On Dirac ensembles in noncommutative geometry

a gauge theory perspective

CP7-lunch seminar

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I Classical Dirac operators

II Dirac operators in noncommutative geometry

III Random ('quantum') Dirac operators and multi-matrix models

IV Yang-Mills-Higgs random matrix theory

I. CLASSICAL DIRAC OPERATORS IN RIEMANNIAN GEOMETRY

$$\begin{tikzcd} G \arrow[r, hook] & P \arrow[d] \\ & M \end{tikzcd}$$

For us M will be a Riemannian closed manifold, $\dim M = d$.
In physics, M is a spacetime (for this first part, still deterministic).
Q: When do Dirac operators exist on M?
A: Only if the *obstruction* to a spin-structure $w_2(M) \in \check{H}^2(M, \mathbb{Z}_2)$ is trivial.

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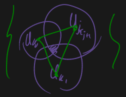
$\mathbb{Z}_2$-Čech cohomology in short,

- $U = \{U_i\}_i$ good open cover of M
- j-simplices are $\sigma = (k_0, \ldots, k_j)$ such that $U_{k_0 k_1 \ldots k_j} = U_{k_0} \cap U_{k_1} \cap \cdots \cap U_{k_j} \neq \varnothing$



$\xrightarrow{f} \mathbb{Z}_2 = \{+,-\}$

- j-cochains, maps $f : \{j\text{-simplices}\} \to \mathbb{Z}_2$ satisfying invariance $\tau^* f = f$ under $\tau \in \mathfrak{S}(j+1)$, form an abelian group $\check{C}^j(U, \mathbb{Z}_2)$
- coboundary maps $\delta^j : \check{C}^j(U, \mathbb{Z}_2) \to \check{C}^{j+1}(U, \mathbb{Z}_2)$ given by

$$(\delta^j f)(k_0, \ldots, k_{j+1}) := f(k_1, \ldots, k_{j+1}) f(k_0, k_1 \ldots, k_{j+1}) \cdots f(k_0, \ldots, k_j)$$

- $\check{H}^j(U, \mathbb{Z}_2) = \ker \delta^j / \mathrm{im}\, \delta^{j-1}$

A more familiar $\mathbb{Z}_2$-Čech cohomology class is the orientability obstruction

- $\{U_j\}_j$ good open cover of M
- pick $s_j : U_j \to F(M)$ sections on the frame bundle $\mathrm{O}(d) \hookrightarrow F(M) \to M$
- on a 1-simplex (k_0, k_1), $s_{k_0} = s_{k_1} G_{k_0 k_1}$

$$\begin{aligned} f(k_0, k_1) &= \det(G_{k_0,k_1}) \\ &= \det(G_{k_1,k_0}) = f(k_1, k_0) \end{aligned}$$

- since $\{G_{j,l}\}_{j.l}$ are transition functions,

$$(\delta^1 f)(j, l, m) = G_{j,l} G_{l,m} G_{m,j} = 1 \quad \Rightarrow \quad [f] \in \check{H}^1(\mathcal{U}, \mathbb{Z}_2)$$

- other choice of sections s'_k yields $G'_{kl} = g_k G_{kl} g_l^{-1}$ and $f' = (\delta^0 h) f$ where $h = \det g_k$
- other choice $\{V\}_j$ of a good open cover yields a cochain complex map $\check{C}*(V, \mathbb{Z}_2) \to \check{C}*(U, \mathbb{Z}_2)$,

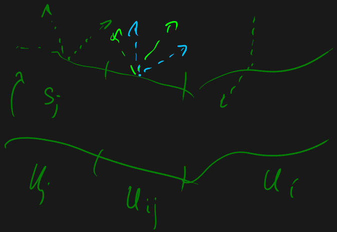
Low Stiefel-Whitney classes (first two floors of Whitehead tower)

THEOREM M is orientable iff the 1st Stiefel-Whitney class $w_1(M) := [f] = 1$.

Proof of 'if'. If $w_1(M) = 1$, $f(k_0, k_1) = \det(G_{k_0,k_1})$ is a 0-coboundary, $f = \delta^0 h$. We can pick sections $\{s_k : U_k \to F(M)\}_k$ and $g_k \in \mathrm{O}(d)$ with $\det g_k = h(k)$, so that the transition functions for $s'_k := s_k \cdot g_k$ satisfy

SO(d)

$$\det(G'_{k_0,k_1}) = \det(g_{k_0}^{-1} G_{k_0,k_1} g_{k_1})$$
$$= [(\delta^0 h) \cdot f](k_0, k_1) = 1. \qquad \square$$



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$$\begin{aligned}\det(G'_{k_0,k_1}) &= \det(g_{k_0}^{-1} G_{k_0,k_1} g_{k_1}) \\ &= [(\delta^0 h) \cdot f](k_0, k_1) = 1. \quad \square\end{aligned}$$

$\mathbb{R}^n$ λ

In similar way, a *spin structure* $(\lambda, P(M))$

$$\begin{tikzcd}
\mathrm{Spin}(d) \times P(M) \arrow[r] \arrow[d, "{(\rho,\lambda)}"'] & P(M) \arrow[d, "\lambda"] \arrow[dr] & \\
 & & M \\
\mathrm{SO}(d) \times F_{\mathrm{SO}}(M) \arrow[r] & F_{\mathrm{SO}}(M) \arrow[ur] &
\end{tikzcd}$$

exists when the SO-frame bundle can be lifted in compatible way with the double cover $\mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(d) \xrightarrow{\rho} \mathrm{SO}(d)$. Transition functions $g_{ij} : U_{ij} \to \mathrm{SO}(d)$ can be lifted to $\underline{\mathrm{Spin}}(d)$-valued $\tilde{g}_{ij}$. For $U_{ijk} \neq \varnothing$, let

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} =: z(i,j,k)\mathrm{id}_{\mathrm{Spin}(d)} \qquad z(i,j,k) \in \mathbb{Z}_2.$$

$$\rho(\;-\!\!-\!\!-\;) = \mathrm{Id}_{SO(d)}$$

THEOREM [A. Haefliger, '56] Orientable M is spin iff its second Stiefel-Whitney class $w_2(M) := [z] = 1$.

The Dirac operator (assume d even)

- if $P(M)$ is a spin structure and $\mathbb{S}$ a rep. of the Clifford algebra $\mathbb{C}\ell(d) \simeq \mathrm{End}(\mathbb{S})$

$$\mathbb{S}(M) = P(M) \times_\alpha \mathbb{S}$$

dim $\frac{d}{2}$

- the Levi-Civita connection ∇^{LC} can be also lifted to the *spin connection* $\nabla^s : \Gamma(\mathbb{S}) \to \Omega^1(M) \otimes \Gamma(\mathbb{S})$

$$\nabla^s c(\omega)\psi = c(\nabla^{\mathrm{LC}}\omega)\psi + c(\omega)\nabla^s\psi$$

$\psi \in \Gamma(\mathbb{S}), \omega \in \Omega^1(M)$

$[\nabla^S, c(\omega)] = c(\nabla^{LC}\omega)$

being c Clifford multiplication, basically $c(\mathrm{d}x^\mu) = \gamma^\mu$

- on the space of square integrable spinors $L^2(M, \mathbb{S})$ there is an (ess.) self-adjoint operator, the *Dirac operator*,

$$D_M = -\mathrm{i}c \circ \nabla^s \overset{\text{loc.}}{=} -\mathrm{i} \sum_{\mu=1}^{d} \gamma^\mu(\partial_\mu + \omega_\mu)$$

and by Leibniz rule

$$[D_M, a] = -\mathrm{i}c(\mathrm{d}a) \quad a \in C^\infty(M)$$

which is bounded

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a representation of A_M
- D_M is self-adjoint and for each $a \in A_M$, $[D_M, a]$ is bounded. D_M has compact resolvent $(D_M + i)^{-1} \in \mathcal{K}(H)$

