

# An invitation to Dirac ensembles in random noncommutative geometry

*Seminario de Física Matemática CCM, Morelia*

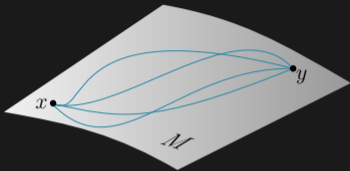
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21 April 2022

# I. MOTIVATING NONCOMMUTATIVE GEOMETRY

*Heuristics:*

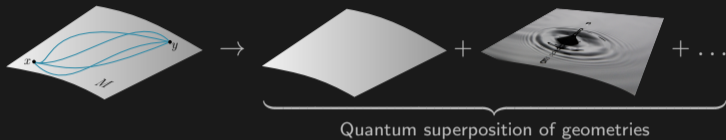
- Path integrals on  $M$



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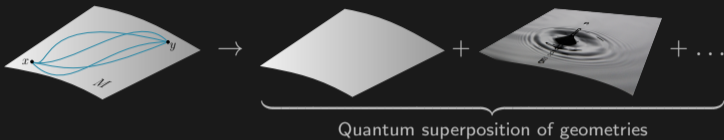
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- Path integrals on  $M$   $\rightarrow$  Path integral of spacetime (Quantum Gravity)



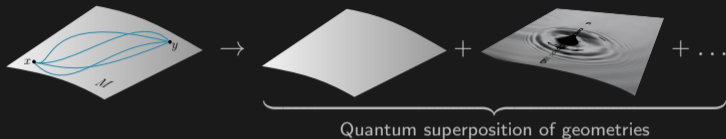
- Quantum Gravity  $\rightarrow$  Random Geometry

$$\mathcal{Z}_{\text{QG}} = \int\!\!\int_{\substack{\text{topologies} \\ \text{geometries}}} e^{\frac{i}{\hbar} S_{\text{EH}}[g]} \mathcal{D}[g] \rightarrow \mathcal{Z} = \int\!\!\int_{\substack{\text{topologies} \\ \text{geometries}}} e^{-\frac{1}{\hbar} S_{\text{EH}}[g]} \mathcal{D}[g]$$

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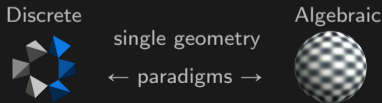
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- To access  $\mathcal{Z}$ , models for 'quantum space' are proposed, e.g.



(↖ from Wikipedia)

# The Spectral Standard Model

$$\begin{aligned}
 & -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \\
 & \frac{1}{2}ig_s^2 (\bar{q}_i^\mu \gamma^\mu q_j^\mu) g_\mu^a + \bar{G}^a \partial^2 G^a + g_s f^{abc} \partial_\mu \bar{G}^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - \\
 & M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_w^2} M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
 & \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
 & \frac{1}{2c_w} M \phi^0 \phi^0 - \beta_h \left[ \frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-) \right] + \frac{2M^4}{g^2} \alpha_h - \\
 & ig_{c_w} [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - Z_\nu^0 (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + \\
 & Z_\mu^0 (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+) ] - ig_{s_w} [\partial_\nu A_\mu (W_\mu^+ W_\nu^- - \\
 & W_\nu^+ W_\mu^-) - A_\nu (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\mu^- - \\
 & W_\nu^- \partial_\nu W_\mu^+) ] - \frac{1}{2}g^2 W_\mu^+ W_\nu^+ W_\mu^- W_\nu^- + \frac{1}{2}g^2 W_\nu^+ W_\mu^+ W_\nu^- W_\mu^- + \\
 & g^2 c_w^2 (Z_\mu^0 W_\nu^+ Z_\nu^0 W_\mu^- - Z_\nu^0 Z_\mu^0 W_\nu^+ W_\mu^-) + g^2 s_w^2 (A_\mu W_\nu^+ A_\nu W_\mu^- - \\
 & A_\nu A_\mu W_\nu^+ W_\mu^-) + g^2 s_w c_w [A_\mu Z_\nu^0 (W_\nu^+ W_\mu^- - W_\mu^- W_\nu^+) - \\
 & 2A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^-] - \frac{1}{8}g^2 \alpha_h [H^4 + \\
 & (\phi^0)^4 + 4(\phi^+ \phi^-)^2 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - \\
 & gM W_\mu^+ W_\mu^- H - \frac{1}{2}g \frac{M}{c_w} Z_\mu^0 Z_\mu^0 H - \frac{1}{2}ig [W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
 & W_\mu^- (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \frac{1}{2}g [W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - \\
 & W_\mu^- (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \frac{1}{2}g \frac{1}{c_w} (Z_\mu^0 (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
 & ig \frac{s_w^2}{c_w} M Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + ig_{s_w} M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - \\
 & ig \frac{1-2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + ig_{s_w} A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \\
 & \frac{1}{4}g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \frac{1}{4}g^2 \frac{1}{c_w^2} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + \\
 & 2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + W_\mu^- \phi^+) -
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + \\
 & W_\mu^- \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W_\mu^+ \phi^- - W_\mu^- \phi^+) - g^2 \frac{s_w}{c_w} (2c_w^2 - \\
 & 1) Z_\mu^0 A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma \partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma \partial \nu^\lambda - \\
 & \bar{u}_j^\lambda (\gamma \partial + m_u^\lambda) u_j^\lambda - \bar{d}_j^\lambda (\gamma \partial + m_d^\lambda) d_j^\lambda + ig_{s_w} A_\mu [- (\bar{e}^\lambda \gamma^\mu e^\lambda) + \\
 & \frac{2}{3} (\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3} (\bar{d}_j^\lambda \gamma^\mu d_j^\lambda)] + \frac{ig}{4c_w} Z_\mu^0 [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + \\
 & (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (\frac{4}{3}s_w^2 - 1 - \gamma^5) u_j^\lambda) + (\bar{d}_j^\lambda \gamma^\mu (1 - \\
 & \frac{8}{3}s_w^2 - \gamma^5) d_j^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (1 + \\
 & \gamma^5) C_{\lambda\kappa}^\dagger d_j^\kappa)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger \gamma^\mu (1 + \\
 & \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m_\lambda^2}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\
 & \frac{g}{2} \frac{m_\lambda^2}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_a^2 (\bar{u}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \\
 & \gamma^5) d_j^\kappa) + m_u^2 (\bar{u}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) d_j^\kappa) + \frac{ig}{2M\sqrt{2}} \phi^- [m_d^2 (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \\
 & \gamma^5) u_j^\kappa) - m_u^2 (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \gamma^5) u_j^\kappa) - \frac{g}{2} \frac{m_h^2}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \\
 & \frac{g}{2} \frac{m_d^2}{M} H (\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda)
 \end{aligned}$$

Num. of generations and  $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightsquigarrow$

NCG

$\rightsquigarrow$  Classical Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* 2007 (Euclidean); J. Barrett *J. Math. Phys.* 2007 (Lorentzian)]

go to sketch of proof  $\triangleright$

# Commutative spectral triples

A spin manifold  $M$  yields  $(A_M, H_M, D_M)$

- $A_M = C^\infty(M)$  is a comm.  $*$ -algebra
- $H_M := L^2(M, \mathbb{S})$  a representation of  $A_M$
- $D_M = -i\gamma^\mu(\partial_\mu + \omega_\mu)$  is self-adjoint
- for each  $a \in A_M$ ,  $[D_M, a]$  is bounded, and in fact  $[D_M, x^\mu] = -ic(dx^\mu) = -i\gamma^\mu$ , being  $c$  Cliff. mult.
- $D_M$  has compact resolvent

[go to more on spin geometry >](#)

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- a  $*$ -algebra  $A$
- a representation  $H$  of  $A$
- a self-adjoint operator  $D$  on  $H$  with compact resolvent and such that  $[D, a]$  is bounded for each  $a \in A$

The 'commutative case' motivates

$$\Omega_D^1(A) := \left\{ \sum_{(\text{finite})} b[D, a] \mid a, b \in A \right\}$$

go to more on spin geometry  $\triangleright$



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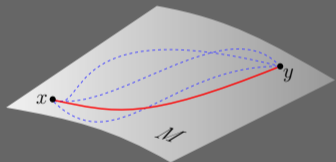
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**RECONSTRUCTION THEOREM:** [A, Connes, JNCG '13] (quite roughly formulated)  
Commutative spectral triples +some more axioms are Riemannian manifolds.

