RANDOM MATRICES & TOPOLOGICAL RECURSION FOR JT-GRAVITY

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Structures CP7-Seminar on the Schwarzian Theory

I'll sketch several applications of matrix integrals, spending the last part of the talk in Jackiew-Teitelboim gravity applications as pioneered by Saad-Shenker-Stanford '19 (sss). The partition function for S_{JT} turns out to decompose in boundary wiggles and bulk. The former$|_{2\gamma=\alpha=1}$ is (Stanford-Witten '17)

$$Z^{g=0,n=1}_{\mathrm{Sch}}(\beta)=\frac{\mathrm{e}^{\frac{\pi^2}{\beta}}}{4\sqrt{\pi}\beta^{3/2}}\,,\qquad Z^{g=0,n=2}_{\mathrm{Sch}}(\beta,b)=\frac{\mathrm{e}^{-\frac{b^2}{4\beta}}}{2\sqrt{\pi\beta}}\,.$$

We focus on the remaining (according to sss, harder) integral over the bulk aided by random matrix theory.

1. Discrete surfaces

- To address the enumeration of surfaces constructed by 'gluings of polygons', we first address an simpler problem: count gluings of a rooted polygon of $2p$ sides. By a gluing, we mean pairings $\pi\in\mathcal{P}_2(2p)$ of its sides. We think of π as chords inside the polygon; 'rooted' means that the polygon is fixed while $\mathbb{Z}_{2p}$ rotates the chord diagram
- from the $(2p-1)!!=(2p)!/2^p p!=\#\mathcal{P}_2(2p)$ gluings, let $c_g(p)$ be the number of those having genus g. Call $Q_p(N)$ the generating series (a polynomial in this case) in the sense

$$Q_p(N)=\frac{1}{N^2}\sum_{g\geqslant 0}c_g(p)N^{2-2g}\,,$$

where the scalings in N (still just a formal variable to be clarified) are by convenience. Notice that for $g>0$, $c_g(p)$ are higher genus generalizations of the Catalan number $c_0(p)=\frac{1}{p+1}\binom{2p}{p}$. For instance, $N^2Q_3(N)=5N^2+10N^0$

- a further step is dropping the restriction of the polygons having $2p$-sides and summing over these

$$1+2zN+2z\sum_{p\geqslant 1}\frac{Q_p(N)}{(2p-1)!!}(Nz)^p=\left(\frac{1+z}{1-z}\right)^N \qquad (1)$$

which is Harer-Zagier '86 formula. This generating function contains all the information, since the coefficient $[z^{p+1},N^{p+1-2g}]\frac{1}{2}$RHS gives the genus-$g$ fraction of gluings of $2p$-agons for arbitrary p

- a matrix integral representation was relevant in one of the many proofs of Eq. 1. With the trace $\mathrm{Tr}(H)=\sum_{a=1}^N H_{a,a}$,

$$Q_p(N)=\textstyle\int_{M_N(\mathbb{C})_{\mathrm{s.a.}}}\frac{1}{N}\mathrm{Tr}(H^{2p})\,\mathrm{d}\mu(H)=:\langle\frac{1}{N}\,\mathrm{Tr}\,H^{2p}\rangle_{\text{Gauß}},$$

where $\mathrm{d}\mu(H)$ is the normalized Gaußian measure $\mathrm{d}\mu(H)=K_N\exp\big[-(N/2)\,\mathrm{Tr}\,H^2\big]\mathrm{d}H$. While in order to get Formula 1 one has to work more, the matrix integral representation is readily obtained via $\langle H_{a,b}H_{c,d}\rangle_{\text{Gauß}}=\frac{1}{N}\delta_{a,d}\delta_{b,c}$ and

$$\langle H_{a_1,b_1}\cdots H_{a_{2p},b_{2p}}\rangle_{\text{Gauß}}=\sum_{\pi\in\mathcal{P}_2(2p)}\prod_{(i,j)\in\pi}\langle H_{a_i,b_i}H_{a_j,b_j}\rangle_{\text{Gauß}}$$