A *spectral triple* (A, H, D) consists of

- a $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$

  The 'commutative case' motivates

$$\Omega^1_D(A) := \left\{ \sum\nolimits_{\text{(finite)}} b[D, a] \mid a, b \in A \right\}$$

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- $H_M := L^2(M, \mathbb{S})$ a representation of A_M
- D_M is self-adjoint and for each $a \in A_M$, $[D_M, a]$ is bounded. D_M has compact resolvent

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- a $*$-algebra A
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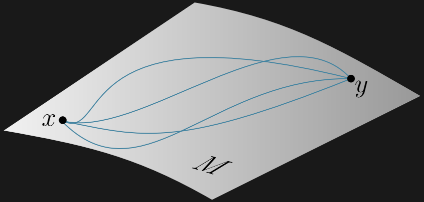
$$\Omega^1_D(A) := \left\{ \sum\nolimits_{(\text{finite})} b[D, a] \mid a, b \in A \right\}$$

Reconstruction Theorem:[A, Connes, *JNCG* '13] Commutative spectral triples$^{+\textit{five axioms}}$ are Riemannian manifolds.

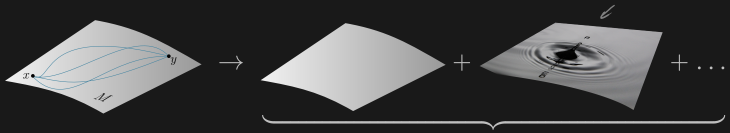
QUANTIZATION & RANDOMNESS

- Path integrals <u>on</u> *M*



Quantization & randomness

- Path integrals on M → Path integral of spacetime (Quantum Gravity)



Quantum superposition of geometries

QUANTIZATION & RANDOMNESS

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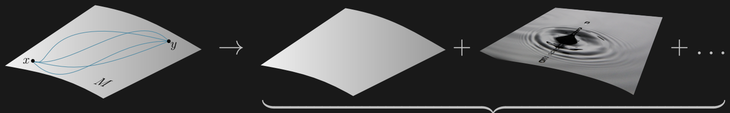


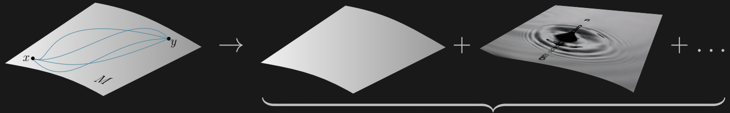
Quantum superposition of geometries

- Quantum Gravity* → Random Geometry

$$\mathcal{Z}_{\mathrm{QG}} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathrm{e}^{\underset{\sim}{\frac{\mathrm{i}}{\hbar}} S_{\mathrm{EH}}[g]} \mathcal{D}[g] \rightarrow \mathcal{Z} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathrm{e}^{-\frac{1}{\hbar} S_{\mathrm{EH}}[g]} \mathcal{D}[g]$$

QUANTIZATION & RANDOMNESS

- Path integrals on M → Path integral of spacetime (Quantum Gravity)



Quantum superposition of geometries

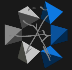
- Quantum Gravity → Random Geometry

$$\mathcal{Z}_{\text{QG}} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathrm{e}^{\frac{i}{\hbar}S_{\text{EH}}[g]}\mathcal{D}[g] \to \mathcal{Z} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathrm{e}^{-\frac{1}{\hbar}S_{\text{EH}}[g]}\mathcal{D}[g]$$

- To access $\mathcal{Z}$, models for 'quantum space' are proposed, e.g.

[Doplicher Fredenhagen Roberts, CMP 75]

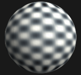
Discrete



single geometry

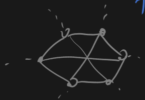
← paradigms →

Algebraic



← $Y^{\ell}_{m=\ell/2}$

(↖ from Wikipedia)

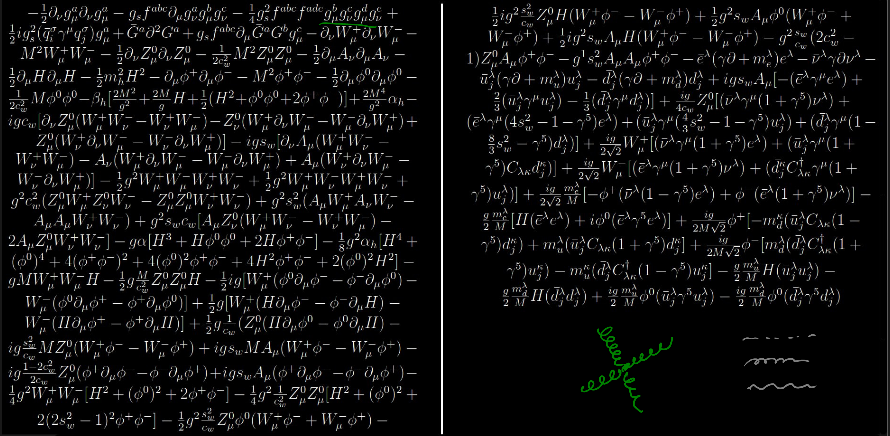


$\simeq C(\mathbb{T}^2)$ / Răzvan

interaction vertex, Random tensors

Standard Model (min. coupled to gravity with right-handed neutrinos)

$\mathcal{L}_{SM} =$



$$\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc} f^{ade}\underline{g^b_\mu g^c_\nu g^d_\mu g^e_\nu} + \\
\tfrac{1}{2} i g_s^2 (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu + \bar{G}^a \partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H \partial_\mu H - \tfrac{1}{2} m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\tfrac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h [\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
i g c_w [\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + \\
Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - i g s_w [\partial_\nu A_\mu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - \\
W^-_\nu \partial_\nu W^+_\mu)] - \tfrac{1}{2} g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2} g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w [A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2 A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha [H^3 + H \phi^0 \phi^0 + 2H \phi^+ \phi^-] - \tfrac{1}{8} g^2 \alpha_h [H^4 + \\
(\phi^0)^4 + 4(\phi^+ \phi^-)^2 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - \\
g M W^+_\mu W^-_\mu H - \tfrac{1}{2} g \tfrac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \tfrac{1}{2} i g [W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \tfrac{1}{2} g [W^+_\mu (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - \\
W^-_\mu (H \partial_\mu \phi^+ - \phi^+ \partial_\mu H)] + \tfrac{1}{2} g \tfrac{1}{c_w} (Z^0_\mu (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - \\
i g \tfrac{s_w^2}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + i g s_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
i g \tfrac{1-2c_w^2}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + i g s_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \\
\tfrac{1}{4} g^2 W^+_\mu W^-_\mu [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \tfrac{1}{4} g^2 \tfrac{1}{c_w^2} Z^0_\mu Z^0_\mu [H^2 + (\phi^0)^2 + \\
2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \tfrac{1}{2} g^2 \tfrac{s_w^2}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) -
\end{gathered}$$