The idea is to replace the metric in  $(M, g)$  by  $D_M$

## Connes' geodesic distance

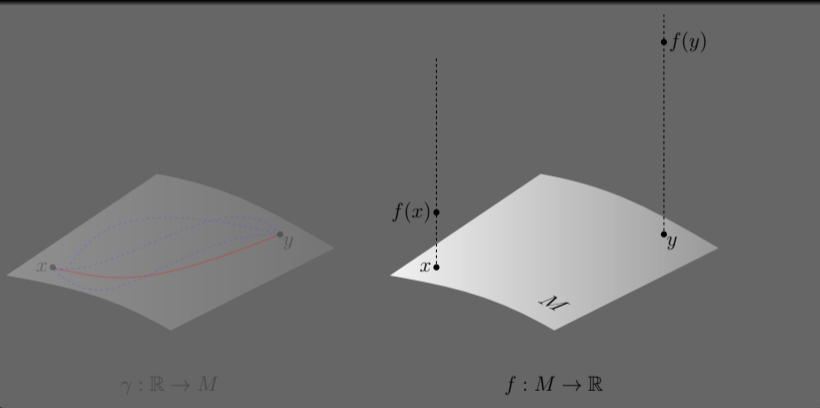


$$\gamma : \mathbb{R} \rightarrow M$$

$$\inf_{\gamma \text{ as above}} \left\{ \int_{\gamma} ds \right\} = d(x, y)$$

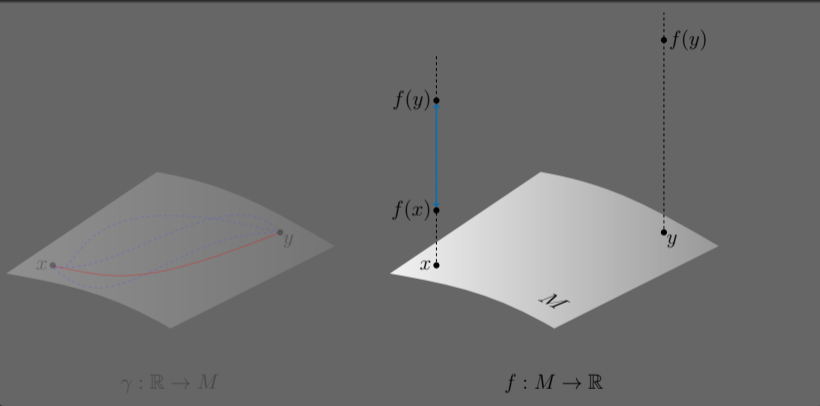
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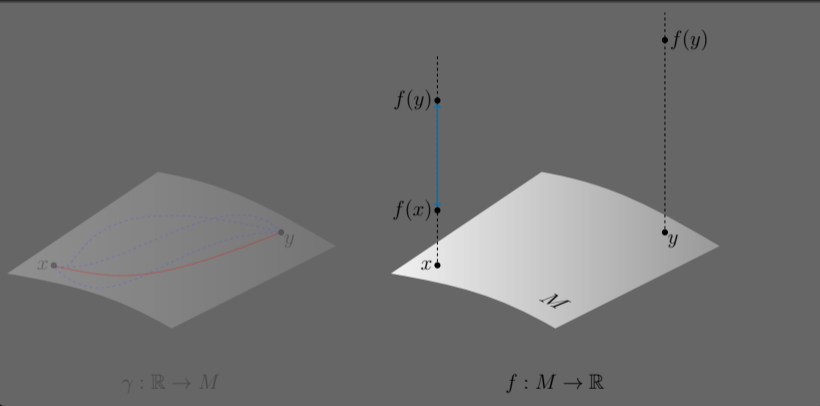
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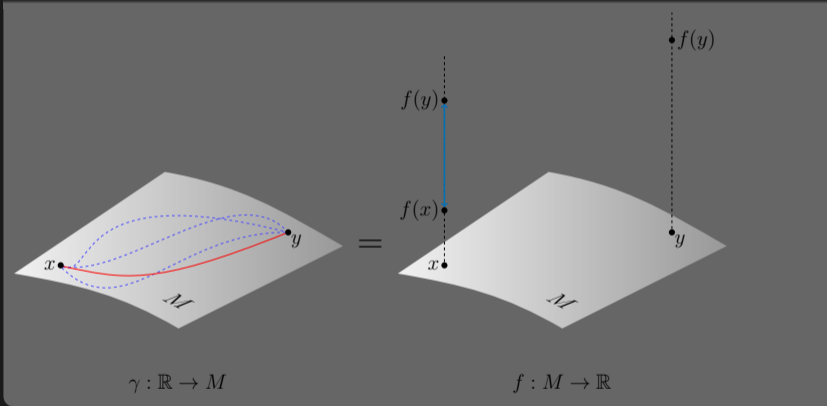
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$$\sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \|D_M f - f D_M\| \leq 1 \}$$

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$$\inf_{\gamma \text{ as above}} \left\{ \int_{\gamma} ds \right\} = d(x, y) = \sup_{f \in C^{\infty}(M)} \left\{ |f(x) - f(y)| : \|D_M f - f D_M\| \leq 1 \right\}$$

go to examples ▽

# NCG toolkit in high energy physics

- On a spectral triple  $(A, H, D)$  the (bosonic) classical action is given by

$$S(D) = \text{Tr}_H f(D/\Lambda) \quad [\text{Chamseddine-Connes } CMP '97]$$

for a bump function  $f$  around the origin and  $\Lambda$  a cut off scale. It's computed with heat kernel expansion [P. Gilkey, *J. Diff. Geom.* '75]

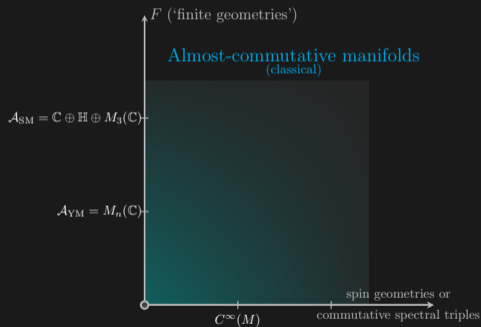
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- Realistic, classical models come from *almost-commutative manifolds*  $M \times F$ , where  $F$  is a finite-dim. spectral triple  $(C^\infty(A_F), H_M \otimes H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$





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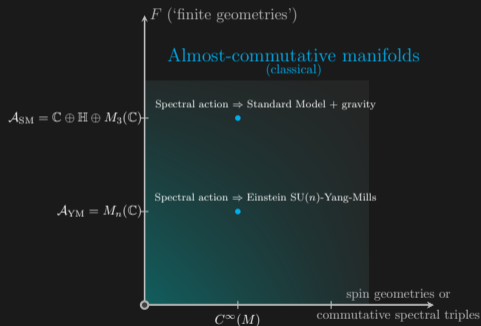
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$$\psi a = J a^* J^{-1} \psi \quad \psi \in H, a \in A$$



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- let's sketch *connections*: if  $S^G$  is a  $G$ -invariant functional on  $M$

$$S^G \rightsquigarrow S^{\text{Maps}(M, G)}$$

$$d \rightsquigarrow d + \mathbb{A} \quad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u \mathbb{A} u^{-1} + u du^{-1} \quad u \in \text{Maps}(M, G)$$

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- given  $(A, H, D)$  and a Morita equivalence  $A \simeq_M B$  (i.e.  $\text{End}_A(E) \cong B$ ) yields new  $(B, E \otimes_A H, D')$ . For  $A = B$ , in fact a tower

$$\{(A, H, D_\omega)\}_{\omega \in \Omega_D^1(A)}$$

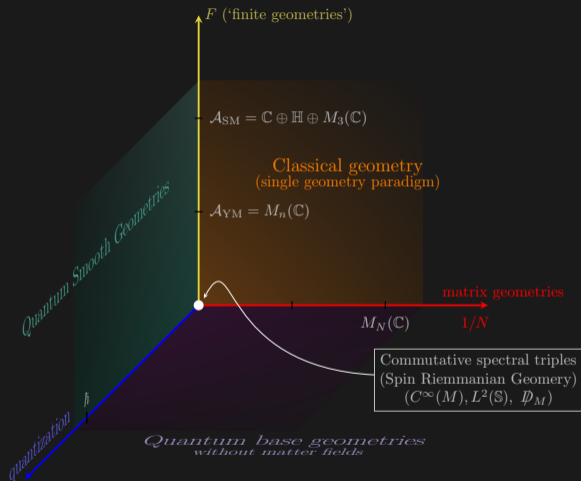
with  $D_\omega = D + \omega$ . In presence of  $J$

$$D_\omega = D + \omega \pm J \omega J^{-1}$$

$$D_\omega \mapsto \text{Ad}(u) D_\omega \text{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u \omega u^* + u [D, u^*] \quad u \in \mathcal{U}(A)$$

# Organization



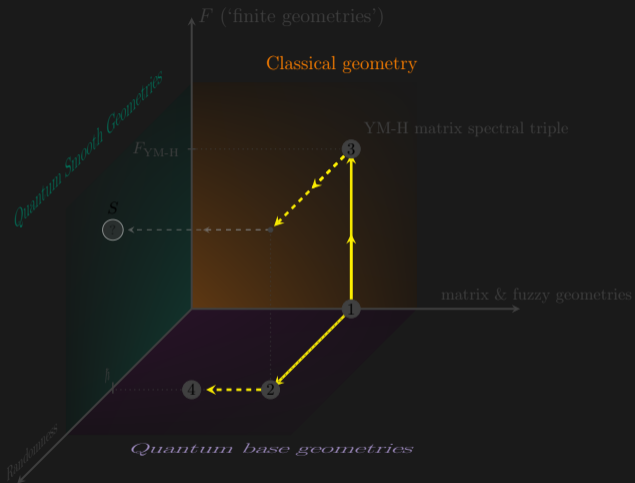
AIM: Make sense of

$$\mathcal{Z} = \int_{\text{DIRAC}} e^{-\frac{1}{\hbar} \text{Tr}_H f(D)} dD$$

- Plane  $(\hbar, 1/N, 0)$  of 'base geometries'
- Plane  $(\hbar, 0, F) = \lim_{N \rightarrow \infty} (\hbar, 1/N, F)$
- Plane  $(0, 1/N, F) = \lim_{\hbar \rightarrow 0} (\hbar, 1/N, F)$  of classical geometries

[CP 2105.01025]

# Organization



## 1 Matrix Geometries

[J. Barrett, *J. Math. Phys.* 2015]

## 2 Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action

[CP 1912.13288]

## 3 Gauge matrix spectral triples

[CP 2105.01025]

## 4 Functional Renormalization [CP 2007.10914] and [CP 2111.02858]

A *fuzzy geometry* (of signature  $(p, q)$  (thus of dim.  $p + q$  and KO-dim  $q - p$ ) consists of

- $A = M_N(\mathbb{C})$
- $H = \mathbb{S} \otimes M_N(\mathbb{C})$ , being  $\mathbb{S}$  a  $\mathcal{Cl}(p, q)$ -module  
... (axioms for  $D$  omitted, go to axioms  $\nabla$ ) ...

