\text{E.1})$$

and the Leibniz rule. From these two rules further properties such as the product rule, chain rule follow. To make the transition to functional derivatives we can think of the variable to be discretized,

$$t \in [a, b] \rightarrow t_i = (t_1, t_2, \dots) = \left(a, a + \frac{b-a}{N}, \dots \right), \quad (\text{E.2})$$

where the final expression on the right hand side just indicates one possible discretization. Indeed for a function of more than one variable we can generalize the idea of the discretization on a suitable lattice. At the end we can then use a suitable limit of increasingly fine lattices.

On this discrete lattice we can now represent the function by its values on the lattice points,

$$q(t) \rightarrow q_i = (q(t_1), q(t_2), \dots) = (q_1, q_2, \dots). \quad (\text{E.3})$$

For the purpose of our derivative we can now treat all q_i as independent variables¹ fulfilling Eq. (E.1). Taking the limit of an increasingly fine lattice we find the natural generalization is given by,

$$\frac{\delta q(t)}{\delta q(t')} = \delta(t - t'). \quad (\text{E.4})$$

Let us briefly consider a couple of explicit examples.

Example 1:

$$F[q] = \int dt' (q(t'))^2 \quad (\text{E.5})$$

This is a functional as it maps a function $q(t)$ onto a single number $F[q]$.

Note also that we have carefully chose t' as the integration variable (which as usual we can choose to name at will) because we will now take the functional derivative with respect to $q(t)$.

$$\begin{aligned} \frac{\delta F[q]}{\delta q(t)} &= \int dt' 2(q(t))\delta(t - t') \\ &= 2q(t). \end{aligned} \quad (\text{E.6})$$

Where in the first step we have used the chain rule and in the second we have performed the integral using the δ -function.

Example 2:

$$F[q] = \int dt' \left[\frac{(\dot{q}(t'))^2}{2} - \frac{(q(t'))^2}{2} \right]. \quad (\text{E.7})$$

¹This is exactly what we have done in the path integral, too.

Using the linearity of the functional integral we can simplify,

$$\begin{aligned}\frac{\delta F[q]}{\delta q(t)} &= \frac{\delta}{\delta q(t)} \left[\int dt' \frac{(\dot{q}(t'))^2}{2} \right] - \frac{\delta}{\delta q(t)} \left[\int dt' \frac{(q(t'))^2}{2} \right] \\ &= \frac{\delta}{\delta q(t)} \left[\int dt' \frac{(\dot{q}(t'))^2}{2} \right] - q(t).\end{aligned}\tag{E.8}$$

In the last step we could make use of our previous result.

Focussing only on the first term we have,

$$\begin{aligned}\frac{1}{2} \frac{\delta}{\delta q(t)} \left[\int dt' \left(\frac{d}{dt} q(t') \right)^2 \right] &= \int dt' \left(\frac{d}{dt} \delta(t' - t) \right) \dot{q}(t') \\ &= - \int dt' \delta(t' - t) \ddot{q}(t') \\ &= -\ddot{q}(t).\end{aligned}\tag{E.9}$$

Here, we have used partial integration in the second to last step and assumed that the boundary terms vanish at infinity.

Putting the two parts together we have,

$$\frac{\delta F[q]}{\delta q(t)} = -\ddot{q}(t) - q(t).\tag{E.10}$$

As a note, let us remark that the functional F is just the action of a harmonic oscillator with $m = \omega = 1$ and $\delta F/\delta q = 0$ gives us the equation of motion.

F. Gaussian path integrals

In this Appendix we discuss the computation of Gaussian path integrals. To that end we first consider a high-dimensional Gaussian integral in n -dimensions with

$$\int d^n q \exp\left[-\frac{1}{2} \mathbf{q}^T A \mathbf{q}\right], \quad (\text{F.1})$$

where $\mathbf{q} = (q_1, \dots, q_n)^T$ or $\mathbf{q} = \sum_k q_k \hat{\mathbf{e}}_k$ with the orthonormal basis $\hat{\mathbf{e}}_k$ and $k = 1, \dots, n$ is the position variable in the n -dimensional space and A is an $n \times n$ matrix. We assume that A can be diagonalised and is positive definite. Then, (F.1) can be simplified in terms of the normalised eigenvectors $\hat{\mathbf{v}}_k$ of A ,

$$A \hat{\mathbf{v}}_k = \lambda_k \hat{\mathbf{v}}_k, \quad \text{with} \quad \hat{\mathbf{v}}_j^T \hat{\mathbf{v}}_k = \delta_{jk}. \quad (\text{F.2})$$