- if one allows connected 'gluings' of several polygons, the natural concept is *combinatorial map* $G=(J,\phi,\tau)$, where $J=\{1,\dots,h\}$ is the set of $h\in 2\mathbb{N}$ half-edges and $\phi,\tau\in\mathfrak{S}_h=\mathrm{S}_h$, being τ free from free ponts and $\tau^2=1$. The faces, edges and vertices of the map are the cycles (denoted $\mathcal{C}$) of ϕ,τ and $\upsilon=\phi\circ\tau$, respectively. Thus $\#\mathcal{C}(\upsilon)-\#\mathcal{C}(\tau)-\#\mathcal{C}(\phi)=\chi(G)=2-2g$. For instance, $J=\{1,\dots,6\},\phi=(162435),\tau=(14)(25)(36)$ describe a map with $\chi($$)=0$, since $\upsilon=(132)(465)$
- to generate maps, one introduces a potential $V(x)=\sum_{0<k\leqslant d}t_kx^k/k$ which yields a new partition function $\mathcal{Z}=C_N\int_{M_N(\mathbb{C})_{\mathrm{s.a.}}}\mathrm{e}^{-NV(H)}\mathrm{d}H$. Maps (with ∂) are counted by

$$\langle\mathrm{Tr}\,H^{\ell_1}\cdots\mathrm{Tr}\,H^{\ell_n}\rangle=:\textstyle\sum_{g\geqslant 0}N^{2-2g-n}\mathcal{T}^{(g)}_{\ell_1,\ell_2,\dots,\ell_n}$$

where the LHS is computed with $\langle P(H)\rangle:=\mathcal{Z}^{-1}\int P(H)\mathrm{e}^{NV(H)}\mathrm{d}H$. These can be obtained when $\partial_\ell:=\partial/\partial t_\ell$ hits the partition function $\mathcal{Z}$,

$$N^2\mathcal{T}_{\ell_1\ell_2}=\ell_1\ell_2\cdot\partial_{\ell_1}\partial_{\ell_2}\mathcal{Z} \qquad (2)$$

- with $\ell=(\ell_2,\dots,\ell_n)$, *Tutte Equations*$\big|_{t_1=0,t_2=-1}$ read

$$\mathcal{T}^{(g)}_{\ell_1+1,\ell}=\sum_{j=3}^d t_j\cdot\mathcal{T}^{(g)}_{\ell_1+j-1,\ell}+\sum_{c=2}^k\ell_c\cdot\mathcal{T}^{(g)}_{\ell_1+\ell_c-1,\ell_2,\dots,\widehat{\ell_c},\dots,\ell_n}+\sum_{\substack{p,q\text{ with}\\ p+q=\ell_1-1}}\Big\{\sum_{\substack{h_1+h_2=g\\ I\cup J=\ell}}\mathcal{T}^{(h_1)}_{p,I}\times\mathcal{T}^{(h_2)}_{q,J}+\mathcal{T}^{(g-1)}_{p,q,\ell}\Big\}$$

- to obtain these (with $t_1,t_2+1\neq 0$) one can use *Schwinger-Dyson equations* (SDE), sketched next: from $\int\mathrm{d}(X\mathrm{e}^{-S/\hbar})=0$, it holds $\int\big[\operatorname{div}X-\frac{1}{\hbar}\mathrm{d}S(X)\big]\mathrm{e}^{-S/\hbar}=0$. So $\langle\operatorname{grad}S(X)\rangle=\hbar\langle\operatorname{div}X\rangle$. For $\hbar\to 0$, the SDE yield $\langle$classical EOM$\rangle$
- using relations like e.g. (2), Tutte Equations can be restated as differential operators, $\mathcal{L}_k$, $k=-1,0,1,2\dots$ that annihilate the partition function $\mathcal{Z}'=\exp(N^2t_0)\mathcal{Z}$, $\mathcal{L}_k\mathcal{Z}'=0$.