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... (axioms for  $D$  omitted, go to axioms  $\nabla$ ) ...
- Gamma-matrices conventions:
  - $(\gamma^\mu)^2 = +1$ ,  
 $\mu = 1, \dots, p, \gamma^\mu$  Hermitian
  - $(\gamma^\mu)^2 = -1$ ,  
 $\mu = 1 + p, \dots, q + p, \gamma^\mu$  anti-Hermitian
  - $\Gamma^I := \gamma^{\mu_1} \dots \gamma^{\mu_r}$  for  $\mu_i = 1, \dots, p + q$ ,  
 $I = (\mu_1, \dots, \mu_r)$

- Characterization of  $D$  in even dimensions:

$$D = \sum_J \Gamma_{\text{s.a.}}^J \otimes \{H_J, \cdot\} + \sum_J \Gamma_{\text{anti.}}^J \otimes [L_J, \cdot]$$

multi-index  $J$  monot. increasing,  $|J|$  odd

[J. Barrett, *J. Math. Phys.* '15],  $H_J^* = H_J, L_J^* = -L_J$

A *fuzzy geometry* (of signature  $(p, q)$  (thus of dim.  $p + q$  and KO-dim  $q - p$ ) consists of

- $A = M_N(\mathbb{C})$
- ' $H = \mathbb{S} \otimes M_N(\mathbb{C})$ , being  $\mathbb{S}$  a  $\mathbb{C}\ell(p, q)$ -module  
... (axioms for  $D$  omitted, go to axioms  $\nabla$ ) ...
- Gamma-matrices conventions:
  - $(\gamma^\mu)^2 = +1$ ,  
 $\mu = 1, \dots, p, \gamma^\mu$  Hermitian
  - $(\gamma^\mu)^2 = -1$ ,  
 $\mu = 1 + p, \dots, q + p, \gamma^\mu$  anti-Hermitian
  - $\Gamma^I := \gamma^{\mu_1} \dots \gamma^{\mu_r}$  for  $\mu_i = 1, \dots, p + q$ ,  
 $I = (\mu_1, \dots, \mu_r)$

- Characterization of  $D$  in even dimensions:

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[J. Barrett, *J. Math. Phys.* '15],  $H_J^* = H_J, L_J^* = -L_J$

- Examples:

- $D_{(1,1)} = \gamma^1 \otimes [L, \cdot] + \gamma^2 \otimes \{H, \cdot\}$

- $D_{(0,4)} = \sum_{\mu} \gamma^\mu \otimes [L_\mu, \cdot] + \gamma^{\hat{\mu}} \otimes \{H_{\hat{\mu}}, \cdot\}$

- [J. Barrett, L. Glaser, *J. Phys. A* 2016]

$$\{H, \cdot\} \mapsto H \otimes 1_N + 1_N \otimes H^T$$

$$[L, \cdot] \mapsto L \otimes 1_N - 1_N \otimes L^T$$

so we will get double traces from

$$\text{Tr}_H = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_{M_N(\mathbb{C})} = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_N^{\otimes 2}$$

**Notation:**  $\text{Tr}_V X$  is the trace on operators  $X : V \rightarrow V$ ,  $\text{Tr}_V 1 = \dim V$ . So  $\text{Tr}_N 1 = N$  but  $\text{Tr}_{M_N(\mathbb{C})}(1) = N^2$ .



Spectral Action:  $\text{Tr}_H D^{2m}$  [C.P. 1912.13288] Chord diags.  $d = 2$  geometries,  $\eta = \text{diag}(\mathbf{e}_1, \mathbf{e}_2)$  go to more CD's  $\triangleright$

$$\text{Tr}_H(D^6) = 2N \sum_{\mu} \overline{\text{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \dots \gamma^{\mu_6})} \times$$

$$+(-1)^2 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_6} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_4} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_5} + (-1)^1 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_5} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_4}$$

$$+(-1)^1 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_6} \eta^{\mu_4 \mu_5} + (-1)^0 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} \eta^{\mu_5 \mu_6} + (-1)^3 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_6} + (-1)^2 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_5} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_6}$$

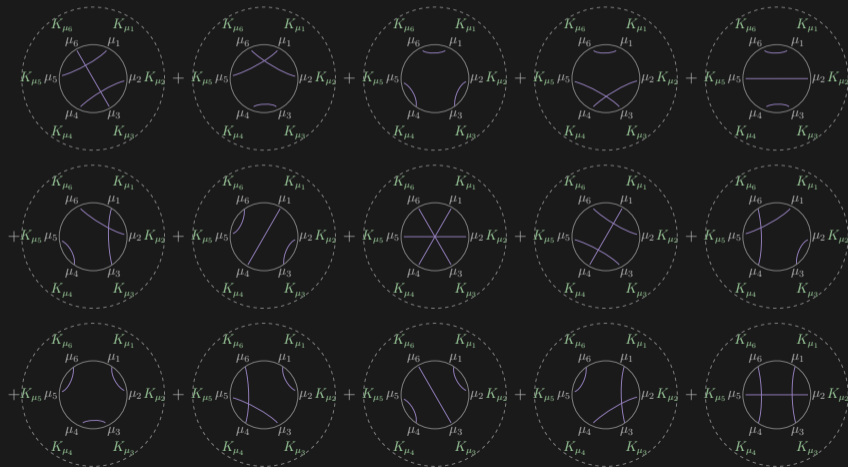
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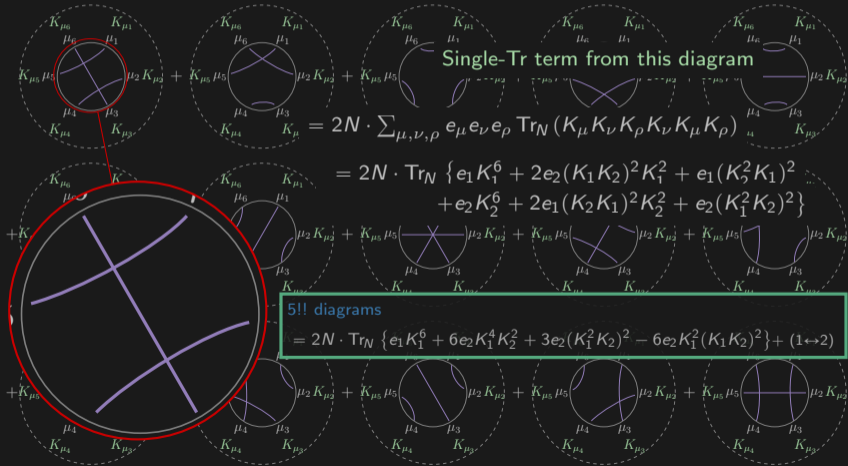
$$\text{Tr}_H(D^6) = 2N \sum_{\mu} \overbrace{\text{Tr}_S(\gamma^{\mu_1} \dots \gamma^{\mu_6})}^{\text{solid circ.}} \times$$

$$\begin{aligned}
 & \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} \\
 & + (-1)^2 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_6} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_4} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_5} + (-1)^1 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_4} \eta^{\mu_3 \mu_5} + (-1)^0 \eta^{\mu_6 \mu_1} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_4} \\
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 & + (-1)^1 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_6} \eta^{\mu_4 \mu_5} + (-1)^0 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} \eta^{\mu_5 \mu_6} + (-1)^3 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_5} \eta^{\mu_3 \mu_6} + (-1)^2 \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_6} \eta^{\mu_3 \mu_5} + (-1)^1 \eta^{\mu_1 \mu_5} \eta^{\mu_2 \mu_3} \eta^{\mu_4 \mu_6} \\
 & + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} + \begin{array}{c} \mu_6 \quad \mu_1 \\ \circ \\ \mu_5 \quad \mu_2 \\ \mu_4 \quad \mu_3 \end{array} \\
 & + (-1)^0 \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} \eta^{\mu_5 \mu_6} + (-1)^1 \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_5} \eta^{\mu_4 \mu_6} + (-1)^0 \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_6} \eta^{\mu_4 \mu_5} + (-1)^1 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} \eta^{\mu_5 \mu_6} + (-1)^2 \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_5} \eta^{\mu_4 \mu_6}
 \end{aligned}$$

$$\text{Tr}_H(D^6) = 2N \sum_{\mu} \overbrace{\text{Tr}_S(\gamma^{\mu_1} \cdots \gamma^{\mu_6})}^{\text{solid circ.}} \times \overbrace{\text{Tr}_N(K_{\mu_1} \cdots K_{\mu_6})}^{\text{dashed circ.}} + \overbrace{\text{Tr}_N P \times \text{Tr}_N Q \text{ terms}}^{(1,5), (2,4), (3,3)\text{-partitions}}$$



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# Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\begin{aligned} \mathcal{Z} &= \int_{\text{DIRAC}} e^{-\text{Tr}_H f(D)} dD \quad (\hbar = 1) \\ &= \int_{M_{p,q}} e^{-N \text{Tr}_N P - \text{Tr}_N^{\otimes 2} (Q_{(1)} \otimes Q_{(2)})} d\mathbb{X}_{\text{LEB}} \end{aligned}$$

- $\mathbb{X} \in M_{p,q}$  = products of  $\mathfrak{su}(N)$  and  $\mathcal{H}_N$
- $d\mathbb{X}_{\text{LEB}}$  is the Lebesgue measure on  $M_{p,q}$
- $P, Q_{(i)}$  in  $\mathbb{C}\langle k \rangle = \mathbb{C}\langle X_1, \dots, X_k \rangle = \mathbb{C}\langle \mathbb{X} \rangle$  are certain noncommutative polynomials
- $\mathcal{Z}_{\text{FORMAL}}$  leads to colored ribbon graphs

$$\bar{g}_1 \text{Tr}_N (ABBBAB) \leftrightarrow \text{Diagram 1}$$

$$\bar{g}_2 \text{Tr}_N^{\otimes 2} (AABABA \otimes AA) \leftrightarrow \text{Diagram 2}$$



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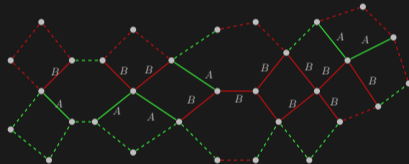
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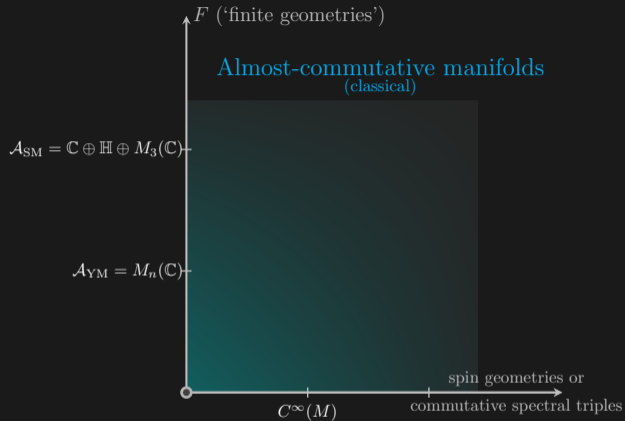
- Multitrace: 'touching interactions' [Klebanov, *PRD* '95], wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01], 'stuffed maps' [G. Borot *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* '14], AdS/CFT [Witten, *hep-th/0112258*]
- Ribbon graphs: Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78]. The ones we get are 'face-worded'



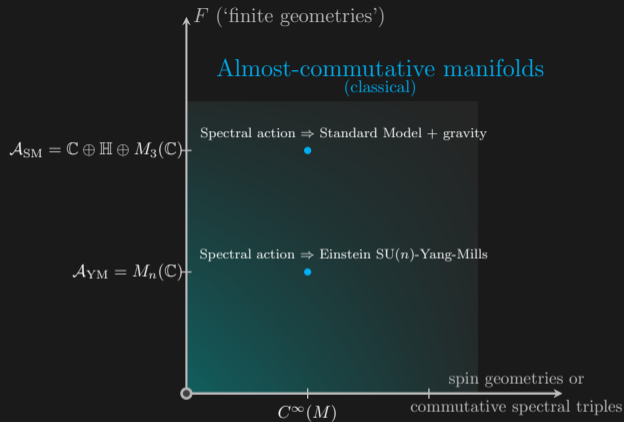
& intersection numbers of  $\psi$ -classes [Kontsevich, *CMP*, '92]

$$\begin{aligned} & \sum_{a_1 + \dots + a_n = \dim_{\mathbb{C}} \bar{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^n \frac{(2a_j - 1)!!}{s_j^{2a_j+1}} \\ &= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\text{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}} \end{aligned}$$