$$\begin{gathered}
\tfrac{1}{2} i g^2 \tfrac{s_w^2}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2} g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + \\
W^-_\mu \phi^+) + \tfrac{1}{2} i g^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - g^2 \tfrac{s_w}{c_w} (2c_w^2 - \\
1) Z^0_\mu A_\mu \phi^+ \phi^- - g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^- - \bar{e}^\lambda (\gamma\partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma\partial \nu^\lambda - \\
\bar{u}^\lambda_j (\gamma\partial + m_u^\lambda) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m_d^\lambda) d^\lambda_j + i g s_w A_\mu [-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \\
\tfrac{2}{3} (\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3} (\bar{d}^\lambda_j \gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w} Z^0_\mu [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + \\
(\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (\tfrac{4}{3} s_w^2 - 1 - \gamma^5) u^\lambda_j) + (\bar{d}^\lambda_j \gamma^\mu (1 - \\
\tfrac{8}{3} s_w^2 - \gamma^5) d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}} W^+_\mu [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + (\bar{u}^\lambda_j \gamma^\mu (1 + \\
\gamma^5) C_{\lambda\kappa} d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}} W^-_\mu [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa} \gamma^\mu (1 + \\
\gamma^5) u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}} \tfrac{m_e^\lambda}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\
\tfrac{g}{2} \tfrac{m_e^\lambda}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}} \phi^+ [-m_d^\kappa (\bar{u}^\lambda_j C_{\lambda\kappa} (1 - \\
\gamma^5) d^\kappa_j) + m_u^\lambda (\bar{u}^\lambda_j C_{\lambda\kappa} (1 + \gamma^5) d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}} \phi^- [m_d^\lambda (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1 + \\
\gamma^5) u^\kappa_j) - m_u^\kappa (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1 - \gamma^5) u^\kappa_j] - \tfrac{g}{2} \tfrac{m_u^\lambda}{M} H (\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2} \tfrac{m_d^\lambda}{M} H (\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2} \tfrac{m_u^\lambda}{M} \phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \tfrac{ig}{2} \tfrac{m_d^\lambda}{M} \phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}$$

Input

$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \rightarrowtail$ NCG $\rightarrowtail$ Classical Lagrangian of the SM

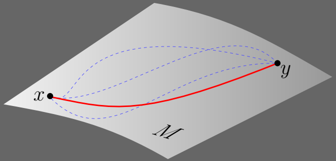
number of families (3) by hand

[Chamseddine-Connes-Marcolli *ATMP* 2007 (Euclidean); J. Barrett *J. Math. Phys.* 2007 (Lorenzian)]

II. DIRAC OPERATORS IN NONCOMMUTATIVE GEOMETRY

The idea is to replace the metric in (M, g) by D_M ←

Connes' geodesic distance



$\gamma : \mathbb{R} \to M$

$$\inf_{\gamma \text{ as above}}\left\{\int_\gamma \mathrm{d}s\right\} = \underset{g}{d}(x, y)$$

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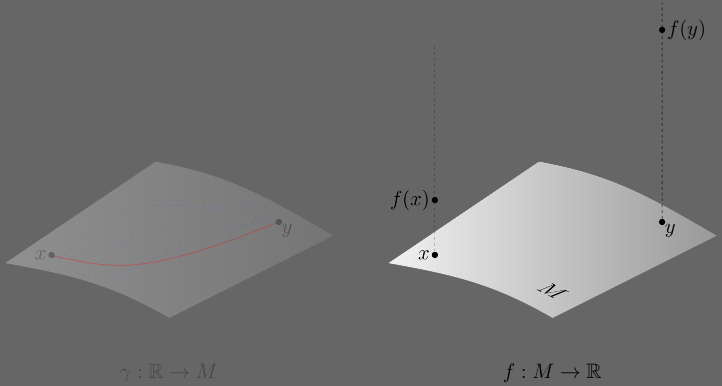

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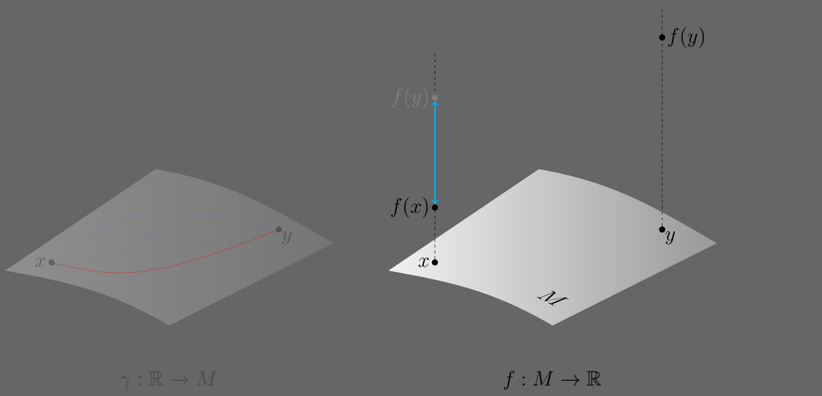

$$|f(x) - f(y)|$$

II. Dirac operators in noncommutative geometry

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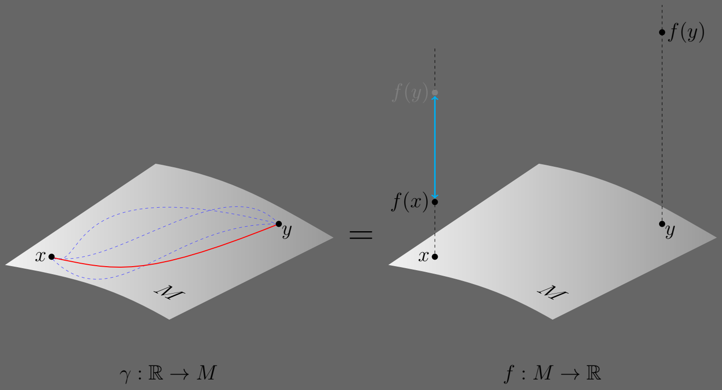


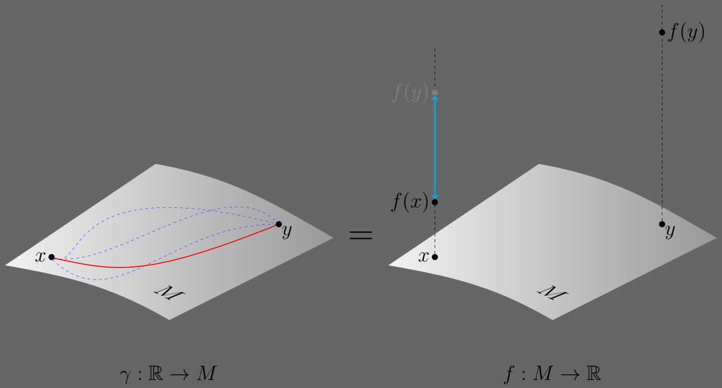
$$\sup_{f\in C^\infty(M)} \{ |f(x) - f(y)| \ : \ ||Df - fD|| \leqslant 1\}$$