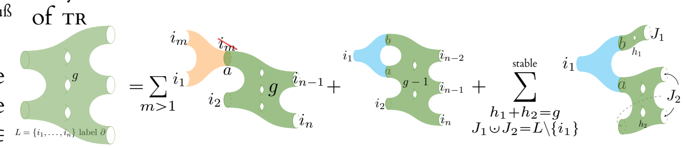
Omitting the cases $k=0,\pm1$, $\mathcal{L}_k$ is given by $\quad (k>1)$

$$\sum_{j=1,\dots,k-1}\frac{j(k-j)}{N^2}\partial_j\partial_{k-j}+\frac{2k}{N^2}\partial_k\partial_0+\sum_{j\in\mathbb{N}}(j+k)t_j\partial_{j+k},$$

which satisfies $[\mathcal{L}_p,\mathcal{L}_q]=(p-q)\mathcal{L}_{p+q}$ (if $\mathcal{L}_{-1,0,1}$ are added) for $p,q\in\mathbb{Z}_{\geqslant -1}$ i.e. non-central $\mathfrak{vir}$

2. Airy structures and Topological Recursion (TR)

Airy structures Kontsevich-Soibelman '17 extract the essence of TR

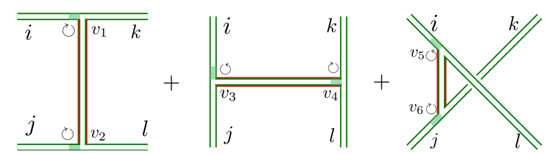


- Let W^* be the vector space with basis $\{t_j\}_{j=1}^d$. If $\hbar$ is a formal variable, an *Airy structure* on W is a family of operators $\{L_k\}_k$ on $\mathrm{Sym}(W^*)[[\hbar,\hbar^{-1}]]$ of the form

$$L_k=\hbar\partial_k-\frac{1}{2}(\mathbf{t},A^k\mathbf{t})-\hbar(\mathbf{t},B^k\partial)-\frac{\hbar^2}{2}(\partial,C^k\partial)-\hbar D_k\,,$$

such that $[L_i,L_j]=\hbar\sum_k f^k_{i,j}L_k$, being $f^k,A^k,B^k,C^k\in M_d(\mathbb{C})$, where A^k and C^k are symmetric, while f^k is skew-symmetric for each k (not a matrix index nor exponent)

- The Lie algebra condition implies that A, seen as a tensor, is fully symmetric; that $f^k_{i,j}=B^i_{j,k}-B^j_{i,k}$; and three IHX-relations described next. To the six vertices one associates letters. Red edges have indices that run. Further, the indices of each letter O at the vertices $O^{\boxed{1}}_{\boxed{2}\boxed{3}}$ is determined in the sense of the arrow, starting at the shaded edge. The IHX-relation for $(v_1,v_2,\dots,v_6)=(B,B,B,B,C,A)$ is that $\sum_{a=1}^d B^i_{j,a}A^a_{k,l}+B^i_{k,a}A^j_{a,l}+B^i_{l,a}A^j_{a,k}$ is $(i\leftrightarrow j)$-symmetric. Similar relations hold for $(v_1,v_2,\dots,v_6)=(B,A,B,B,B,A)$ and (C,B,B,C,C,B). The name 'Airy' comes from the $W=\mathbb{C}$ case applied to the next theorem.



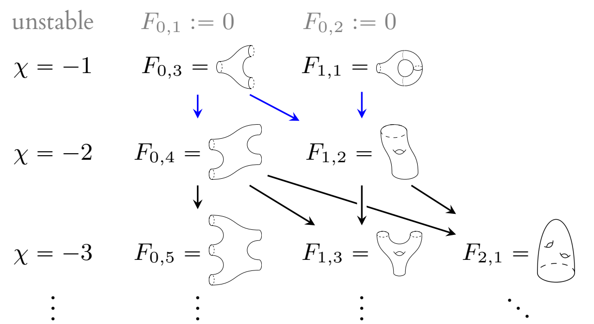
(Other diagrams describing D appear, but are omitted here)