A general vector \mathbf{q} can be expressed in terms of the basis $\{\hat{\mathbf{v}}_k\}$ with $k = 1, \dots, n$, and this basis is obtained from the basis $\hat{\mathbf{e}}_i$ by a rotation,

$$\mathbf{q} = \sum_{k=1}^n c_k \hat{\mathbf{v}}_k, \quad \text{and} \quad d^n q = d^n c. \quad (\text{F.3})$$

with the new variable $\mathbf{c} = (c_1, \dots, c_n)^T$ and the Jacobi determinant $\det \partial \mathbf{q} / \partial \mathbf{c} = 1$. With (F.2) and (F.3) we find

$$\begin{aligned} \int d^n q e^{-\frac{1}{2} \mathbf{q}^T A \mathbf{q}} &= \int d^n c e^{-\frac{1}{2} \sum_k \lambda_k c_k^2} = \int d^n c \prod_{k=1}^n e^{-\frac{1}{2} \lambda_k c_k^2} = \prod_{k=1}^n \left[\int dc_k e^{-\frac{1}{2} \lambda_k c_k^2} \right] = \prod_{k=1}^n \sqrt{\frac{2\pi}{\lambda_k}} \\ &= (2\pi)^{n/2} \frac{1}{\sqrt{\det A}}. \end{aligned} \quad (\text{F.4})$$

G. Feynman rules for QCD in the covariant gauge

In this Appendix we depict the Feynman rules for QCD in the covariant gauge with the classical action (14.3),

$$S[A, q, \bar{q}] = \int_x \frac{1}{2} \text{tr}_f F_{\mu\nu}^2 + \frac{1}{2\xi} \int_x (\partial_\mu A_\mu^a)^2 - \int_x \bar{c}^a \partial_\mu D_\mu^{ab} c^b - \sum_{f=1}^{N_f} \int_x \bar{q}_f (\not{D} + m_f) q_f, \quad (\text{G.1})$$

with the covariant derivative and field-strength components (13.14) and (13.20),

$$D_\mu = \partial_\mu - igA_\mu, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c. \quad (\text{G.2})$$

$$\begin{array}{c} a \quad b \\ \text{-----} \quad \text{-----} \\ p_\mu \quad k_\nu \end{array} = \delta^{ab} \frac{1}{p^2} \left(\delta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right) (2\pi)^d \delta(p+k)$$

$$\begin{array}{c} a \quad b \\ \text{-----} \quad \text{-----} \\ p \quad k \end{array} = \delta^{ab} \frac{1}{p^2} (2\pi)^d \delta(p+k)$$

$$\begin{array}{c} A \quad B \\ \text{-----} \quad \text{-----} \\ p \quad k \end{array} = \delta^{AB} \frac{1}{i\not{p} + m} (2\pi)^d \delta(p+k)$$

$$\begin{array}{c} a \quad k_{1,\mu} \\ \downarrow \\ c \quad k_{3,\rho} \quad b \quad k_{2,\nu} \end{array} = ig f^{abc} \left[(k_2 - k_1)_\rho \delta_{\mu\nu} + (k_1 - k_3)_\nu \delta_{\mu\rho} + (k_3 - k_2)_\mu \delta_{\nu\rho} \right] (2\pi)^d \delta(k_1 + k_2 + k_3)$$

$$\begin{array}{c} a \quad k_{1,\mu} \quad b \quad k_{2,\nu} \\ \swarrow \quad \searrow \\ d \quad k_{4,\sigma} \quad c \quad k_{3,\rho} \end{array} = g^2 \left[f^{eab} f^{ecd} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) + f^{eac} f^{ebd} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) + f^{ead} f^{ebc} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma}) \right] (2\pi)^d \delta \left(\sum_{i=1}^4 k_i \right)$$

$$\begin{array}{c} a \quad k_\mu \\ \downarrow \\ b \quad q \quad c \quad p \end{array} = ig f^{abc} p_\mu (2\pi)^d \delta(p - q - k)$$

$$\begin{array}{c} a \quad k_\mu \\ \downarrow \\ q \quad p \end{array} = -ig \gamma_\mu T^a (2\pi)^d \delta(p - q - k)$$

Figure G.1.: Feynman rules of QCD in a general covariant gauge in d dimensions.