### III. YANG-MILLS-HIGGS MATRIX THEORY



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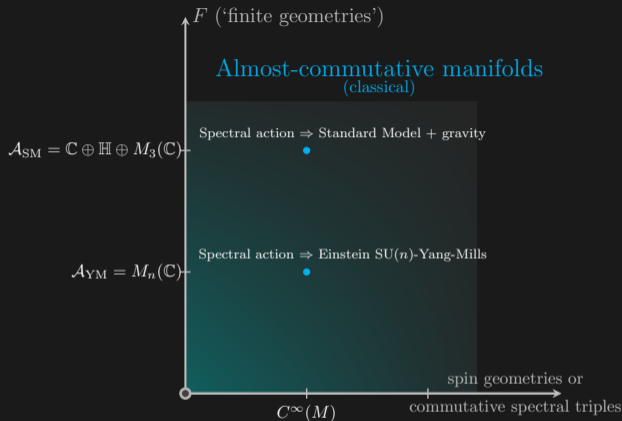


$$\mathcal{Z}_{\text{AC}} \stackrel{?}{=} \int_{\text{DIRAC}} e^{-\frac{1}{\hbar} \text{Tr} f(D)} dD$$

(hard for almost-commutative manifolds)



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(hard for almost-commutative manifolds)

**DEFINITION** [CP 2105.01025] We define a *gauge matrix spectral triple*  $G_f \times F$  as the spectral triple product of a fuzzy geometry  $G_f$  with a finite geometry  $F = (A_F, H_F, D_F)$ ,  $\dim A_F < \infty$ .

LEMMA-DEFINITION [CP 2105.01025] Consider a gauge matrix spectral triple  $G_f \times F$  with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and  $G_f$  Riemannian ( $d = 4$ ) fuzzy geometry on the algebra  $M_N(\mathbb{C})$ , whose **fluctuated** Dirac op. is

$$D_\omega = \sum_{\mu=0}^3 \overbrace{\gamma^\mu \otimes (\ell_\mu + a_\mu) + \gamma^{\hat{\mu}} \otimes (x_\mu + s_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}},$$

The **field strength** is given by

$$\mathcal{F}_{\mu\nu} := [\underbrace{\ell_\mu + a_\mu}_{d_\mu}, \ell_\nu + a_\nu] =: [\mathbf{F}_{\mu\nu}, \cdot]$$

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**THEOREM** For a Yang-Mills–Higgs matrix spectral triple on a 4-dimensional flat ( $x = 0 = s$ ) Riemannian ( $p = 0$ ) fuzzy base, the Spectral Action for a real polynomial  $f(x) = \frac{1}{2} \sum_{i=1}^4 a_i x^i$  reads

$$\frac{1}{4} \text{Tr}_H f(D) = S_{\text{YM}}^\ell + S_{\text{H}}^\ell + S_{\text{g-H}}^\ell + S_\vartheta^\ell,$$

where each sector is defined as follows:

$$S_{\text{YM}}^\ell(\ell, a) := -\frac{a_4}{4} \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}),$$

$$S_{\text{g-H}}^\ell(\ell, a, \Phi) := -a_4 \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}}(\mathcal{D}_\mu \Phi \mathcal{D}^\mu \Phi),$$

$$S_{\text{H}}^\ell(\Phi) := \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}} f_e(\Phi),$$

$$S_\vartheta^\ell(\ell, a) := \text{Tr}_{M_{N \otimes n}^{\mathbb{C}}} f_e(\vartheta^{1/2}). \quad \vartheta = d_\mu d^\mu$$

Moreover, one obtains positivity for each of the following functionals, independently:

$$S_\vartheta^\ell, S_{\text{YM}}^\ell, S_{\text{H}}^\ell \geq 0, \quad \text{if } a_4 \geq 0.$$

## MEANING

RANDOM MATRIX CASE, FLAT  $d = 4$  RIEM.

## SMOOTH OPERATOR

Tr = TRACE OF OPS.  $M_N \otimes M_n \rightarrow M_N \otimes M_n$

Derivation

$$\ell_\mu = [L_\mu \otimes \mathbf{1}_n, \cdot]$$

 $\partial_i$ 

Gauge potential

$$a_\mu = [A_\mu, \cdot]$$

 $\mathbb{A}_i$ 

Higgs field

 $\Phi$  $h$ 

Covariant Derivative

$$d_\mu = \ell_\mu + a_\mu$$

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$$[d_\mu, d_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\neq 0} + [\ell_\mu, a_\nu] - [\ell_\nu, a_\mu] + [a_\mu, a_\nu]$$

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Higgs potential

$$\text{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$$

$$\int_M (f_2 |h|^2 + f_4 |h|^4) \text{vol}$$

Gauge-Higgs coupling

$$- \text{Tr}(d_\mu \Phi d^\mu \Phi)$$

$$- \int_M |\mathbb{D}_i h|^2 \text{vol}$$

Yang-Mills action

$$-\frac{1}{4} \text{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu})$$

$$-\frac{1}{4} \int_M \text{Tr}_{\text{su}(n)}(\mathbb{F}_{ij} \mathbb{F}^{ij}) \text{vol}$$

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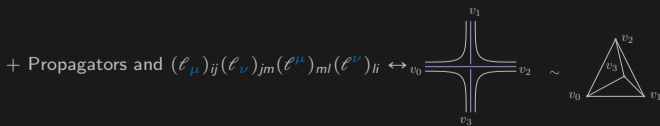
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## Motivation from '2D-Quantum Gravity'

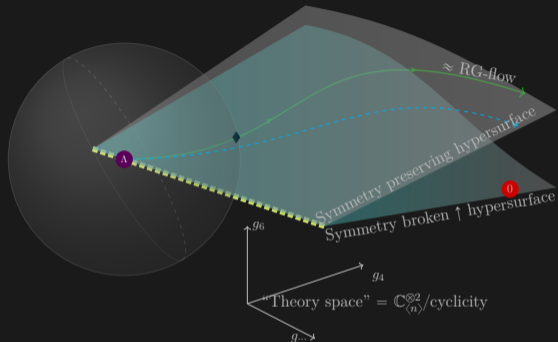
discrete surfaces	↔	matrix integrals $\mathcal{Z}(\lambda)$ <small>[B. Eynard, <i>Counting Surfaces</i> '16]</small>
smooth surface	↔	$\langle \text{area} \rangle$ finite & infinitesimal mesh $a$ $\langle \text{area} \rangle_g \sim \frac{a^2(2-2g)}{\lambda/\lambda_c - 1}$
all topologies	↔	$\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \underbrace{\mathcal{Z}_g(\lambda)}_{(\lambda_c - \lambda)^{(2-2g)/\theta}}$
↑		
double-scaling limit		$N(\lambda_c - \lambda)^{1/\theta} = C$
lin. RG-flow near a fixed point	↔	$\lambda(N) = \lambda_c + (N/C)^{-\theta}$ $\theta = -(\partial\beta/\partial\lambda) _{\lambda_c}$ <small>[Eichhorn-Kosłowski, PRD, '13]</small>



# IV. FRG FOR MULTIMATRIX MODELS WITH MULTITRACES

## Motivation from '2D-Quantum Gravity'

discrete surfaces	$\leftrightarrow$	matrix integrals $\mathcal{Z}(\lambda)$ [B. Eynard, <i>Counting Surfaces</i> '16]
smooth surface	$\leftrightarrow$	$\langle \text{area} \rangle$ finite & infinitesimal mesh $a$ $\langle \text{area} \rangle_g \sim \frac{a^2(2-2g)}{\lambda/\lambda_c - 1}$
all topologies	$\leftrightarrow$	$\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \mathcal{Z}_g(\lambda)$ $(\lambda_c - \lambda)^{(2-2g)/\theta}$
$\uparrow$		
double-scaling limit		$N(\lambda_c - \lambda)^{1/\theta} = C$
lin. RG-flow near a fixed point	$\leftrightarrow$	$\lambda(N) = \lambda_c + (N/C)^{-\theta}$ $\theta = -(\partial\beta/\partial\lambda) _{\lambda_c}$ [Eichhorn-Koslowski, PRD, '13]



- Chosen bare action  $S = \Gamma_{N=\Lambda}$
- Full effective action  $\Gamma = \Gamma_{N=0}$
- ◆ Interpolating action  $\Gamma_{N=\Lambda-\rho}$  (projected & truncated)
- RG-flow with truncation and projection
- ⋯ Moduli of Dirac operators  $\leftrightarrow$  theory space
- - - → RG-flow without truncation nor projection
- $g_{\dots}$  Rest of coupling constants

# Two approaches

## 1. Mathematical construction:

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE,  
“ $\dot{\Gamma} = \frac{1}{2} \text{STr} \{ \dot{R}_N / [\Gamma^{(2)} + R_N] \}$ ”
- its proof determines the algebra that governs geometric series in the Hessian of  $\Gamma$  (this expansion is originally from [D. Benedetti, K. Groh, P. F. Machado and F. Saueressig, *JHEP* 2011])
- to determine the scalings of the couplings with  $Z$  and  $N$  we use [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13], but the proof of the FRGE dictates an algebra not reported there
- fixed-point solution to  $\beta$ -equations for a sextic truncation (48 running operators)

- for the unique real solution  $g^*$  leading to a single relevant direction (positive e.v. of  $-(\partial\beta_i/\partial g_j)_{i,j}|_{g^*}$ ) yields an  $R_N$ -dependent

$$g_{A^4}^* = 1.002 \times (g_{A^4}^* |_{[\text{Kazakov-Zinn-Justin, Nucl. Phys. B '99}]})$$

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## 2. Pragmatic approach: [CP 2111.02858]

- write down Wetterich Equation (*Generatio Sponanea*)
- assume an expansion of its rhs in unitary-invariant operators ( $\neq$  exact RG)
- impose the one-loop structure
- determine from it the ‘algebra of functional renormalization’; it is unique and the one reported in [CP 2007.10914]



