Counting maps

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Motivating the Topological Recursion

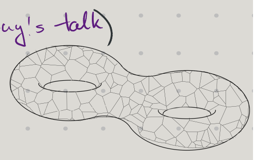
Based on <u>Eynard's</u> Book (Springer '16)* free also at his site

but also ..
- Eynard - Kimura - Ribault (Lect on Random Matrices)
- Video lectures: Eynard, Orantin Lectures (YouTube)
- E. García Failde's PhD thesis (Univ. classes, OCW, ...)

arXiv:2002.00316

Outline

Ch 1 Intro to maps (Today's talk)
& Tutte's equations



Ch 2. Matrix integrals & loop eqs'

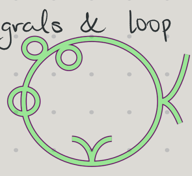
Comb

Feyn

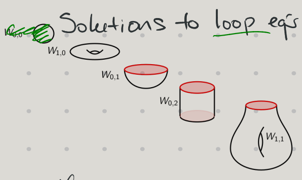
Virasoro

↕

Loop / Schwinger - Dyson eqs

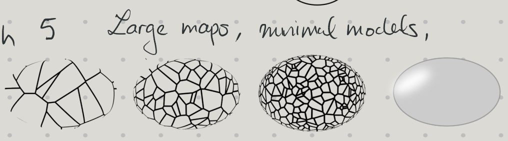


Ch 3 Solutions to loop eq's



$W^{(g)}_n$

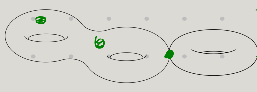
Ch 5 Large maps, minimal models,



Criticality

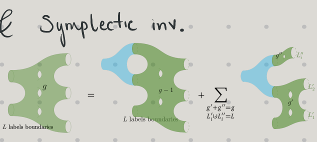
Minimal models (CFT, Liouville th)

Ch 6 Counting Riemannian Surfaces

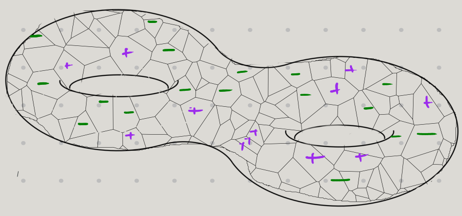
$\mathcal{M}_{g,n} = \{ \Sigma_{g,n} \mid$  $\}$

Ch 7 TR & Symplectic inv.

$W^{(g)}_n \sim$



Ch 8 Ising Model ~ 2MM



$\mathrm{Tr}\, \{ A^2 + B^2 + V(A) + U(B) + AB \}$

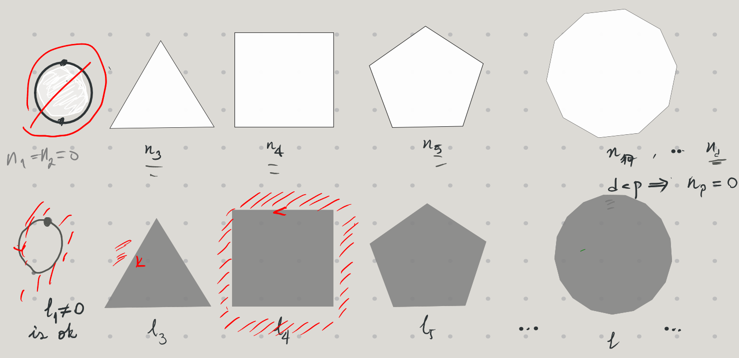
Ch 1. Intro to maps & Tutte's equations

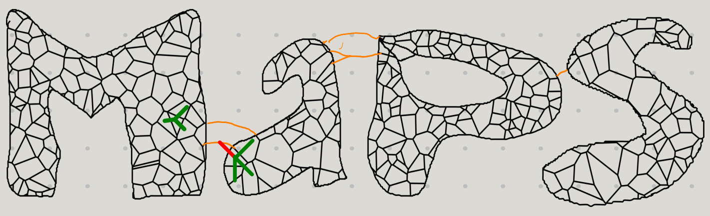
Tutte [Census-paper series, '62,'63] counted *maps*. For instance, the number of planar triangulations with $2n$ faces (one of which marked) is "rooted"

$$\mathcal{T}_{k=1}^{(g=0)} = \frac{1}{t_3} \sum_{n=1}^{\infty} \underbrace{\frac{2^{3n+1}\Gamma(3n/2+1)}{(n+2)! \cdot \Gamma(n/2+1)} t_3^{2n}}$$

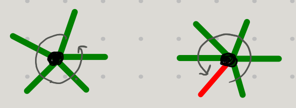
Extract from formal matrix models (combinatorics)
($\neq$ random matrices) harder

Intuitively, gluings of oriented polygons





· Connected maps $\Rightarrow$ log to retract

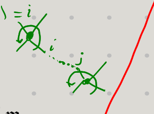


cyclic order

Def. A ~~labeled~~ *map* is a triad $\mathfrak{m} = (H, \nu, \varepsilon)$ satysfing :

- H is a set of even cardinality $D = \#H \in 2\mathbb{N}$
- $\nu \in \mathrm{Sym}(H) = \mathfrak{S}_D$ is a permutation in this set
- ε is an involution without fixed points in there