H. Computational details of the one-loop computation of the running coupling

In this Appendix we collect some computational details of the derivation of the one-loop running coupling in [Section 14.2](#). Specifically we details the gauge group contractions and reparametrisations required as a preparation for using the master integrals in dimensional regularisation in [Appendix D](#).

8.1. One-loop gluon propagator

The one-loop vacuum polarisation $\Pi(p)$ of the gluon propagator is given in [\(14.52\)](#) as a sum of the different diagrams depicted in [Figure 14.2](#).

First, the diagram $\Pi_{3\text{gl}}$ has been discussed, which is depicted in [Figure 14.5](#). Using the Feynman rules in [Appendix G](#), we are led to [\(14.54\)](#), which we recall here for the sake of convenience,

$$\begin{aligned} [\Pi_{3\text{gl}}(p)]_{\mu\nu}^{ab} &= \frac{1}{2} g^2 \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} f^{acd} \left[\delta_{\mu\rho}(p-l)_\sigma + \delta_{\rho\sigma}(2l+p)_\mu + \delta_{\sigma\mu}(-2p-l)_\rho \right] \\ &\quad \times \frac{\delta_{\rho\rho'}}{l^2} f^{bcd} \left[\delta_{\nu\rho}(-p+l)_{\sigma'} + \delta_{\rho'\sigma'}(-2l-p)_\nu + \delta_{\sigma'\nu}(2p+l)_{\rho'} \right] \frac{\delta_{\sigma\sigma'}}{(l+p)^2}, \end{aligned} \quad (\text{H.1})$$

The gauge group contraction of the two structure constants in [\(H.1\)](#) is nothing but the trace of two generators in the adjoint representation,

$$f^{acd} f^{bcd} = (-i f^a)^{cd} (-i f^b)^{dc} = \text{tr}_{ad} t^a t^b, \quad \text{with} \quad (t_{ad}^a)^{bc} = -i f^{abc}, \quad (\text{H.2})$$

where t_{ad}^a are the generators in the adjoint representation of the gauge group $\text{SU}(N_c)$, which is $(N_c^2 - 1) \times (N_c^2 - 1)$ -dimensional. The t_{ad}^a satisfy the Lie-algebra as $f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$. In a general representation R we have

$$\text{tr}_R t^a t^b = C(R) \delta^{ab}, \quad \text{with} \quad C(f) = \frac{1}{2}, \quad C(ad) = N_c. \quad (\text{H.3})$$

where $C(R)$ the Dynkin index of the representation, and the fundamental and adjoint representations are labelled by $R = f, ad$. The second Casimir operator of a group is defined by,

$$t_R^a t_R^a = C_2(R) \mathbb{1}_R, \quad \text{with} \quad C_2(ad) = C(ad) = N_c, \quad C_2(f) = \frac{N_c^2 - 1}{N_c}. \quad (\text{H.4})$$

Collecting all these results, the diagram takes the form

$$[\Pi_{3\text{gl}}(p)]_{\mu\nu}^{ab} = \frac{1}{2} g^2 \mu^{2\epsilon} N_c \int \frac{d^d l}{(2\pi)^d} \left[\delta_{\nu\rho} \cdots \right] \left[\delta_{\nu\rho} \cdots \right] \frac{1}{l^2} \frac{1}{(l+p)^2}, \quad (\text{H.5})$$

where the expressions in the square brackets in [\(H.5\)](#) are that in [\(14.54\)](#). With the Feynman trick [\(D.1\)](#) we convert the loop integral in [\(H.5\)](#) into the form, that is simply a sum of the master integrals in [Appendix D](#). Specifically we use