# Handwaving Functional Renormalization for $k$ -matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time  $t = \log N$ ,  $1 \ll N < \mathcal{N}$ . E.g. for  $k = 2$  and with bare action

$$S[A, B] = \mathcal{N} \operatorname{Tr}_{\mathcal{N}} \left\{ \frac{1}{2} A^2 + \frac{1}{2} B^2 + g_{A^4} \frac{1}{4} A^4 + g_{B^4} \frac{1}{4} B^4 + \frac{1}{2} g_{ABAB} ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates  $\mathcal{N} \operatorname{Tr}_{\mathcal{N}} (ABBA)$ .

$$\Gamma_N[A, B] = \operatorname{Tr}_N \left\{ \underbrace{\frac{Z_A}{2} A^2 + \frac{Z_B}{2} B^2 + \bar{g}_{A^4} \frac{1}{4} A^4 + \bar{g}_{B^4} \frac{1}{4} B^4 + \frac{1}{2} \bar{g}_{ABAB} ABAB}_{\text{operators from the bare action (but with 'running couplings')}} + \underbrace{\frac{1}{2} \bar{g}_{ABBA} ABBA + \frac{1}{2} \bar{g}_{A|A} \operatorname{Tr}_N (A) \times A + \dots}_{\text{radiative corrections}} \right\}$$

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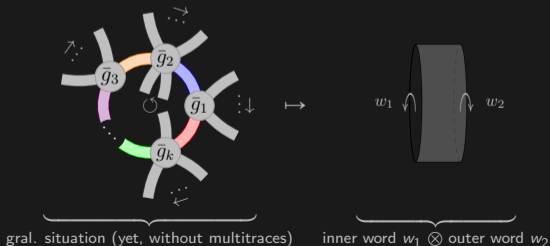
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We are interested in *one-loop graphs*. These are graphs  $G$  whose 1-dim skeleton  $G^\circ = G / \{\text{interaction vertices collapsed to points}\}$  has  $b_1(G^\circ) = 1$ . The *effective vertex*  $O_G^{\text{eff}}$  of such Feynman graph is formed by reading off each word  $w_i$  traveling around all ribbon edges (propagators) by both sides:

$$O_G^{\text{eff}} = \underbrace{\text{Tr}_N(w_1) \times \text{Tr}_N(w_2) \times \dots \times \text{Tr}_N(w_s)}_{\text{from vertices contracted with propagators}} \times \underbrace{\text{Tr}_N(U_1) \times \text{Tr}_N(U_2) \dots \times \text{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$



- *nc-derivative*  $\partial_A : \mathbb{C}\langle k \rangle \rightarrow \mathbb{C}\langle k \rangle^{\otimes 2}$  sums over 'replacements of  $A$  by  $\otimes$ ' [Rota-Sagan-Stein+Voiculescu]:

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$$\partial_B \partial_A \left( \begin{array}{c} A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad B \\ B \quad \bullet \quad A \\ B \quad \bullet \quad A \end{array} \right) \quad \text{go to examples of nc-Hessians } \nabla$$

$$= 1_N \otimes \left( \begin{array}{c} A \quad A \\ B \quad \bullet \quad B \\ B \quad \bullet \quad A \\ B \quad \bullet \quad A \end{array} + \begin{array}{c} A \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad B \\ B \quad \bullet \quad A \end{array} + \begin{array}{c} A \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad B \end{array} \right)$$

$$+ \left( \begin{array}{c} A \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad B \end{array} + \begin{array}{c} A \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad B \\ B \quad \bullet \quad A \end{array} + \begin{array}{c} A \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad A \\ B \quad \bullet \quad B \end{array} \right) \otimes 1_N + \dots$$

in ellipsis  $\sum_{\text{cuts}} \begin{array}{c} A \quad A \\ B \quad \bullet \quad B \\ B \quad \bullet \quad A \\ B \quad \bullet \quad A \end{array} \rightarrow BAA \otimes ABB$



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- products of traces  $\Rightarrow$  extend by  $\boxtimes$ ,  $\mathcal{A}_k = (\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle}) \oplus (\mathbb{C}_{\langle k \rangle} \boxtimes \mathbb{C}_{\langle k \rangle})$



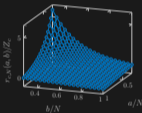
$$\text{Hess}_{a,b}(\text{Tr} P \cdot \text{Tr} Q)$$

$$= \text{Tr} P \cdot \text{Hess}_{a,b}[\text{Tr} Q] + (\partial_{X_a} \text{Tr} P) \boxtimes (\partial_{X_b} \text{Tr} Q) + (P \leftrightarrow Q)$$

- Wetterich Eq. governs the functional RG

$$t = \log N$$

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2} \text{STr} \left\{ \frac{\partial_t R_N}{\text{Hess} \Gamma_N[\mathbb{X}] + R_N} \right\}$$



$$\stackrel{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \dots, \eta_n)$$

$$\times \underbrace{\frac{1}{2} (-1)^k \text{STr} \{ (\text{Hess} \Gamma_N^{\text{INT}}[\mathbb{X}])^{*k} \}}_{\text{regulator-independent part}}$$

- $\text{STr} = \text{Tr}_k \otimes \text{Tr}_{\mathcal{A}_n}$ . Tadpoles

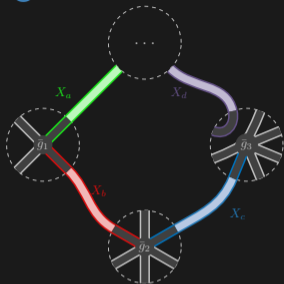


imply

$$\text{Tr}_{\mathcal{A}_n}(P \otimes Q) = \text{Tr}_N P \cdot \text{Tr}_N Q, \text{Tr}_{\mathcal{A}_n}(P \boxtimes Q) = \text{Tr}_N(PQ)$$

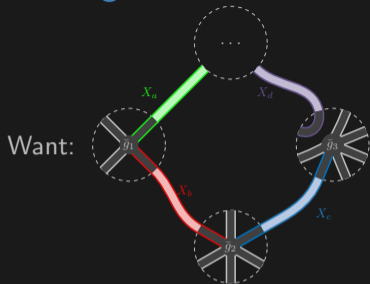
# Finding $\star$

Want:



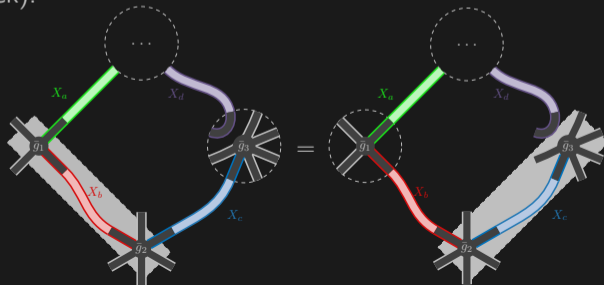
$$\subset \text{Hess}_{a,b} O_1 \star \text{Hess}_{b,c} O_2 \star \text{Hess}_{c,d} O_3 \star \dots \star \text{Hess}_{*,a} O_\ell$$

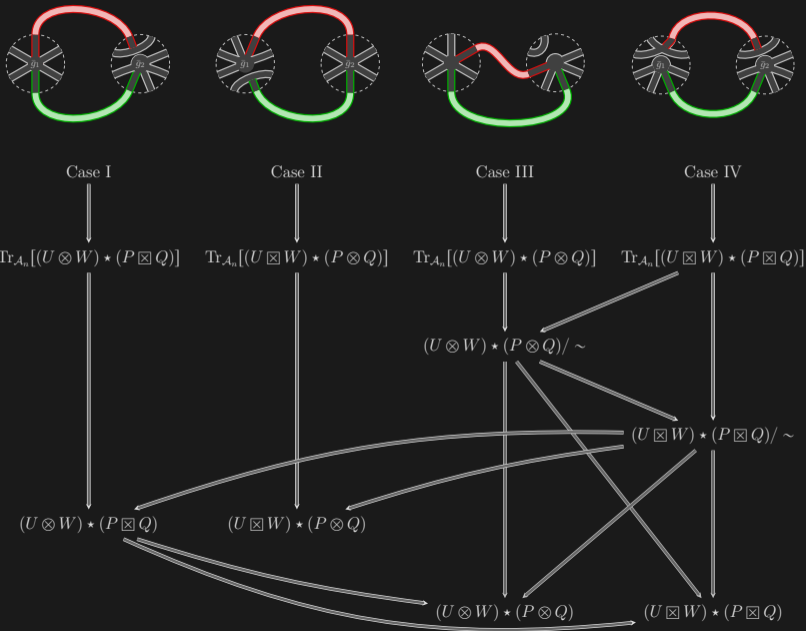
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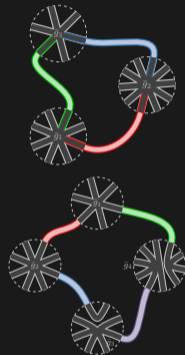
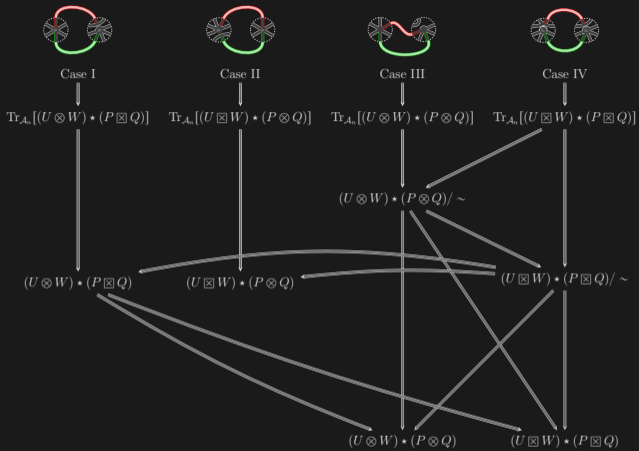
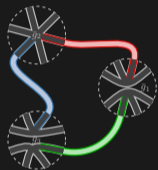