- Thm (Kontsevich-Soibelman) *There exists a unique* $\hbar^{-1}F\in\mathrm{Sym}(W^*)[[1/\hbar]]$ *such that* $\{L_j\mathrm{e}^F=0\}_{j=1,\dots,d}$

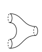
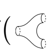
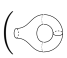
Proof sketch of uniqueness (cf. Borot's '17 lectures). Expand $F=\sum_{g\geqslant 0}\hbar^{1-g}\sum_{n\geqslant 1}\sum_{I,\#I=n}F_{g,n}[I]t_I/n!$, with $I=(i_1,\dots,i_n)\in\{1,\dots,d\}^n$ and $t_I=t_{i_1}\cdots t_{i_n}$ in multi-index notation, and read off the coefficient of $\hbar^g \times t_{i_2} \cdots t_{i_n}/(n-1)!$ in $\exp(-F) L_{i_1} \exp(F) = 0$. This yields $F_{0,3}[i_1, i_2, i_3] = A^{i_1}_{i_2,i_3}$ and $F_{1,1}[i_1] = D_{i_1}$ for $\chi = -1$, while for higher $-\chi$, $F_{g,n}[i_1, i_2, \ldots, i_n]$ is determined by recursion and equals

$$\sum_{m=2}^{n} B^{i_1}_{i_m,a} F_{g,n-1}[a, i_2, \ldots, \widehat{i_m}, \ldots, i_n] \quad (j_q := \#J_q,\ q=1,2)$$
$$+ \frac{1}{2} C^{i_i}_{a,b} \Big\{ F_{g-1,n+1}[a, b, i_2, \ldots, i_n] \tag{3}$$
$$+ \sum_{\substack{h_1+h_2=g \\ J_1 \cup J_2 = \{i_2,\ldots,i_n\}}} F_{h_1,1+j_1}[a, J_1] \times F_{h_2,1+j_2}[b, J_2] \Big\}$$

- adj. 'topological' explained by excisions of 'pair of pants' (Y)



- the boundaries above are not oriented; but, parenthetically, the ABCD-terms could stem from a TQFT $\mathcal{F} : \mathsf{Bord}_2 \to \mathsf{Vect}_{\mathbb{C}}$

$A = \mathcal{F}\big($  $\big)$, $B = \mathcal{F}\big($  $\big)$, $C = \mathcal{F}\big($  $\big)$, $D = \mathcal{F}\big($  $\big)$

3. The volume of the moduli space $\mathcal{M}_{g,n}(L)$

- $\mathscr{T}_{g,n}(L) = \{$metrics on $\Sigma_{g,n}$: length of boundary $b_j = L_j\}/\{$conformal maps$\}$, with $L = (L_1, \ldots, L_n)$
- $\Gamma_{g,n} = \{\mathrm{Diff}(\Sigma_{g,n})$ that keep labels$\}/\{$isotopies to $\mathrm{id}_{\Sigma_{g,n}}\}$
- $\mathcal{M}_{g,n}(L) = \mathscr{T}_{g,n}(L)/\Gamma_{g,n}$ = Teichmüller/mapping class
- decomposition of a stable surface $\Sigma_{g,n}$ in simple closed curves yields p Y-pieces, each having Euler number -1, so $p = -\chi(\Sigma_{g,n})$. From the $3p$ geodesic boundaries, n are *not* glued, so there are $\frac{1}{2}(p-n) = 3g+n-3 := d_{g,n}$ inner pairings of cycles, whose lengths ℓ_j can coincide. The twisting angle θ_j of one cycle with respect to the other is another parameter
- $\{\ell_j, \theta_j\}_{j=1,\ldots,3g+n-3}$ are in fact the *Fenchel-Nielsen* coordinates of $\mathscr{T}_{g,n}$. The form $\omega_{\mathrm{WP}} = \sum_j \mathrm{d}\ell_j \wedge \mathrm{d}\theta_j$ is $\Gamma_{g,n}$-invariant, as shown by Wolpert '85, and $\omega_{\mathrm{WP}}^{\wedge d_{g,n}}/d_{g,n}!$ defines the volume form of $\mathcal{M}_{g,n}(L)$ and $V_{g,n}(L) = \mathrm{vol}[\mathcal{M}_{g,n}(L)]$