$$\frac{1}{l^2} \frac{1}{(l+p)^2} = \int_0^1 d\alpha \frac{1}{\alpha(l+p)^2 + (1-\alpha)l^2} = \int_0^1 d\alpha \frac{1}{(k^2 + \Delta)^2}, \quad (\text{H.6})$$

with $k = l + \alpha p$ and

$$\Delta = \alpha(1 - \alpha)p^2. \quad (\text{H.7})$$

We can also shift the l -integration: $\int d^d l = \int d^d k$. It is only left to rewrite the numerator in (14.54) in terms of k : $l = k - \alpha p$. With this reparametrisation terms with a linear dependence on l_μ in the numerator vanish identically due to the angular integration, which also enforces, that terms with $k_\mu k_\nu$ are proportional to $\delta_{\mu\nu}$,

$$\int d^d k \frac{k_\mu}{(k^2 + \Delta^2)^2} = 0, \quad \text{and} \quad \int d^d k \frac{k_\mu k_\nu}{(k^2 + \Delta^2)^2} = \frac{1}{d} \int d^d k \frac{k^2}{(k^2 + \Delta^2)^2}. \quad (\text{H.8})$$

Collecting all these results, we arrive at

$$\left[\Pi_{3\text{gl}}(p) \right]_{\mu\nu}^{ab} = \delta^{ab} N_c g^2 \mu^{2\epsilon} \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^2} \frac{N_{\mu\nu}}{2}, \quad (\text{H.9a})$$

with

$$\begin{aligned} N_{\mu\nu} &= \left[\delta_{\mu\rho}(p - l)_\sigma + \delta_{\rho\sigma}(2l + p)_\mu + \delta_{\mu\sigma}(-2p - l)_\rho \right] \left[\delta_{\nu\rho}(-p + l)_\sigma + \delta_{\rho\sigma}(-2l - p)_\nu + \delta_{\sigma\nu}(2p + l)_\rho \right] \\ &= \left[\delta_{\mu\rho}(p(1 + \alpha) - k)_\sigma + \delta_{\rho\sigma}(2k + p(1 - 2\alpha))_\mu + \delta_{\mu\sigma}(-k - (2 - \alpha)p)_\rho \right] \\ &\quad \times \left[\delta_{\nu\rho}(-(1 + \alpha)p + k)_\sigma + \delta_{\rho\sigma}(-2k - p(1 - 2\alpha))_\nu + \delta_{\nu\sigma}(k + (2 - \alpha)p)_\rho \right], \end{aligned} \quad (\text{H.9b})$$

where we have used that $l = k - \alpha p$. Now we use that the linear terms in (H.9b) do not contribute to the integral in (H.9a) due to (H.8). Dropping these terms leads us to

$$N_{\mu\nu} \simeq \left[\delta_{\mu\nu} \left[-p^2 \left((1 + \alpha)^2 + (2 - \alpha)^2 \right) - 2k^2 \right] + p_\mu p_\nu \left((2 - d)(1 - 2\alpha)^2 + 2(1 + \alpha)(2 - \alpha) \right) + k_\mu k_\nu (6 - 4d) \right]. \quad (\text{H.10})$$

With (H.8) for the terms quadratic in $k_\mu k_\nu$ this leads us to

$$\begin{aligned} \left[\Pi_{3\text{gl}}(p) \right]_{\mu\nu}^{ab} &= \frac{\delta^{ab}}{2} N_c g^2 \mu^{2\epsilon} \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^2} \\ &\quad \times \left[-\delta_{\mu\nu} \left(p^2 \left[(1 + \alpha)^2 + (2 - \alpha)^2 \right] + 6k^2 \left[1 - \frac{1}{d} \right] \right) + p_\mu p_\nu \left[(2 - d)(1 - 2\alpha)^2 + 2(1 + \alpha)(2 - \alpha) \right] \right]. \end{aligned} \quad (\text{H.11})$$