II. DIRAC OPERATORS IN NONCOMMUTATIVE GEOMETRY

The idea is to replace the metric in (M, g) by D_M

$$D_M^2 = -g^{\mu\nu}\nabla_\mu\nabla_\nu + \dots$$

Connes' geodesic distance



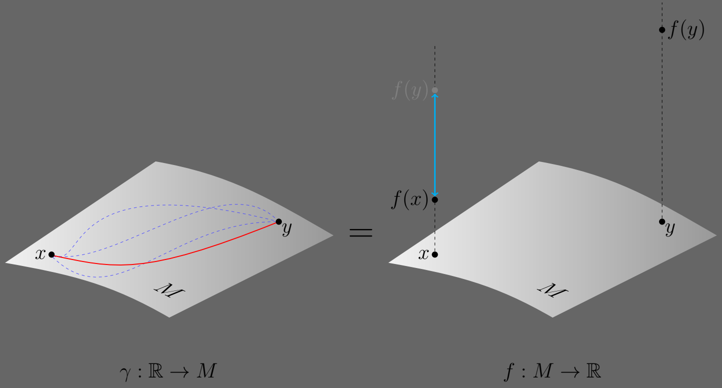
$$\inf_{\gamma \text{ as above}}\left\{\int_\gamma \mathrm{d}s\right\} = d(x,y) = \sup_{f \in C^\infty(M)} \left\{\, |f(x) - f(y)| \;:\; \|Df - fD\| \leqslant 1 \right\}$$

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{f(x) - f(y)}{d_g(x-y)}$$

II. DIRAC OPERATORS IN NONCOMMUTATIVE GEOMETRY

The idea is to replace the metric in (M, g) by D_M

Connes' geodesic distance

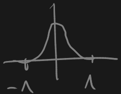


$$\inf_{\gamma \text{ as above}} \{ \int_\gamma \mathrm{d}s \} = d(x,y) \; = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \; : \; ||Df - fD|| \leqslant 1 \}$$

NCG toolkit in high energy physics (1.5 of 2)

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

  $$S_b(D) = \mathrm{Tr}_H\, f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

  for a bump function f around the origin and Λ a cut off scale. It's computed with heat kernel expansion [P. Gilkey, *J. Diff. Geom.* '75]

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- Realistic, classical models come from *almost-commutative manifolds* $M \times F$, where F is a finite-dim. spectral triple

$$(C^\infty(A_F), H_M \underline{\otimes} H_F, D_M \otimes 1_F + \gamma_5 \otimes D_F)$$

$C^\infty(M)\otimes A_F$ chirality

- applications require (A, H, D) to have a *reality* $J : H \to H$ antiunitary$^{+\textit{some axioms}}$, implementing a right A-action on H,

$$\psi a = J a^* J^{-1} \psi \qquad \psi \in H, a \in A$$

- let's sketch *connections*: if S is a functional on M

$$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

$$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \qquad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

$$\mathbb{A}' = u\mathbb{A}u^{-1} + u\,\underline{\underline{\mathrm{d}u}}^{-1} \qquad u \in \mathrm{Maps}(M, G)$$

NCG toolkit in high energy physics (1.5 of 2)

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E module | $H' = E \underset{A}{\otimes} H$; D

- given (A, H, D) and a Morita equivalence $A \simeq_{\mathrm{M}} B$ (i.e. $\mathrm{End}_A(E) \cong B$) yields new (B, H', D'). For $A = B$, in fact a tower

$$\{(A, H, D_\omega)\}_{\underbrace{\omega \in \Omega^1_D(A)}_{1\ \text{fms.}}} \qquad D'(\eta \otimes a\psi)$$

fluctuated D

  with $D_\omega := D + \omega$. In presence of J $= D'(\eta a \otimes \psi)$

$$D_\omega = D + \omega \pm J\omega J^{-1} \qquad \nabla : E \to E \otimes \Omega^1(M)$$

Gauge inv of S_b : $D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* \qquad \nabla(1) = \omega$

$$\omega \mapsto u\omega u^* + u[\underbrace{D, u^*}_{1\text{-}\mathrm{fms}}] \qquad \underline{u \in \mathcal{U}(A)}$$

NCG toolkit in high energy physics (2 of 2)

- The fermionic part is not treated here but is essentially given by

$$S_f(D) = \frac{1}{2}\langle J\psi \mid D\psi\rangle \quad \leftarrow$$

where ψ are (Graßman valued) fermions, J implements charge-conjugation (J fixes the spin structure)

$$\psi \in L^2(\$) \otimes \mathcal{H}_F$$

$$\|$$

lists all particles,

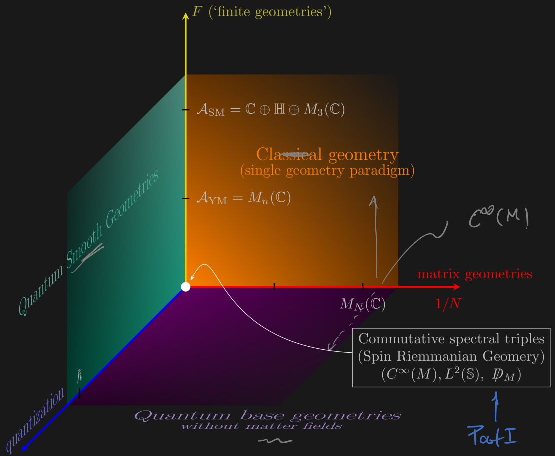
Dirac F_{SM} operator

$$D_F = \left(\begin{array}{cccc|ccccc|cccc|ccccc}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Upsilon}_d & 0 & 0 &
\end{array}\right) \in M_{96}(\mathbb{C})_{\text{s.a.}}$$

χ

$\sim 10^4$ zeroes from geometry.

III. Random Dirac operators & multi-matrix models



Aim: Make sense of

$$\mathcal{Z} = \int_{\text{Dirac}} e^{-\frac{1}{\hbar}\mathrm{Tr}_H f(D)} \mathrm{d}D$$

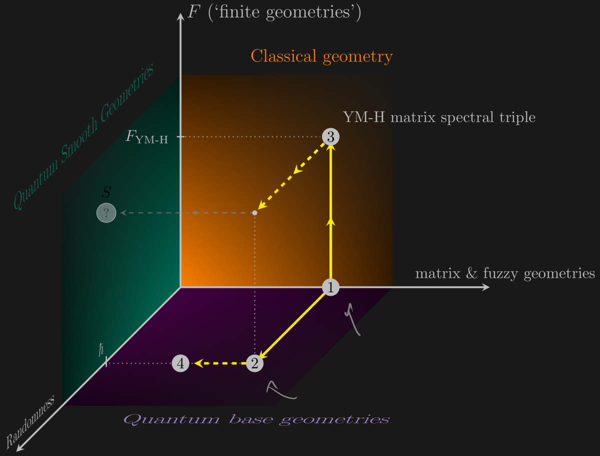
on $(A, H, \cdot)$

- *Plane* $(\hbar, 1/N, 0)$ *of 'base geometries'*
- *Plane* $(\hbar, 0, F) = \lim_{N\to\infty}(\hbar, 1/N, F)$
- *Plane* $(0, 1/N, F) = \lim_{\hbar\to 0}(\hbar, 1/N, F)$ *of classical geometries*

[CP 2105.01025]

Part II ● × F = almost-comm.