- Mirzakhani '06 TR states that $V_{g,n+1}(L_0, L)$ equals

$$\sum_{m>0} \int_{\mathbb{R}_+} B_{\mathrm{Mirz}}(L_0, L_m, \ell) V_{g,n}(\ell, L_1, \ldots, \widehat{L_m}, \ldots, L_n)\, \mathrm{d}\ell$$
$$+ \frac{1}{2} \int_{\mathbb{R}^2_+} C_{\mathrm{Mirz}}(L_0, \ell, \ell') \Big[V_{g-1,n+2}(\ell, \ell', L_1, \ldots, L_n)$$
$$+ \sum_{\substack{h_1+h_2=g \\ J_1 \cup J_2 = \{L_j\}_{j=0}^n}} V_{h_1,1+j_1}(\ell, J_1) V_{h_2,1+j_2}(\ell', J_2) \Big]\, \mathrm{d}\ell \mathrm{d}\ell',$$

where $B_{\mathrm{Mirz}}(L_1, L_2, L_3)$ and $C_{\mathrm{Mirz}}(L_1, L_2, L_3)$ are given by

$$\frac{L_3}{L_1} \log \frac{[1 + \mathrm{e}^{(L_3+L_2-L_1)/2}][1 + \mathrm{e}^{(L_3-L_2-L_1)/2}]}{[1 + \mathrm{e}^{(L_3+L_2+L_1)/2}][1 + \mathrm{e}^{(L_3-L_2+L_1)/2}]}$$

and $\quad 2\frac{L_2 L_3}{L_1} \log \frac{1 + \mathrm{e}^{(L_3+L_2-L_1)/2}}{1 + \mathrm{e}^{(L_3+L_2+L_1)/2}}$, respectively

- keeping the coefficients of the volumes as amplitudes,

$$V_{g,n}(L) = \sum_{a_1,\ldots,a_n \geqslant 0} F_{g,n}[a_1, \ldots, a_n] \prod_{j=1}^{n} L_j^{2a_j},$$

Mirzakhani's TR and (3) lead to the Airy structure:

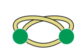
$$B^i_{j,k} = \frac{(2k+1)!}{(2i+1)!(2j+1)!} (2j+1)\, \theta_{k-j-i} \tag{4a}$$
$$C^i_{j,k} = \frac{(2j+1)!(2k+1)!}{(2i+1)!} \theta_{k+j+1-i}, \tag{4b}$$

where $\sum_{k+1\geqslant 0} z^{2k}\theta_k/\mathrm{d}z = 4\pi/\sin(2\pi z)\mathrm{d}z^2 =: 1/y(z)\mathrm{d}z^2$. But we need the initial A and D terms

- maps, or ribbon graphs, are useful to compute also intersection numbers. Kontsevich's '93 remarkable formula helps

$$\sum_{\substack{a_j \in \mathbb{Z}_{\geqslant 0}, \text{for all } j \\ a_1+\ldots+a_n = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^{n} \frac{(2a_j-1)!!}{\lambda_j^{2a_j+1}}$$
$$= \sum_{G \text{ trivalent, of topology } (g,n)} \frac{2^{2g-2+n}}{\#\mathrm{Aut}(G)} \prod_{e \in E(G)} \frac{1}{\lambda_{L(e)} + \lambda_{R(e)}}$$

- there is a unique $(1,1)$-graph:

$G =$  $\quad \Rightarrow \quad \frac{1}{\lambda^3} \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{2^1}{\#\mathrm{Aut}(G)} \frac{1}{(2\lambda)^3}$

$$\mathrm{Aut}(G) = \{\psi \in \mathfrak{S}_6 : \text{ commuting with } \phi \text{ and } \tau\}$$
$$= \{\mathrm{id}, (123)(456), (132)(465), \tau, \phi, \phi^{-1}\}$$

then $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24$, so $D_1 = 1/24$. Penner '85 computed $V_{1,1}(0) = \zeta(2)$ implying $D_0 = \pi^2/6$. Else $D_k = 0$ for $k > 1$

- four graphs of $(0,3)$-type $\Rightarrow A^{i_1}_{i_2,i_3} \neq 0$ iff $i_* = 0$ $(A^0_{0,0} = 1)$

4. Back to JT-Gravity (No longer addressed in the talk)

- Recall $\mathrm{e}^{-S_{\mathrm{JT}}} = (\mathrm{e}^{S_0})^{\chi(M)} \mathrm{e}^{\frac{1}{2}\int_M \varphi(R+2) + \int_{\partial M}(K-1)}$. Whilst in the path-integral the green term leads to 'wiggles', the red one yields a 'functional $\int \mathrm{e}^{\mathrm{i}xy}\mathrm{d}x \sim \delta(y)$', thus—unlike $V_{0,1}$, $V_{0,2}$—the volume of all such stable M does exists
- in random matrix theory $\chi(M)$ is the exponent of N = matrix size, but 'double scaling' in such a way that density of eigenvalues keeps normalized implies N depends on e^{-S_0}, which shall replace $1/N$ as expansion parameter
- the relation to random matrix theory reads

$$R_{\substack{g\\n}}(E_1, \ldots, E_n) = \int_{\mathbb{R}_+} V_{g,n}(L) \prod_{j=1}^{n} \frac{(-L_j)}{2z_j} \mathrm{e}^{-L_j z_j} \mathrm{d}L_j,$$

where $z_j^2 = -E_j$ and if $R(E) = \langle \mathrm{Tr}[1/(E-H)] \rangle$ and

$$\langle R(E_1) \cdots R(E_n) \rangle_c = \sum_{g \in \mathbb{Z}_{\geqslant 0}} (\mathrm{e}^{-S_0})^{2-2g-n} R_{\substack{g\\n}}(E_1, \ldots, E_n)$$

and the LHS is the connected part of the double-scaled ensemble with leading eigenvalue density

$$\rho_0(E) = \frac{\sinh(2\pi E^{1/2})}{(2\pi)^2}, \qquad y = -\pi \mathrm{i} \rho_0(E)$$

which, in terms of z (with $z^2 = -E$), yields the function that generates the Mirzakhani-Airy structure (4)

- from inverting $R(E) = -\int_0^\infty \mathfrak{z}(\beta)\mathrm{e}^{\beta E}\mathrm{d}\beta$ one can compute the correlators $\langle \mathfrak{z}(\beta_1) \cdots \mathfrak{z}(\beta_n) \rangle_{c,\mathrm{JT}}$ where $\mathfrak{z}(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta H})$ and the random matrix is thought of as Hamiltonian at ∂M
- Eynard-Orantin TR-theory: in terms of residues of $W_{\substack{g\\n}}(z_1, \ldots, z_n) = (-2z_1) \cdots (-2z_n) R_{\substack{g\\n}}(-z_1^2, \ldots, z_n^2)$ is

$$W_{\substack{g\\n}}(z_1, K) = \operatorname*{Res}_{z \to 0} \Big\{ \frac{1}{(z_1^2 - z^2)} \frac{1}{4y(z)} \Big[W_{\substack{g-1\\n+1}}(z, -z, K)$$
$$\#K = n-1 \qquad + \sum_{\substack{J_1 \cup J_2 = K \\ h_1+h_2=g}}^{\text{stable}} W_{\substack{h_1\\1+j_1}}(z, J_1) \times W_{\substack{h_1\\1+j_2}}(-z, J_2) \Big] \Big\}$$