In the final step we use the master integrals (D.2) and (D.4) and arrive at

$$\begin{aligned} \left[\Pi_{3\text{gl}}(p) \right]_{\mu\nu}^{ab} &= \frac{\delta^{ab}}{2} N_c g^2 \frac{1}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{\mu^{2\epsilon}}{\Delta^{2-\frac{d}{2}}} \left\{ -\delta_{\mu\nu} p^2 \left[\Gamma(2 - d/2) \left((1 + \alpha)^2 + (2 - \alpha)^2 \right) \right. \right. \\ &\quad \left. \left. + \Gamma(1 - d/2) 3(d - 1)\alpha(1 - \alpha) \right] + p_\mu p_\nu \Gamma(2 - d/2) \left[(2 - \alpha)(1 - 2\alpha)^2 + 2(1 + \alpha)(2 - \alpha) \right] \right\}. \end{aligned} \quad (\text{H.12})$$

with Δ being defined in (H.7).

I. Wilson loop in QED

In this appendix we discuss the case of an electron-positron pair e^+e^- . Then the static potential is the standard Coulomb potential. Indeed in the static limit there is no self-interaction of the photon and the expectation value of the Wilson loop is simply given by the sum of boxes with n photon exchanges from positions x_i to y_i where one integrates over x_i and y_i on the contour $C[L, T]$. This is depicted in Fig. I.1.

In other words we have

$$W[L, T] = e^{-\frac{e^2}{2} \int_{C[L, T]} dx_\mu \int_{C[L, T]} dy_\nu \langle A_\mu(x) A_\nu(y) \rangle_{\text{sub}}}, \quad (\text{I.1})$$

where we have used that $\langle A_{\mu_1} \cdots A_{\mu_{n+1}} \rangle = 0$. The subscript $\langle \cdots \rangle_{\text{sub}}$ refers to the necessary subtraction of infinite selfenergies related to close loops with endpoints $x = y$. Moreover, all correlation functions decay in products of two-point functions (Wick-theorem), schematically we have $\langle A_1 \cdots A_{2n} \rangle = \langle A_1 A_2 \rangle \cdots \langle A_{2n-1} A_{2n} \rangle + \cdots$, and there are $(2n-1)(2n-3) \cdots$ combinations. Upon contour integration all combinations give the same contribution and overall we have the n th order term in the propagator

$$\frac{(2n-1)(2n-3) \cdots}{(2n)!} 2^n \left(-\frac{e^2}{2} \right)^n \left(\int_C dx_\mu \int_C dy_\nu \langle A_\mu(x) A_\nu(y) \rangle \right)^n = \frac{1}{n!} \left(-\frac{e^2}{2} \int_C dx_\mu \int_C dy_\nu \langle A_\mu(x) A_\nu(y) \rangle \right)^n, \quad (\text{I.2})$$

for a general contour C , leading to the Gaußian expression eq. (I.1). This leaves us with the task of computing

$$\begin{aligned} \int_C dx_\mu \int_C dy_\nu \langle A_\mu(x) A_\nu(y) \rangle &= \int_C dx_\mu \int_C dy_\nu \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2} \left(\delta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right) e^{ip(x-y)} \\ &= \int_C dx_\mu \int_C dy_\mu \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2} e^{ip(x-y)} \\ &= \int_C dx_\mu \int_C dy_\mu \frac{1}{4\pi^2} \frac{1}{(x-y)^2}. \end{aligned} \quad (\text{I.3})$$

To be explicit, we picked a covariant gauge in eq. (I.3). However, we have already proven that the *closed* Wilson line is gauge invariant which now is explicit as the ξ -dependent term drops out with the help of

$$\int_C dx_\mu p_\mu e^{ipx} = -i \int_C dx_\mu \partial_\mu^x e^{ipx} = 0, \quad (\text{I.4})$$

which eliminates all longitudinal contributions for closed loops. Note that this is not valid for open Wilson