III. RANDOM DIRAC OPERATORS & MULTI-MATRIX MODELS



1 Matrix Geometries
[J. Barrett, *J. Math. Phys.* 2015]

2 Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action [CP 1912.13288]

3 Gauge matrix spectral triples
[CP 2105.01025]

4 Functional Renormalization [CP 2007.10914] and [CP 2111.02858]

Matrix or Fuzzy Geometries

DEFINITION (“condensed” from [J. Barrett, *J. Math. Phys.* 2015]).
A *fuzzy geometry* of *signature* $(p, q) \in \mathbb{Z}^2_{\geqslant 0}$ is given by

- a simple matrix algebra $\mathcal{A}$ – we take always $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian $\mathbb{C}\ell(p, q)$-module $\mathbb{S}$ with a *chirality* γ. That is a linear map $\gamma : \mathbb{S} \to \mathbb{S}$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$
- a Hilbert space $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w)\mathrm{Tr}_N(R^* S)$ for each $R, S \in M_N(\mathbb{C})$, being $(\cdot, \cdot)$ the inner product of $\mathbb{S}$
- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}$, $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

  $\ldots$

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- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}$, $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$
- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

$s \equiv q - p \pmod 8$	0	1	2	3	4	5	6	7
ϵ	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ'	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ''	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$

- a real structure $J = C \otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on $\mathbb{S}$ satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon'\gamma^\mu C$ for all the gamma matrices $\mu = 1, \ldots, p + q$.
- a self-adjoint operator D on $\mathcal{H}$ satisfying the *order-one condition*

$$[\,[D, \rho(a)]\,, J\rho(b)J^{-1}] = 0 \qquad \text{for all } a, b \in \mathcal{A}$$

- a chirality $\Gamma = \gamma \otimes 1_{\mathcal{A}}$ for $\mathcal{H}$, where γ is the chirality of $\mathbb{S}$. The signs above impose:

$$J^2 = \epsilon\,, \qquad JD = \epsilon' DJ\,, \qquad J\Gamma = \epsilon''\Gamma J\,.$$

Characterization of Fuzzy Geometries

A fuzzy geometry of signature (p, q) (thus of dim. $p + q$ and KO-dim $q - p$) consists of

- $\mathcal{A} = M_N(\mathbb{C})$
- $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$, being $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

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  . . . (axioms for D omitted ⊳⊳) . . .
- Gamma-matrices conventions:
  - $(\gamma^\mu)^2 = +1$, $\mu = 1, \ldots, p, \gamma^\mu$ Hermitian
  - $(\gamma^\mu)^2 = -1$, $\mu = 1 + p, \ldots, q + p, \gamma^\mu$ anti-Hermitian
  - $\Gamma^I := \gamma^{\mu_1} \cdots \gamma^{\mu_r}$ for $\mu_i = 1, \ldots, p + q$, $I = (\mu_1, \ldots, \mu_r)$

- Characterization of D in even dimensions:

$$D = \sum_I \Gamma^I_{\text{s.a.}} \otimes \{H_I, \ \cdot \ \} + \sum_I \Gamma^I_{\text{anti.}} \otimes [L_I, \ \cdot \ ]$$

  with multi-index I monot. increasing, $|I|$ odd [J. Barrett, *J. Math. Phys.* '15]

- Examples:
  - $D_{(1,1)} = \gamma^1 \otimes [L, \ \cdot \ ] + \gamma^2 \otimes \{H, \ \cdot \ \}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \ \cdot \ ] + \gamma^{\hat{\mu}} \otimes \{H_{\hat{\mu}}, \ \cdot \ \}$
- [J. Barrett, L. Glaser, *J. Phys. A* 2016]

$$\{H, \ \cdot \ \} \mapsto H \otimes 1_N + 1_N \otimes H^T$$
$$[L, \ \cdot \ ] \mapsto L \otimes 1_N - 1_N \otimes L^T$$

so we will get double traces from $\text{Tr}_H = \text{Tr}_{\mathbb{S}} \otimes \text{Tr}_N^{\otimes 2}$

Spectral Action: Traces of D^{2m} [C.P. 1912.13288] Chord Diagrams for $d = 2$ geometries, $\eta = \mathrm{diag}(e_1, e_2)$

$$\mathrm{Tr}_H(D^6) = 2N \sum_{\boldsymbol{\mu}} \overline{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \cdots \gamma^{\mu_6})} \times$$

$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6}$ $+(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4}$ $+(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5}$ $+(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5}$ $+(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$

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$+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6}$ $+(-1)^1\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\mu_6}$ $+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_6}\eta^{\mu_4\mu_5}$ $+(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\eta^{\mu_5\mu_6}$ $+(-1)^2\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_5}\eta^{\mu_4\mu_6}$

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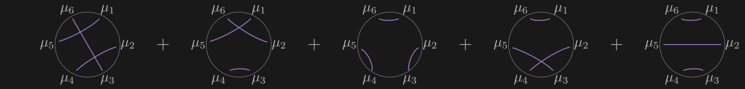

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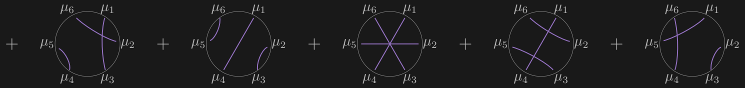

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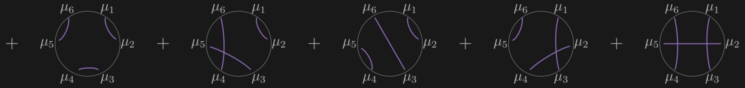

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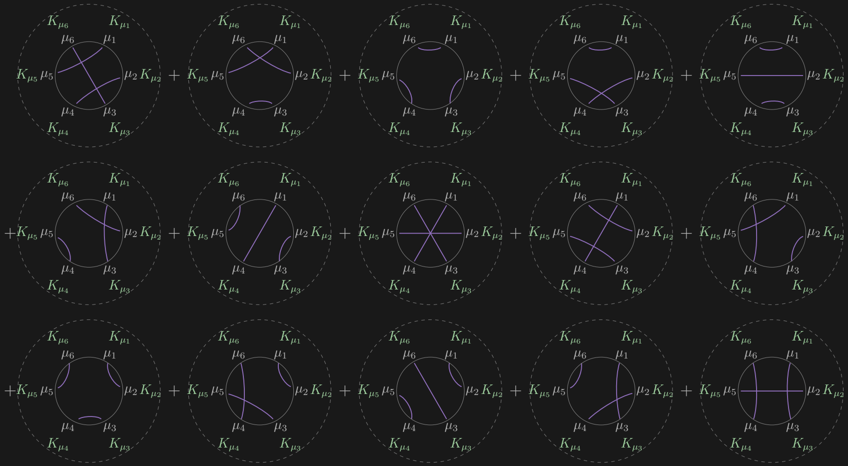
$$\mathrm{Tr}_H(D^6) = 2N\sum_{\boldsymbol{\mu}} \overbrace{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1}\cdots\gamma^{\mu_6})}^{\text{solid circ.}} \times \overbrace{\mathrm{Tr}_N\left(K_{\mu_1}\cdots K_{\mu_6}\right)}^{\text{dashed circ.}} + \overbrace{\mathrm{Tr}_N P \times \mathrm{Tr}_N Q \text{ terms}}^{(1,5),(2,4),(3,3)\text{-partitions}}$$