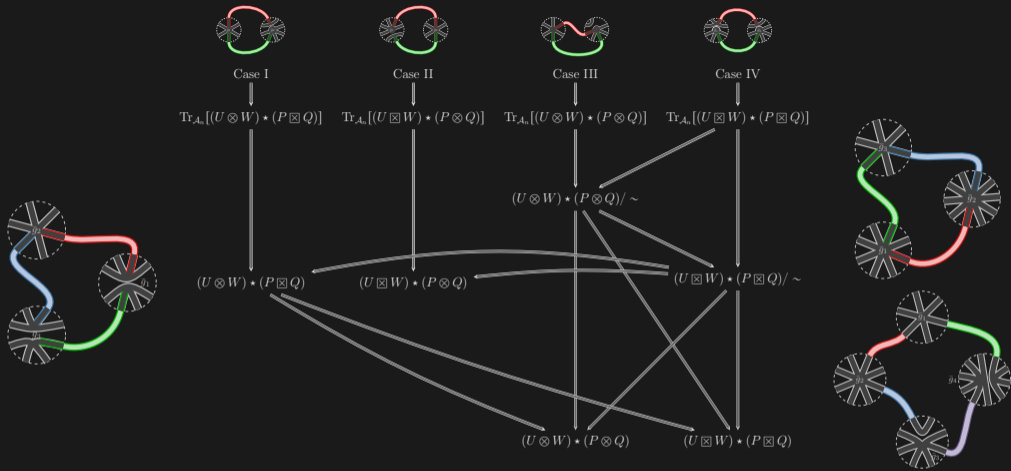
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Associativity (trivial check):









**THM.** [CP 2111.02858] If the RG-flow is computable in terms of  $U(N)$ -invariants, the algebra of Functional Renormalization is  $M_k(\mathcal{A}_{N,k}, \star)$  where  $\mathcal{A}_{N,k} = (\mathbb{C}_{\langle k \rangle}^{(N)} \otimes \mathbb{C}_{\langle k \rangle}^{(N)}) \oplus (\mathbb{C}_{\langle k \rangle}^{(N)} \boxtimes \mathbb{C}_{\langle k \rangle}^{(N)})$  whose product in homogeneous elements reads:

$$\begin{aligned}
 (U \otimes W) \star (P \otimes Q) &= PU \otimes WQ, \\
 (U \boxtimes W) \star (P \otimes Q) &= U \boxtimes PWQ, \\
 (U \otimes W) \star (P \boxtimes Q) &= WPU \boxtimes Q, \\
 (U \boxtimes W) \star (P \boxtimes Q) &= \text{Tr}(WP)U \boxtimes Q.
 \end{aligned}$$

## Example: Hermitian 'free' 3-matrix model

Consider two operators  $O_1 = \frac{\bar{g}_1}{2} [\text{Tr}_N (\frac{A^2}{2})]^2$  and  $O_2 = \bar{g}_2 \text{Tr}_N (ABC)$ . We compute  $g_1 g_2^2$ -coefficients:

$$\text{Hess}_{I,J} O_1 = \delta_I^J \delta_I^A \bar{g}_1 \left\{ \underbrace{\text{Tr}_N (A^2/2) \cdot [1_N \otimes 1_N]}_{\text{filled ribbon}} + \underbrace{A \boxtimes A}_{\text{white ribbon}} \right\},$$

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$$\text{Hess } O_2 = \bar{g}_2 \begin{bmatrix} 0 & C \otimes 1_N & B \otimes 1_N \\ 1_N \otimes C & 0 & A \otimes 1_N \\ 1_N \otimes B & 1_N \otimes A & 0 \end{bmatrix} \Rightarrow [\text{Hess } O_2]^{*2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C \otimes C + B \otimes B}^{\text{diagrams}} & B \otimes A & C \otimes A \\ A \otimes A + C \otimes C & A \otimes B & C \otimes B \\ B \otimes C & A \otimes C & B \otimes B + A \otimes A \end{bmatrix}.$$



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Extracting coefficients

$$[\bar{g}_1 \bar{g}_2] \text{STr} \{ \text{Hess } O_1 \star [\text{Hess } O_2]^{\star 2} \} = \text{Tr}_N (A^2/2) \times [(\text{Tr}_N C)^2 + (\text{Tr}_N B)^2] + \text{Tr}_N (ACAC + ABAB),$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of  $\left\{ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right\}$  with any of  $\left\{ \begin{array}{c} \text{---} \\ \times \\ \text{---} \end{array} \right\}$ . This is a toy example in [CP '21]; in [CP '20] 48 such operators run  $\Rightarrow$  (48 less friendly Hessians)<sup>3</sup>. [go to nc-Hessians examples >](#)

## CONCLUSION

- spectral triple  $\equiv$  spin without commutativity of the ‘algebra of functions’
- spin  $M \times \{\text{finite spectral triple}\} \equiv$  almost-commutative  
(reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry  $\approx$  finite spectral triple +  $\mathbb{C}\ell$ -action  
( $\mathcal{Z}_{\text{FUZZY}} = \int_{\text{DIRAC}} e^{-\frac{1}{\hbar} \text{Tr}_H f(D)} dD$  is a multimatrix model with multitraces)
- $G_\ell \times F =$  fuzzy  $\times$  finite = gauge matrix spectra triple  
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thank you!

## THE CLASSICAL DIRAC OPERATOR IN RIEMANNIAN GEOMETRY

For us  $M$  will be a Riemannian closed manifold,  $\dim M = d$ .

In physics,  $M$  is a spacetime (for this first part, still deterministic).

Q: When do Dirac operators exist on  $M$ ?

A: Only if the *obstruction* to a spin-structure  $w_2(M) \in \check{H}^2(M, \mathbb{Z}_2)$  is trivial.

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$\mathbb{Z}_2$ -Čech cohomology in short,

- $U = \{U_i\}_i$  good open cover of  $M$
- $j$ -simplices are  $\sigma = (k_0, \dots, k_j)$  such that  $U_{k_0 k_1 \dots k_j} = U_{k_0} \cap U_{k_1} \cap \dots \cap U_{k_j} \neq \emptyset$
- $j$ -cochains, maps  $f : \{j\text{-simplices}\} \rightarrow \mathbb{Z}_2$  satisfying invariance  $\tau^* f = f$  under  $\tau \in \mathfrak{S}(j+1)$ , form an abelian group  $\check{C}^j(U, \mathbb{Z}_2)$
- coboundary maps  $\delta^j : \check{C}^j(U, \mathbb{Z}_2) \rightarrow \check{C}^{j+1}(U, \mathbb{Z}_2)$  given by

$$(\delta^j f)(k_0, \dots, k_{j+1}) := f(k_1, \dots, k_{j+1})f(k_0, k_1, \dots, k_{j+1}) \cdots f(k_0, \dots, k_j)$$

- $\check{H}^j(U, \mathbb{Z}_2) = \ker \delta^j / \text{im } \delta^{j-1}$

A more familiar  $\mathbb{Z}_2$ -Čech cohomology class is the orientability obstruction

- $\{U_j\}_j$  good open cover of  $M$
- pick  $s_j : U_j \rightarrow F(M)$  sections on the frame bundle  $O(d) \hookrightarrow F(M) \rightarrow M$
- on a 1-simplex  $(k_0, k_1)$ ,  $s_{k_0} = s_{k_1} G_{k_0 k_1}$

$$\begin{aligned} f(k_0, k_1) &= \det(G_{k_0, k_1}) \\ &= \det(G_{k_1, k_0}) = f(k_1, k_0) \end{aligned}$$

- since  $\{G_{j,l}\}_{j,l}$  are transition functions,

$$(\delta^1 f)(j, l, m) = G_{j,l} G_{l,m} G_{m,j} = 1$$

- other choice of sections  $s'_k$  yields  $G'_{kl} = g_k G_{kl} g_l^{-1}$  and  $f' = (\delta^0 h)f$  where  $h = \det g_k$
- other choice  $\{V\}_j$  of a good open cover yields a cochain complex map  $\check{C}^*(V, \mathbb{Z}_2) \rightarrow \check{C}^*(U, \mathbb{Z}_2)$ ,

## Low Stiefel-Whitney classes (first two floors of Whitehead tower)

**THEOREM**  $M$  is orientable iff the 1st Stiefel-Whitney class  $w_1(M) := [f] = 1$ .

*Proof of 'if'.* If  $w_1(M) = 1$ ,  $f(k_0, k_1) = \det(G_{k_0, k_1})$  is a 0-coboundary,  $f = \delta^0 h$ . We can pick sections  $\{s_k : U_k \rightarrow F(M)\}_k$  and  $g_k \in O(d)$  with  $\det g_k = h(k)$ , so that the transition functions for  $s'_k := s_k \cdot g_k$  satisfy

$$\begin{aligned} \det(G'_{k_0, k_1}) &= \det(g_{k_0}^{-1} G_{k_0, k_1} g_{k_1}) \\ &= [(\delta^0 h) \cdot f](k_0, k_1) = 1. \quad \square \end{aligned}$$

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In similar way, a *spin structure*  $\lambda$

$$\begin{array}{ccc} \text{Spin}(d) \times P(M) & \longrightarrow & P(M) \\ \downarrow & & \downarrow \lambda \\ \text{SO}(d) \times F_{\text{SO}}(M) & \longrightarrow & F_{\text{SO}}(M) \end{array} \begin{array}{c} \nearrow \\ \nearrow \\ \nearrow \end{array} M$$

exists when the SO-frame bundle can be lifted in compatible way with the double cover  $\mathbb{Z}_2 \rightarrow \text{Spin}(d) \xrightarrow{\rho} \text{SO}(d)$ . Transition functions  $g_{ij} : U_{ij} \rightarrow \text{SO}(d)$  can be lifted to Spin( $d$ )-valued  $\tilde{g}_{ij}$ . For  $U_{ijk} \neq \emptyset$ , let

$$\tilde{g}_{ij} \tilde{g}_{jk} \tilde{g}_{ki} =: z(i, j, k) \text{id}_{\text{Spin}(d)}$$

**THEOREM** [A. Haefliger, '56] Orientable  $M$  is spin iff its second Stiefel-Whitney class  $w_2(M) := [z] = 1$ .



# Classical Dirac operators (assume $d$ even)

- $M$  (spacetime) will be a closed, Riemannian manifold
- if  $M$  is spin, there is a vector bundle  $\mathbb{S}$  with fibers satisfying  $\text{End}(\mathbb{S}_x) \cong \mathbb{C}\ell(d)$  ( $x \in M$ ). The sections  $\Gamma(\mathbb{S})$  are spinors
- the Levi-Civita connection  $\nabla^{\text{LC}}$  can be also lifted to the *spin connection*  $\nabla^s : \Gamma(\mathbb{S}) \rightarrow \Omega^1(M) \otimes \Gamma(\mathbb{S})$

$$\nabla^s c(\omega)\psi = c(\nabla^{\text{LC}}\omega)\psi + c(\omega)\nabla^s\psi$$
$$\psi \in \Gamma(\mathbb{S}), \omega \in \Omega^1(M)$$

being  $c$  Clifford multiplication, basically  $c(dx^\mu) = \gamma^\mu$

- on the space of square integrable spinors  $L^2(M, \mathbb{S})$  there is an (ess.) self-adjoint operator, the *Dirac operator*,

$$D_M = -ic \circ \nabla^s \stackrel{\text{loc.}}{=} -i \sum_{\mu=1}^d \gamma^\mu (\partial_\mu + \omega_\mu)$$

and by Leibniz rule

$$[D_M, a] = -ic(da) \quad a \in C^\infty(M)$$

which is bounded

back to 'spectral triples'  $\llcorner$

# Matrix or Fuzzy Geometries

DEFINITION ("condensed" from [J. Barrett, *J. Math. Phys.* 2015]).