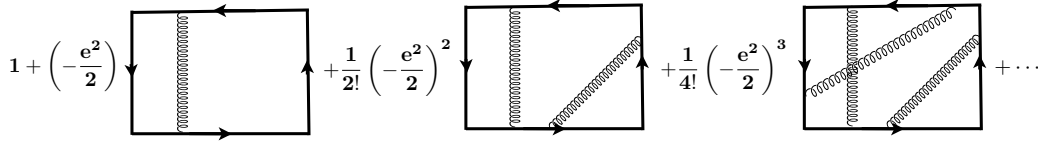


Figure I.1.: Perturbative expansion of the Wilson loop expectation value for e^+e^- .

lines. Finally we are interested in the large T -limit in (I.1), see also (15.125), where we have

$$\begin{aligned}
 V_{e^+e^-}(L) &= - \lim_{T \rightarrow \infty} \frac{1}{T} \log W[L, T] = \lim_{T \rightarrow \infty} \frac{1}{T} \frac{e^2}{2} \lim_{T \rightarrow \infty} \int_{C[L, T]} dx_\mu \int_{C[L, T]} dy_\mu \left(\frac{1}{4\pi^2} \frac{1}{(x-y)^2} \right)_{\text{sub}} \\
 &= - \lim_{T \rightarrow \infty} \frac{1}{T} e^2 \lim_{T \rightarrow \infty} \int_{t_0}^{t_1} dx_0 \int_{t_0}^{t_1} dy_0 \left(\frac{1}{4\pi^2} \frac{1}{(x_0 - y_0)^2 + L^2} \right) \\
 &= - \lim_{T \rightarrow \infty} \frac{1}{T} \frac{e^2}{4\pi} \lim_{T \rightarrow \infty} \int_{t_0}^{t_1} dx_0 \int_{t_0 - x_0}^{t_1 - x_0} dy_0 \left(\frac{1}{\pi} \frac{1}{y_0^2 + L^2} \right) \\
 &= - \lim_{T \rightarrow \infty} \frac{1}{T} 2 \frac{e^2}{4\pi} \int_0^T dx_0 \arctan\left(\frac{x_0}{L}\right) \\
 &= - \frac{e^2}{4\pi} \frac{1}{L}. \tag{I.5}
 \end{aligned}$$

Equation (I.5) is the Coulomb potential as expected. This has to be compared with the lattice result in the strong coupling expansion that shows an area law.

J. Derivation of functional flow equations

For the derivation of general functional flow equations we start with the infrared regularised generating functional

Infrared regularised generating functional

$$Z_k[J] = \int D\hat{\phi} e^{-S[\hat{\phi}] - \Delta S_k[\hat{\phi}] + \int_x J(x)\hat{\phi}(x)}, \quad (\text{J.1})$$

with the infrared cutoff term

$$\Delta S_\Lambda[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \phi(-p) R_k(p^2) \phi(p), \quad \text{with} \quad \lim_{\frac{p^2}{k^2} \rightarrow 0} R_k(p) \geq k^2, \quad \lim_{\frac{p^2}{k^2} \rightarrow \infty} R_k(p) = 0. \quad (\text{J.2})$$

Commonly used infrared regulators are the sharp infrared regulator $R^{(\text{sharp})}$ defined in (16.5), the flat or Litim regulator $R^{(\text{flat})}$ and the standard exponential regulator $R^{(\text{exp})}$,

$$R_k^{(\text{sharp})}(p) = k^2 \left(\frac{1}{\theta(p^2 - k^2)} - 1 \right), \quad R_k^{(\text{flat})}(p) = (k^2 - p^2)\theta(k^2 - p^2), \quad R_k^{(\text{exp})}(p) = k^2 e^{-\frac{p^2}{k^2}}. \quad (\text{J.3})$$

All the choices in (J.3) have in common that they suppress the propagation of infrared modes with $p^2 \lesssim k^2$ and vanish for $p^2/k^2 \rightarrow \infty$, thus leaving the ultraviolet physics untouched.