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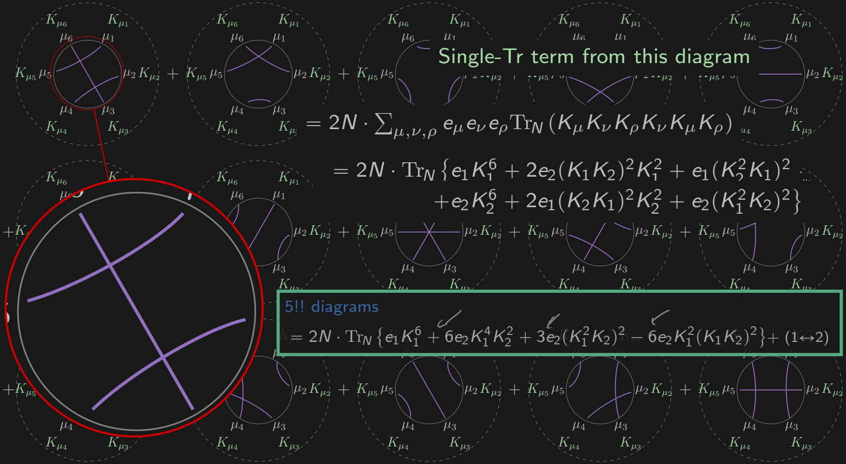


Single-Tr term from this diagram

$$= 2N\cdot\sum_{\mu,\nu,\rho} e_\mu e_\nu e_\rho \mathrm{Tr}_N\left(K_\mu K_\nu K_\rho K_\nu K_\mu K_\rho\right)$$

$$= 2N\cdot\mathrm{Tr}_N\{e_1K_1^6 + 2e_2(K_1K_2)^2K_1^2 + e_1(K_2^2K_1)^2$$
$$+e_2K_2^6 + 2e_1(K_2K_1)^2K_2^2 + e_2(K_1^2K_2)^2\}$$

$K_i = \{ \begin{matrix} H \\ L \end{matrix}$

5!! diagrams

$$= 2N\cdot\mathrm{Tr}_N\{e_1K_1^6 + 6e_2K_1^4K_2^2 + 3e_2(K_1^2K_2)^2 - 6e_2K_1^2(K_1K_2)^2\} + (1\leftrightarrow 2)$$

Multi-matrix models with multi-traces & ribbon graphs

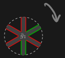
- This holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\textsc{Dirac}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$

$$= \int_{M_{p,q}} \mathrm{e}^{-N \mathrm{Tr}_N P - \mathrm{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\textsc{Leb}}$$

- $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
- $\mathrm{d}\mathbb{X}_{\textsc{Leb}}$ is the Lebesgue measure on $M_{p,q}$
- P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle X_1, \ldots, X_k \rangle = \mathbb{C}\langle \mathbb{X} \rangle$ are certain noncommutative polynomials
- $\mathcal{Z}_{\textsc{formal}}$ leads to colored ribbon graphs

$$\bar{g}_1 \mathrm{Tr}_N (ABBBAB) \quad \leftrightarrow$$



$$\bar{g}_2 \, \mathrm{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$$

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  - $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
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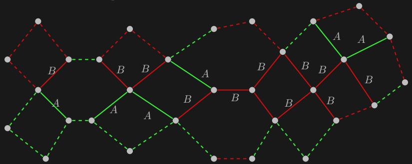
  - $\mathcal{Z}_{\mathrm{FORMAL}}$ leads to colored ribbon graphs

$\bar{g}_1 \mathrm{Tr}_N (ABBBAB) \quad \leftrightarrow$ 

$\bar{g}_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$ 

Ribbon graphs can be useful!

- Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78]. The ones we get are 'face-worded'

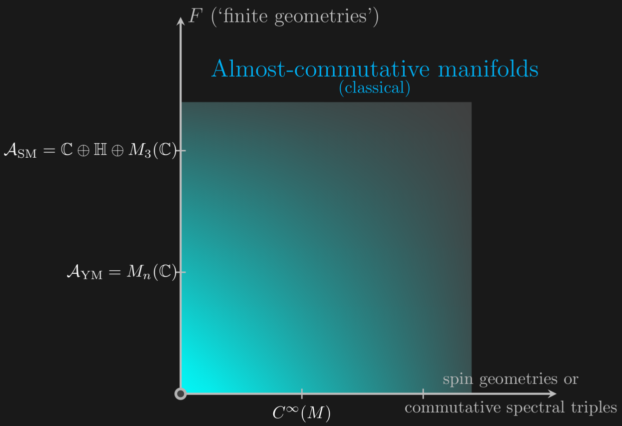


- Intersection numbers of ψ-classes [Kontsevich, *CMP*, '92]

$$\sum_{a_1+\ldots+a_n=\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^{n} \frac{(2a_j-1)!!}{s_j^{2a_j+1}}$$

$$= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\mathrm{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}}$$

IV. YANG-MILLS-HIGGS MATRIX THEORY



F ('finite geometries')

Almost-commutative manifolds

(classical)

$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

$C^\infty(M)$

spin geometries or commutative spectral triples

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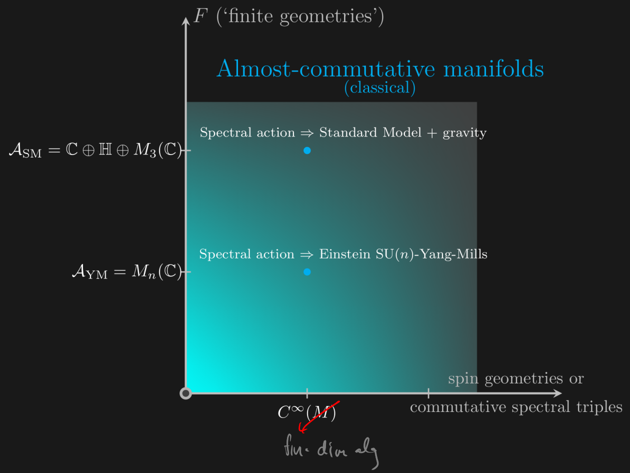

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Spectral action ⇒ Standard Model + gravity

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$C^\infty(\cancel{M})$

fin. dim alg

$$\mathcal{Z}_{\text{AC}} \overset{?}{=} \int_{\text{Dirac ops.}} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)} \mathrm{d}D$$

(hard for AC-manifolds)

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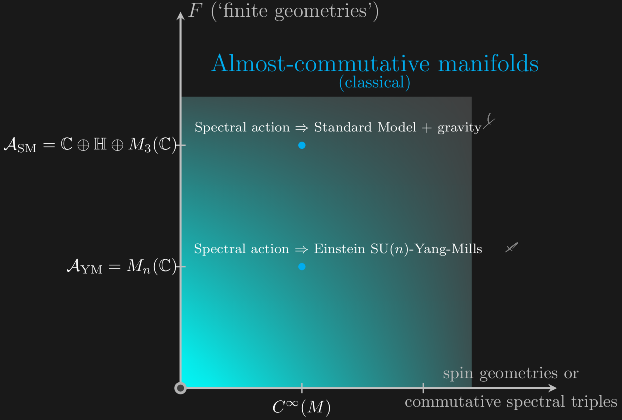

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DEFINITION [CP 2105.01025] We define a *gauge matrix spectral triple* $G_{\not{f}} \times F$ as the spectral triple product of a fuzzy geometry $G_{\not{f}}$ with a finite geometry $F = (\mathcal{A}_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$, $\dim \mathcal{A}_F < \infty$.