A *fuzzy geometry* of *signature*  $(p, q) \in \mathbb{Z}_{\geq 0}^2$  is given by

- a simple matrix algebra  $\mathcal{A}$  – we take always  $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian  $\mathcal{Cl}(p, q)$ -module  $\mathbb{S}$  with a *chirality*  $\gamma$ . That is a linear map  $\gamma : \mathbb{S} \rightarrow \mathbb{S}$  satisfying  $\gamma^* = \gamma$  and  $\gamma^2 = 1$
- a Hilbert space  $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$  with inner product  $\langle v \otimes R, w \otimes S \rangle = (v, w) \text{Tr}_N(R^* S)$  for each  $R, S \in M_N(\mathbb{C})$ , being  $(\cdot, \cdot)$  the inner product of  $\mathbb{S}$
- a left- $\mathcal{A}$  representation  $\rho(a)(v \otimes R) = v \otimes (aR)$  on  $\mathcal{H}$ ,  $a \in \mathcal{A}$  and  $v \otimes R \in \mathcal{H}$

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- three signs  $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$  determined through  $s := q - p$  by the following table:

$s \equiv q - p \pmod{8}$	0	1	2	3	4	5	6	7
$\epsilon$	+	+	-	-	-	-	+	+
$\epsilon'$	+	-	+	+	+	-	+	+
$\epsilon''$	+	+	-	+	+	+	-	+

- a real structure  $J = C \otimes *$ , where  $*$  is complex conjugation and  $C$  is an anti-unitarity on  $\mathbb{S}$  satisfying  $C^2 = \epsilon$  and  $C\gamma^\mu = \epsilon'\gamma^\mu C$  for all the gamma matrices  $\mu = 1, \dots, p + q$ .

- a self-adjoint operator  $D$  on  $\mathcal{H}$  satisfying the *order-one condition*

$$[[D, \rho(a)], J\rho(b)J^{-1}] = 0 \quad \text{for all } a, b \in \mathcal{A}$$

- a chirality  $\Gamma = \gamma \otimes 1_{\mathcal{A}}$  for  $\mathcal{H}$ , where  $\gamma$  is the chirality of  $\mathbb{S}$ . The signs above impose:

$$J^2 = \epsilon, \quad JD = \epsilon'DJ, \quad J\Gamma = \epsilon''\Gamma J.$$

Two-matrix model from  $\text{Tr}_H D^6$ ,  $\eta = \text{diag}(e_1, e_2)$ ,  $K_i^* = e_i K_i$  [CP '19]

$$\begin{aligned} \mathcal{S}_6[K_1, K_2] = 2 \cdot \text{Tr}_N \{ & e_1 K_1^6 + 6e_2 K_1^4 K_2^2 - 6e_2 K_1^2 (K_1 K_2)^2 + 3e_2 (K_1^2 K_2)^2 \\ & + e_2 K_2^6 + 6e_1 K_2^4 K_1^2 - 6e_1 K_2^2 (K_2 K_1)^2 + 3e_1 (K_2^2 K_1)^2 \} \end{aligned}$$

# Two-matrix model from $\text{Tr}_H D^6$ , $\eta = \text{diag}(e_1, e_2)$ , $K_i^* = e_i K_i$ [CP '19]

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and the double-trace part is

$$\mathcal{B}_6[K_1, K_2] = 6 \text{Tr}_N (K_1) \left\{ 2 \text{Tr}_N (K_1^5) + 2 \text{Tr}_N (K_1 K_2^4) + 6e_1 e_2 \text{Tr}_N (K_1^3 K_2^2) - 2e_1 e_2 \text{Tr}_N (K_1^2 K_2 K_1 K_2) \right\} \\ + 6 \text{Tr}_N (K_2) \left\{ 2 \text{Tr}_N (K_2^5) + 2 \text{Tr}_N (K_2 K_1^4) + 6e_1 e_2 \text{Tr}_N (K_2^3 K_1^2) - 2e_1 e_2 \text{Tr}_N (K_2^2 K_1 K_2 K_1) \right\} \\ + 48 \text{Tr}_N (K_1 K_2) \cdot [e_1 \text{Tr}_N (K_1^3 K_2) + e_2 \text{Tr}_N (K_2^3 K_1)] \\ + 6 \text{Tr}_N (K_1^2) \cdot \left\{ e_2 [8 \text{Tr}_N (K_1^2 K_2^2) - 2 \text{Tr}_N (K_2 K_1 K_2 K_1)] + e_1 [5 \text{Tr}_N (K_1^4) + \text{Tr}_N (K_2^4)] \right\} \\ + 6 \text{Tr}_N (K_2^2) \cdot \left\{ e_1 [8 \text{Tr}_N (K_1^2 K_2^2) - 2 \text{Tr}_N (K_1 K_2 K_1 K_2)] + e_2 [5 \text{Tr}_N (K_2^4) + \text{Tr}_N (K_1^4)] \right\} \\ + 4(5[\text{Tr}_N (K_1^3)]^2 + 6e_1 e_2 \text{Tr}_N (K_1 K_2^2) \text{Tr}_N (K_1^3) + 9[\text{Tr}_N K_1^2 K_2]^2 \\ + 5[\text{Tr}_N (K_2^3)]^2 + 6e_1 e_2 \text{Tr}_N (K_1^2 K_2) \text{Tr}_N (K_2^3) + 9[\text{Tr}_N K_1 K_2^2]^2).$$

## Sketch of the Standard Model derivation from NCG

One starts with the  $M \times_{s.t.} F$  and  $\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$

- $F = (\mathcal{A}_{LR}, \mathcal{M}_F^{\#\text{generations}}, D_F)$ ,  $\mathcal{M}_F$  an  $\mathcal{A}_{LR}$ -module
- $\mathcal{M}_F$  has to be of the form  $\mathcal{M}_F = \mathcal{E} \otimes \mathcal{E}^\circ$ , with

$$\mathcal{E} = (2_L \otimes 1^\circ) \oplus (2_R \otimes 1^\circ) \oplus (2_L \otimes 3^\circ) \oplus (2_L \otimes 3^\circ), \quad \dim_{\mathbb{C}} \mathcal{E} = 16$$

- Thus the  $\mathcal{H}_F \cong \mathbb{C}^{32 \times 3}$ . The  $96 \times 96$  matrix  $D_F$  can have off-diagonal elements only for the maximal subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

- Lie group part of  $SU(\mathcal{A}_F) = U(1) \times SU(2) \times SU(3)$

# Sketch of the Standard Model derivation from NCG

With  $Q : \mathbb{C} \hookrightarrow \mathbb{H}$ ,  $Q_\lambda = \text{diag}(\lambda, \bar{\lambda})$  and  $Q_\lambda |\pm\rangle = \pm\lambda |\pm\rangle$ ,

• Weak hypercharge:

$Y$	$\nu$	$e$	$u$	$d$
	$ +\rangle \otimes 1^\circ$	$ -\rangle \otimes 1^\circ$	$ +\rangle \otimes 3^\circ$	$ -\rangle \otimes 3^\circ$
$L$	$-1$	$-1$	$+1/3$	$+1/3$
$R$	$0$	$-2$	$+4/3$	$-2/3$

- $SU(2)$ -adjoint action is 2 on  $\mathcal{H}_L$  or trivial in the  $\mathcal{H}_R$  sector
- $SU(3)$ -adjoint action is the color action on  $\mathcal{H}_q$  and trivial on  $\mathcal{H}_\ell$

$$\text{Lie}(SU(\mathcal{A}_F)) = U(1)_Y \times SU(2)_L \times SU(3)_{\text{color}}$$

- All  $D_F$  such that  $(\mathcal{A}_F, \mathcal{H}_F, D_F)$  is a spectral triple are

$$D_F(\Upsilon_R, \Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d)$$

the moduli of such Dirac operators has dimension  $31 = \text{num. Yukawa couplings in } \nu\text{MSM}$ .







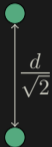
• **Example 1: (Finite spectral triples)** [Exercise from W. van Suijlekom's book]

-  $\mathcal{A} = \mathbb{C}^3$

-  $\mathcal{H} = \mathbb{C}^3 \hookrightarrow \mathcal{A}$ , in defining representation

-  $D = \begin{pmatrix} 0 & 1/d & 0 \\ 1/d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (0 \neq d \in \mathbb{R})$

looks like



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for  $d_{ij} = \sup_{a \in \mathcal{A}} \{|a(i) - a(j)| : \|[D, a]\| \leq 1\}$

$= \sup_{a \in \mathcal{A}} \{|a(i) - a(j)| : |a(1) - a(2)|^2 \leq d^2\} \quad \forall i, j = 1, 2, 3$

• **Example 2: (Finite spectral triples)**

-  $\mathcal{A} = M_3(\mathbb{C})$

-  $\mathcal{H} = \mathbb{C}^3$

-  $D = 0$

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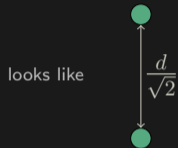
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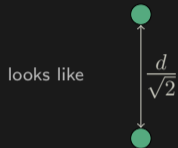
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$$\begin{aligned} \text{for } d_{ij} &= \sup_{a \in \mathcal{A}} \{|a(i) - a(j)| : \|[D, a]\| \leq 1\} \\ &= \sup_{a \in \mathcal{A}} \{|a(i) - a(j)| : |a(1) - a(2)|^2 \leq d^2\} \quad \forall i, j = 1, 2, 3 \end{aligned}$$