The flow of the generating functional is obtained from (J.1) by a scale-derivative with respect to the logarithmic cutoff scale $t = \log k/k_{\text{ref}}$ with some reference scale k_{ref} . We find

Flow equation for the partition function

$$\partial_t Z[J] = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t R_k(p^2) \int \mathcal{D}\phi \phi(-p) \hat{\phi}(p) e^{-S[\hat{\phi}] - \Delta S_k[\hat{\phi}] + \int_x J(x)\hat{\phi}(x)} = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t R_k(p^2) Z^{(2)}[J], \quad (\text{J.4})$$

the flow of Z_k is proportional to the full two-point function $\langle \hat{\phi}(p) \hat{\phi}(-p) \rangle_J$ in the presence of an external current indicated by the subscript J . These expectation values include the disconnected terms

$$\langle \hat{\phi}(p) \rangle_J \langle \hat{\phi}(-p) \rangle_J = \frac{Z_k^{(1)}[J](p) Z_k^{(1)}[J](-p)}{Z_j[J] Z_j[J]}, \quad (\text{J.5})$$

where we have used the notation

$$F^{(n)}[\varphi](x_1, \dots, x_n) = \frac{\delta^n F[\varphi]}{\delta \varphi(x_1) \cdots \delta \varphi(x_n)}, \quad (\text{J.6})$$

for functionals $F[\varphi]$ with the argument $\varphi = J, \phi, \dots$. Equation (J.4) reflects the fact that the partition function $Z_j[J]$ generated full correlation functions including their disconnected parts. Moreover, the expectation values are not normalised. We get rid of part of this redundancy if we consider the infrared regularised Schwinger functional $W_k[J]$ with

$$W_k[J] = \ln Z_k[J], \quad (\text{J.7})$$

in analogy of (7.18). Its flow can be read of from (J.4), to wit

Flow equation for the Schwinger functional

$$\partial_t W_k[J] = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t R_k(p^2) \left[W_k^{(2)}[J](p, -p) + W_k^{(1)}[J](-p) W_k^{(1)}[J](p) \right], \quad (\text{J.8})$$

The flow resembles the Polchinski equation and indeed W_k defined in (J.7) and S_{eff} defined in (16.51) are both given by the logarithm of the partition function Z_k , we simply have to translate their arguments. This is given by

$$S_{\text{eff}}[\phi] = W_k[J = (G_k^{(0)})^{-1} \phi]. \quad (\text{J.9})$$

whose proof we leave to the interested reader. The right hand side of (J.8) is proportional to the sum of the connected two-point function, the propagator $W_k^{(2)}$ and the one-point function squared, $(W_k^{(1)})^2$. The latter is simply the mean field squared and in terms of the infrared regularised 1PI effective action

$$\Gamma_k^{(1\text{PI})}[\phi] = \sup_J \left(\int_x J(x) \phi(x) - W_k[J] \right), \quad (\text{J.10})$$

the flow (J.8) reads

$$\partial_t \Gamma_k^{(1\text{PI})}[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t R_k(p^2) \left[\frac{1}{\Gamma_k^{(1\text{PI})}[\phi]}(p, -p) + \phi(p) \phi(-p) \right]. \quad (\text{J.11})$$

where we have used the relations (11.4) and (11.11). A last step is done for computational convenience: the second term on the right hand side is simply

$$\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \partial_t R_k(p) \phi(p) \phi(-p) = \partial_t \Delta S_k[\phi], \quad (\text{J.12})$$

the flow of the mean field cutoff term. This is a trivial term but will numerically dominate the flow. Therefore, we rewrite the flow (J.11) in terms of the modified effective action $\Gamma_k[\phi]$ defined with

$$\Gamma_k[\phi] = \Gamma_k^{(1\text{PI})}[\phi] - \Delta S_k[\phi]. \quad (\text{J.13})$$

This leads us to our final expression for the *Wetterich* equation

Flow equation for the 1PI effective action

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} G_k[\phi](p, -p) \partial_t R_k(p), \quad \text{with} \quad G_k[\phi] = \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}. \quad (\text{J.14})$$

This concludes our derivation of functional flow equations. We note that all different forms are indeed used in conceptual as well as numerical applications. In most numerical applications the numerical self-stabilising property of the flow equation for the effective action is key for stable flows, the flow for the Schwinger functional is in most cases best suited for conceptual considerations such as the derivation and use of symmetry identities. Finally, using the flow of Z_k is convenient, if the partition function encounters cuts and zeros. Then, taking its logarithm is at best computationally inconvenient.

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