DEFINITION [CP 2105.01025] Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F, J_F = \mathrm{c.c.}\,, \gamma_F = 1)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on the algebra $M_N(\mathbb{C})$

$$D_\omega = \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}} + \sum_\mu \overbrace{\gamma^\mu \otimes (\ell_\mu + \mathscr{a}_\mu) + \gamma^{\hat{\mu}} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}}\,,$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\underbrace{\ell_\mu + \mathscr{a}_\mu}_{\mathscr{d}_\mu}, \ell_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu},\ \cdot\ ]$$

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LEMMA The gauge group $\mathrm{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \quad \text{for all } u \in \mathrm{G}(A)$$

THEOREM. For a Yang-Mills–Higgs matrix spectral triple on a 4-dimensional flat ($\mathscr{x} = 0 = \mathscr{s}$) Riemannian ($p = 0$) fuzzy base, the Spectral Action for a real polynomial $f(x) = \frac{1}{2}\sum_{i=1}^4 a_i x^i$ reads

$$\frac{1}{4}\,\mathrm{Tr}_H\, f(D) = S^\ell_{\mathrm{YM}} + S^\ell_{\mathrm{H}} + S^\ell_{\text{g-H}} + S^\ell_\vartheta,$$

where each sector is defined as follows:

$$\begin{aligned} S^\ell_{\mathrm{YM}}(\mathscr{l}, \mathscr{a}) &:= -\frac{a_4}{4}\,\mathrm{Tr}_{M^{\mathbb{C}}_{N\otimes n}}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}), \\ S^\ell_{\text{g-H}}(\mathscr{l}, \mathscr{a}, \Phi) &:= -a_4\,\mathrm{Tr}_{M^{\mathbb{C}}_{N\otimes n}}\left(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi\right), \\ S^\ell_{\mathrm{H}}(\Phi) &:= \mathrm{Tr}_{M^{\mathbb{C}}_{N\otimes n}}\, f_{\mathrm{e}}(\Phi), \\ S^\ell_\vartheta(\mathscr{l}, \mathscr{a}) &:= \mathrm{Tr}_{M^{\mathbb{C}}_{N\otimes n}}\, f_{\mathrm{e}}\big(\vartheta^{1/2}\big). \end{aligned}$$

Moreover, one obtains positivity for each of the following functionals, independently:

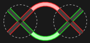
$$S^\ell_\vartheta,\ S^\ell_{\mathrm{YM}},\ S^\ell_{\mathrm{H}} \geqslant 0, \qquad \text{if } a_4 \geqslant 0.$$

Meaning	Random matrix case (Riemannian signature)	Smooth operator
Derivation	$\mathscr{l}_\mu = [L_\mu, -]$	$\sim \partial_i$
Gauge potential	$\mathscr{a}_\mu = [A_\mu, -]$	$\mathbb{A}_i$
Higgs field	Φ	h
Covariant Derivative	$\mathscr{d}_\mu = \mathscr{l}_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[\mathscr{d}_\mu, \mathscr{d}_\nu] = \overbrace{[\mathscr{l}_\mu, \mathscr{l}_\nu]}^{\not\equiv 0} + [\mathscr{l}_\mu, \mathscr{a}_\nu] - [\mathscr{l}_\nu, \mathscr{a}_\mu] + [\mathscr{a}_\mu, \mathscr{a}_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} + \partial_i\mathbb{A}_j - \partial_j\mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Higgs lagrangian	$\mathrm{Tr}(f_2\Phi^2 + f_4\Phi^4)$	$\int_M \left(f_2\lvert h\rvert^2 + f_4\lvert h\rvert^4\right)\mathrm{vol}$
Gauge-Higgs coupling	$\mathrm{Tr}(\vartheta\,\Phi^2)$	$-\int_M \lvert\mathbb{D}_i h\rvert^2\mathrm{vol}$
Yang-Mills action	$-\frac{1}{4}\,\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$	$-\frac{1}{4}\int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$
	$\mathrm{Tr}(\,d_\mu d^\mu)$	$\longrightarrow$?

Handwaving Functional Renormalization for k-matrix models (w/multi-trace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$. E.g. for $k = 2$ and with bare action

$$S[A,B] = N\mathrm{Tr}_N \left\{ \frac{1}{2}A^2 + \frac{1}{2}B^2 + g_{A^4}\frac{1}{4}A^4 + g_{B^4}\frac{1}{4}B^4 + \frac{1}{2}g_{ABAB}ABAB \right\}$$

radiative corrections 'generate' the *effective vertices*. For instance  generates $N\mathrm{Tr}_N\,(ABBA)$.

$$\Gamma_N[A,B] = \mathrm{Tr}_N \left\{ \overbrace{\frac{Z_A}{2}A^2 + \frac{Z_B}{2}B^2 + \bar{g}_{A^4}\frac{1}{4}A^4 + \bar{g}_{B^4}\frac{1}{4}B^4 + \frac{1}{2}\bar{g}_{ABAB}ABAB}^{\text{operators from the bare action (but with 'running couplings')}} + \overbrace{\frac{1}{2}\bar{g}_{ABBA}ABBA + \frac{1}{2}\bar{g}_{A|A}\mathrm{Tr}_N\,(A)\times A + \ldots}^{\text{radiative corrections}} \right\}$$

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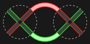
radiative corrections ‘generate’ the *effective vertices*. For instance  generates $N\mathrm{Tr}_N(ABBA)$.

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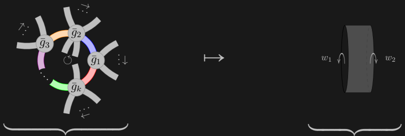
We are interested in *one-loop graphs* (graphs G whose 1-dim skeleton G° has $b_1(G^\circ) = 1$). The *effective vertex* O^{eff}_G of such Feynman graph is formed by reading off each word w_i traveling around all ribbon edges (propagators) by both sides:

$$\begin{aligned} O^{\text{eff}}_G = & \overbrace{\mathrm{Tr}_N(w_1)\times\mathrm{Tr}_N(w_2)\times\cdots\times\mathrm{Tr}_N(w_s)}^{\text{from vertices contracted with propagators}} \\ & \times \underbrace{\mathrm{Tr}_N(U_1)\times\mathrm{Tr}_N(U_2)\dots\times\mathrm{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}} \end{aligned}$$



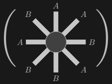
gral. situation (yet, without multi-traces)

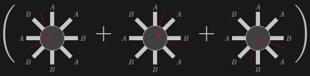
inner word $w_1 \otimes$ outer word w_2

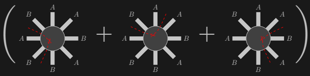
- *nc-derivative* $\partial_A : \mathbb{C}_{\langle k \rangle} \to \mathbb{C}_{\langle k \rangle}^{\otimes 2}$ sums over 'replacements of A by $\otimes$'[Rota-Sagan-Stein+Voiculescu]:

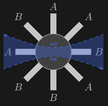
$$\partial_A(PAAR) = P \otimes AR + PA \otimes R\,, \quad \text{but}$$
$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

- for $W \in \mathbb{C}_{\langle k \rangle}$, Hess $\mathrm{Tr}_N\, W \in M_k(\mathbb{C}_{\langle k \rangle} \otimes \mathbb{C}_{\langle k \rangle})$ is the *nc-Hessian* [CP 2007.10914], whose entries are $\mathrm{Hess}_{b,a}\, \mathrm{Tr}\, W = (\partial_{X_b} \circ \partial_{X_a}) \mathrm{Tr}_N\, W$. These are computed by 'cuts': e.g. $W = ABAABABB$