• **Example 2: (Finite spectral triples)**

- $\mathcal{A} = M_3(\mathbb{C}) = \mathbb{C}[\mathcal{G}_\sim]$
- $\mathcal{H} = \mathbb{C}^3$
- $D = 0$

$$\mathbf{3} = \{1, 2, 3\} / \sim$$

# Chord Diagrams of 10 points (1/4)

(appear in  $D^{10}|_{d=2}, D^4|_{d=4}, D^2|_{d=6}$  )



# Chord Diagrams of 10 points (2/4)





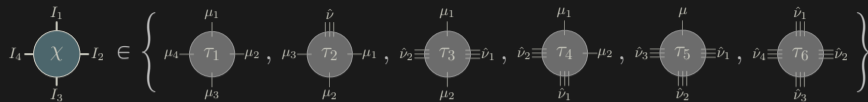
## Chord Diagrams of 10 points (3/4)



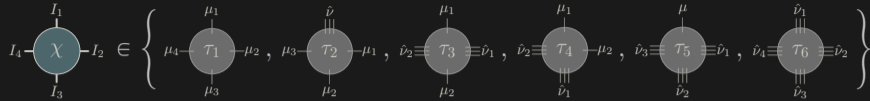
## Chord Diagrams of 10 points (4/4)



Non-vanishing ones...  $\frac{1}{4} \text{Tr} D^4 = NS_4^{\text{Riemann}} + \sum_i \text{Tr}_N^{\otimes 2}(A_i \otimes B_i)$



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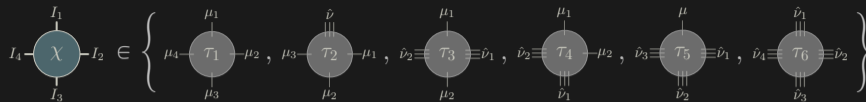


$$\begin{aligned}
 S_4^{\text{Riemann}} = \text{Tr}_N & \left\{ 2 \sum_{\mu} (L_{\mu}^4 + H_{\mu}^4) + 4 \sum_{\mu < \nu} (2L_{\mu}^2 L_{\nu}^2 + 2H_{\mu}^2 H_{\nu}^2 - L_{\mu} L_{\nu} L_{\mu} L_{\nu} - H_{\mu} H_{\nu} H_{\mu} H_{\nu}) \right. \\
 & - \sum_{\mu \neq \nu} [2(L_{\mu} H_{\nu})^2 + 4L_{\mu}^2 H_{\nu}^2] + \sum_{\mu} [2(L_{\mu} H_{\mu})^2 - 4L_{\mu}^2 H_{\mu}^2] \\
 & + 8 \left[ H_1 (L_2 [L_3, L_4] + L_3 [L_4, L_2] + L_4 [L_2, L_3]) \right. \\
 & \quad + H_2 (L_1 [L_3, L_4] + L_3 [L_4, L_1] + L_4 [L_1, L_3]) \\
 & \quad + H_3 (L_1 [L_2, L_4] + L_2 [L_4, L_1] + L_4 [L_1, L_2]) \\
 & \quad \left. + H_4 (L_1 [L_2, L_3] + L_2 [L_3, L_1] + L_3 [L_1, L_2]) \right] + 8 [H \leftrightarrow L] \left. \right\}
 \end{aligned}$$

[C.P. '19]

$L_{\mu}, H_{\mu}$  are random matrices!

Non-vanishing ones...  $\frac{1}{4} \text{Tr} D^4 = N S_4^{\text{Riemann}} + \sum_i \text{Tr}_N \otimes^2 (A_i \otimes B_i)$



$$\begin{aligned}
 S_4^{\text{Riemann}} = \text{Tr}_N \bigg\{ & 2 \sum_{\mu} (L_{\mu}^4 + H_{\mu}^4) + 4 \sum_{\mu < \nu} (2L_{\mu}^2 L_{\nu}^2 + 2H_{\mu}^2 H_{\nu}^2 - L_{\mu} L_{\nu} L_{\mu} L_{\nu} - H_{\mu} H_{\nu} H_{\mu} H_{\nu}) \\
 & - \sum_{\mu \neq \nu} [2(L_{\mu} H_{\nu})^2 + 4L_{\mu}^2 H_{\nu}^2] + \sum_{\mu} [2(L_{\mu} H_{\mu})^2 - 4L_{\mu}^2 H_{\mu}^2] \\
 & + 8 \left[ H_1 (L_2 [L_3, L_4] + L_3 [L_4, L_2] + L_4 [L_2, L_3]) \right. \\
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 & \quad \left. + H_4 (L_1 [L_2, L_3] + L_2 [L_3, L_1] + L_3 [L_1, L_2]) \right] + 8 [H \leftrightarrow L] \bigg\}
 \end{aligned}$$

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$L_{\mu}, H_{\mu}$  are random matrices!

- Analogy  $[L_{\mu}, \cdot] \rightarrow \partial_{\mu}$   $\{H_{\mu}, \cdot\} \rightarrow \omega_{\mu}$  [J. Barrett, L. Glaser, *J. Phys. A* 2016]
- Obtained for any signature, also the  $A_i, B_i$  noncommutative-polynomials [C.P. '19]
- Lorentzian signature [L. Glaser, *J. Phys. A* '17]

OPERATOR      ITS NONCOMMUTATIVE HESSIAN

$$\text{Tr}(A) \text{Tr}(A^3) \quad 3 \cdot \begin{pmatrix} \text{Tr} A \cdot (A \otimes 1 + 1 \otimes A) & 0 \\ +1 \boxtimes A^2 + A^2 \boxtimes 1 & \\ 0 & 0 \end{pmatrix}$$

$$\text{Tr}(ABAB) \quad 2 \cdot \begin{pmatrix} B \otimes B & (1 \otimes BA + AB \otimes 1) \\ (1 \otimes AB + BA \otimes 1) & A \otimes A \end{pmatrix}$$

$$\text{Tr} A \text{Tr}(AAABB) \quad \left( \begin{array}{l} (\text{Tr}(A)1 \otimes (ABB) + \text{Tr}(A)1 \otimes (BBA) + \text{Tr}(A)(ABB) \otimes 1 + \text{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \text{Tr}(A)A \otimes B^2 + \text{Tr}(A)B^2 \otimes A) \quad \text{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \text{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 \quad \text{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{array} \right)$$

**Table:** Some Hessians order operators. Here  $\text{Tr} = \text{Tr}_N$ .

OPERATOR      ITS NONCOMMUTATIVE HESSIAN

$$\text{Tr}(A) \text{Tr}(A^3) \quad 3 \cdot \begin{pmatrix} \text{Tr} A \cdot (A \otimes 1 + 1 \otimes A) & 0 \\ +1 \boxtimes A^2 + A^2 \boxtimes 1 & \\ 0 & 0 \end{pmatrix}$$

$$\text{Tr}(ABAB) \quad 2 \cdot \begin{pmatrix} B \otimes B & (1 \otimes BA + AB \otimes 1) \\ (1 \otimes AB + BA \otimes 1) & A \otimes A \end{pmatrix}$$

$$\text{Tr} A \text{Tr}(AAABB) \quad \left( \begin{array}{l} (\text{Tr}(A)1 \otimes (ABB) + \text{Tr}(A)1 \otimes (BBA) + \text{Tr}(A)(ABB) \otimes 1 + \text{Tr}(A)(BBA) \otimes 1 + 1 \boxtimes (AABB) + 1 \boxtimes (ABBA) + 1 \boxtimes (BBAA) + (AABB) \boxtimes 1 + (ABBA) \boxtimes 1 + (BBAA) \boxtimes 1 + \text{Tr}(A)A \otimes B^2 + \text{Tr}(A)B^2 \otimes A) \quad \text{Tr}(A)(1 \otimes (AAB) + (BAA) \otimes 1 + A \otimes AB + B \otimes A^2 + A^2 \otimes B + BA \otimes A) + 1 \boxtimes (A^3 B) + 1 \boxtimes (BA^3) \\ \text{Tr}(A)(1 \otimes (BAA) + (AAB) \otimes 1 + A \otimes BA + B \otimes A^2 + A^2 \otimes B + AB \otimes A) + (A^3 B) \boxtimes 1 + (BA^3) \boxtimes 1 \quad \text{Tr}(A)(1 \otimes (A^3) + (A^3) \otimes 1) \end{array} \right)$$

**Table:** Some Hessians order operators. Here  $\text{Tr} = \text{Tr}_N$ .

# $\beta$ -functions of 'free' two-matrix models, signature $\eta = \text{diag}(e_1, e_2)$

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$

$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$

$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$

$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$h_2(4a_4^2 + 4c_{22}^2) + a_4(2\eta + 1)$$

$$-h_1(24a_6e_a + 4c_{42}e_b + 4d_{02|4}e_b + 4d_{2|4}e_a) = \beta(a_4)$$

$$h_2(4b_4^2 + 4c_{22}^2) + b_4(2\eta + 1)$$

$$-h_1(24b_6e_b + 4c_{24}e_a + 4d_{02|04}e_b + 4d_{2|04}e_a) = \beta(b_4)$$

$$-h_1(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22})$$

$$+ h_2(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2) + c_{22}(2\eta + 1) = \beta(c_{22})$$

$$8e_a e_b c_{1111} c_{22} h_2 + c_{1111}(2\eta + 1)$$

$$+ h_1(4e_a c_{1311} + 4e_b c_{3111} + 2e_a d_{02|1111} + 2e_b d_{2|1111}) = \beta(c_{1111})$$



$$2h_2(6a_4a_6 + e_a e_b c_{22}c_{42}) + a_6(3\eta + 2) = \beta(a_6)$$

$$2h_2(6b_4b_6 + e_a e_b c_{22}c_{24}) + b_6(3\eta + 2) = \beta(b_6)$$

$$4h_2\{a_4c_{3111} + e_a e_b [c_{22}(c_{1311} + 2c_{3111}) - c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) = \beta(c_{3111})$$

$$2h_2[2a_4c_{2121} + e_a e_b (-2c_{1111}c_{3111} + 4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) = \beta(c_{2121})$$

$$2h_2[a_4c_{24} + 3b_4c_{24} + 2e_a e_b (c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) - c_{1111}c_{1311})] + c_{24}(3\eta + 2) = \beta(c_{24})$$

$$4h_2\{b_4c_{1311} + e_a e_b [c_{22}(2c_{1311} + c_{3111}) - c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) = \beta(c_{1311})$$

$$2h_2[2b_4c_{1212} + e_a e_b (c_{22}(4c_{1212} + c_{42}) - 2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) = \beta(c_{1212})$$

$$2h_2[3a_4c_{42} + 2e_a e_b (3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} + c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) = \beta(c_{42})$$