$\partial_B \partial_A$


$= 1_N \otimes$


$+$

$\otimes 1_N + \ldots$

in ellipsis $\sum_{cuts}$

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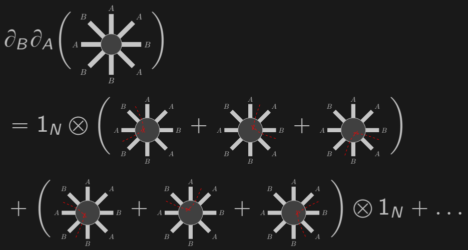
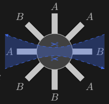


in ellipsis $\sum_{\text{cuts}}$  $\to BAA \otimes ABB$

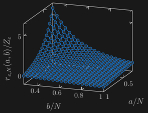
- products of traces $\Rightarrow$ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle}) \oplus (\mathbb{C}_{\langle k\rangle} \boxtimes \mathbb{C}_{\langle k\rangle})$

$$\operatorname{Hess}_{a,b}(\operatorname{Tr} P \cdot \operatorname{Tr} Q) = \operatorname{Tr} P \cdot \operatorname{Hess}_{a,b}[\operatorname{Tr} Q]$$
$$+ \partial_{X_a} \operatorname{Tr} P \boxtimes \partial_{X_b} \operatorname{Tr} Q + (P \leftrightarrow Q)$$

- Wetterich Eq. governs the functional RG

$t = \log N$

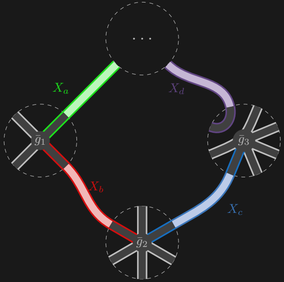
$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2}\operatorname{STr}\left\{\frac{\partial_t R_N}{\operatorname{Hess}\Gamma_N[\mathbb{X}] + R_N}\right\}$$



$$\overset{\text{assume}}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \ldots, \eta_n)$$
$$\times \underbrace{\frac{1}{2}(-1)^k \operatorname{STr}\left\{(\operatorname{Hess}\Gamma_N^{\mathrm{INT}}[\mathbb{X}])^{\star k}\right\}}_{\text{regulator-independent part}}$$

- $\operatorname{STr} = \operatorname{Tr}_k \otimes \operatorname{Tr}_{\mathcal{A}_n}$. Tadpoles imply $\operatorname{Tr}_{\mathcal{A}_n}(P \otimes Q) = \mathrm{Tr}_N P \cdot \mathrm{Tr}_N Q$, $\operatorname{Tr}_{\mathcal{A}_n}(P \boxtimes Q) = \mathrm{Tr}_N(PQ)$

Finding $\star$



Want: $\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \dots \star \mathrm{Hess}_{*,a}\, O_\ell$

Finding $\star$

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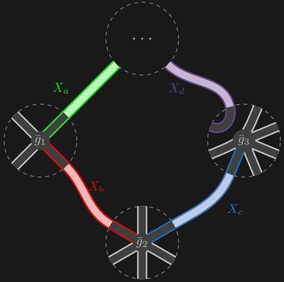
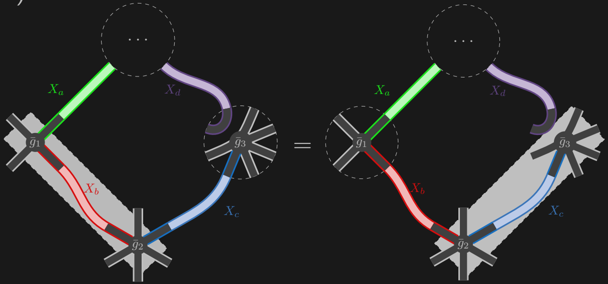
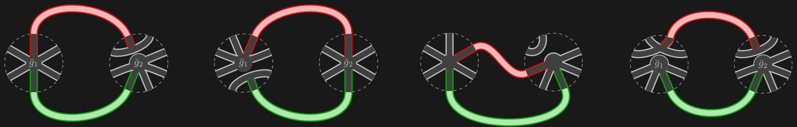


$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \dots \star \mathrm{Hess}_{*,a}\, O_\ell$

Associativity (trivial check):





$$\begin{tikzcd}
\text{Case I} \arrow[d] & \text{Case II} \arrow[d] & \text{Case III} \arrow[d] & \text{Case IV} \arrow[d] \\
\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \boxtimes Q)] \arrow[ddd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \otimes Q)] \arrow[ddd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \otimes Q)] \arrow[d] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \boxtimes Q)] \arrow[dl] \arrow[dd] \\
 & & (U \otimes W) \star (P \otimes Q)/\sim \arrow[dr] \arrow[ddd] \arrow[dddr] & \\
 & & & (U \boxtimes W) \star (P \boxtimes Q)/\sim \arrow[dlll, bend right] \arrow[dll] \arrow[ddl] \arrow[dd] \\
(U \otimes W) \star (P \boxtimes Q) \arrow[drr] \arrow[drrr, bend right] & (U \boxtimes W) \star (P \otimes Q) & & \\
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\end{tikzcd}$$

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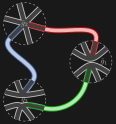

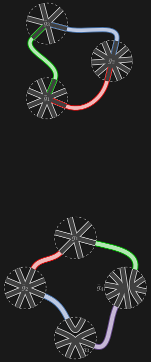

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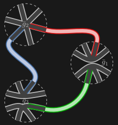

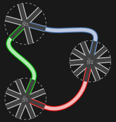

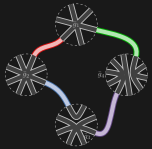

Thm. [CP 2111.02858] The algebra of Functional Renormalization is $M_k(\mathcal{A}_{N,k},\star)$ where $\mathcal{A}_{N,k} = (\mathbb{C}^{(N)}_{\langle k\rangle}\otimes\mathbb{C}^{(N)}_{\langle k\rangle})\oplus(\mathbb{C}^{(N)}_{\langle k\rangle}\boxtimes\mathbb{C}^{(N)}_{\langle k\rangle})$ which in homogeneous elements reads:

$$\begin{aligned}
(U\otimes W)\star(P\otimes Q) &= PU\otimes WQ\,,\\
(U\boxtimes W)\star(P\otimes Q) &= U\boxtimes PWQ\,,\\
(U\otimes W)\star(P\boxtimes Q) &= WPU\boxtimes Q\,,\\
(U\boxtimes W)\star(P\boxtimes Q) &= \operatorname{Tr}(WP)\,U\boxtimes Q\,.
\end{aligned}$$

Conclusion

In this talk the following was sketched:

- $w_1(M)$ and $w_2(M)$ trivial $\Rightarrow$ M spin
- spin minus commutativity $=:$ spectral triple
- spin $M \times \{\text{finite spectral triple}\} =:$ almost-commutative

  (reproduces classical Standard Model, but hard to quantize)
- finite spectral triple with $\mathbb{C}\ell$-action $\approx$ *fuzzy* or *matrix* geometry

  ($\mathcal{Z}_{\text{FUZZY}} = \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\mathrm{Tr}_H f(D)} \mathrm{d}D$ is a multi-matrix model with multi-traces)
- $G_\ell \times F=$ fuzzy $\times$ finite $=$ gauge matrix spectra triple

  (is $\mathrm{PU}(n)$-Yang-Mills-Higgs-like if F is over $M_n(\mathbb{C})$; partition function is a k-matrix model, k large)
- ~~the functional renormalization for the NCG-related multi-matrix~~ models

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thank you!