$\varepsilon^2 = 1$ ↳ $\nexists i$ $\varepsilon(i) = i$



$\sim$

- the cycles of ν are the vertices of $\mathfrak{m}$
- the cycles of ε are the edges of $\mathfrak{m}$
- the cycles of $\varphi^{-1} = \underbrace{\varepsilon \ \nu}_{\nu \circ \varepsilon}$ are the faces of $\mathfrak{m}$

N.B. Eynard gives the def. in terms of $(H, \sigma_1, \sigma_1) = (H, \varphi = \varepsilon \circ \nu^{-1}, \varepsilon)$

mod iso of H (half-edges renaming) $\psi : H \xrightarrow{\sim} H'$ $\mathfrak{m}' \sim \mathfrak{m}$

such that
- $\psi \circ \nu \circ \psi^{-1} = \nu'$
- $\psi \circ \varepsilon \circ \psi^{-1} = \varepsilon'$

$\Rightarrow$ (edges, vert., faces) are thus preserved !

We use the same notation for both : $[\mathfrak{m}] = \mathfrak{m}$

Def. A ~~labeled~~ *map* (with ∂) is a triad $\mathfrak{m} = (H, \nu, \varepsilon)$ satysfing :

- H is a set of even cardinality $D = \#H \in 2\mathbb{N}$ $H = H_{in} \cup H^{\partial}$
- $\nu \in \mathrm{Sym}(H) = \mathfrak{S}_D$ is a permutation in this set
- ε is an involution without fixed points in there

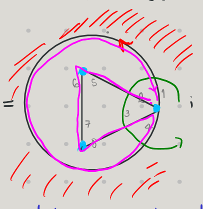
- the cycles of ν are the vertices of $\mathfrak{m}$
- the cycles of ε are the edges of $\mathfrak{m}$
- the cycles of $\varphi^{-1} = \underbrace{\varepsilon \ \nu}$ are the faces of $\mathfrak{m}$ are having at most one element of H^{∂}

~~*N.B.* Eynard gives the def. in terms of $(H, \sigma_1, \sigma_1) = (H, \varphi = \varepsilon \circ \nu^{-1}, \varepsilon)$~~

n-gon a $\varphi^{-1} = (\ \ )(\overset{\leftarrow n \rightarrow}{\dots})$

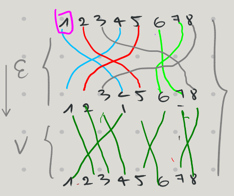
(cycle of length n in φ^{-1}

Ex. How many polygons, of given degree, does next map have?

$\mathfrak{m} =$  ; find $\mathfrak{m} = (H, \nu, \varepsilon)$

$H^\partial = \{1\}$

The permutations are given by...



$\nu = (1234)(56)(78)$

$* \ \varepsilon = (14)$ r-gon square triangle

$\varphi^{-1} = \boxed{(1)}(2684)(375)$

$\boxed{\varphi^{-1} = \nu \circ \varepsilon}$

for each $y, x \in H$

cartographic gr.

$\parallel$

Def. A map $\mathfrak{m}$ is *connected* if $\langle \nu, \epsilon \rangle$ acts transitively in the set of half-edges. The *Euler characteristic* of $\mathfrak{m}$ is

$$\chi(\mathfrak{m}) = \#\text{vertices} - \#\overset{\text{edges}}{\cancel{\text{vertices}}} + \#\text{unmarked faces} = 2 - 2g - k$$

$$\pi_0(\mathfrak{m}) = \#\,\text{orbits of } \langle \epsilon, \nu \rangle \qquad |\mathcal{C}(\varphi)|$$

Ex: find a genus 1 map (Hint □ "identifying ∂")

Aut (m) and symmetry factors

Def. An *automorphism* of a map $\mathfrak{m}$ is a bijection ψ of the set $H_{in} \cup H^\partial$ of half-edges that preserves ϵ and ν and fixes the boundary

$$\psi|_{H^\partial} = \mathrm{Id}|_{H^\partial} \qquad \psi \cdot \sigma \cdot \psi^{-1} = \sigma \in \langle \nu, \epsilon \rangle$$

The number of automorphisms of a map is its symmetry factor.

$$\Downarrow \qquad [\psi, \sigma] = \cancel{id}$$

Prop. *Maps with boundary have trivial automorphism group*

Proof Take $x \in H^\partial$ and ψ aut. $\Downarrow$ $\psi(x) = x$;

take $y \in H$

$y = (\nu\, \varepsilon\, \nu\, \varepsilon \dots) x$

$\psi(y) = \psi(\nu^1 \varepsilon^0 \nu^1 \varepsilon^1 \cdots \varepsilon^1 \cdots \nu^0)(x)$

$= (\nu^1 \varepsilon^0 \cdots \varepsilon\, \nu^1 \cdots \nu^0)\underbrace{\psi(x)}_{x}$

$= y \qquad \psi = \mathrm{id}$ ☒

Maps for a few vertices

$\mathbb{M}^{(g=0)}_{k=1}(v=1) := \{ \bullet \}$

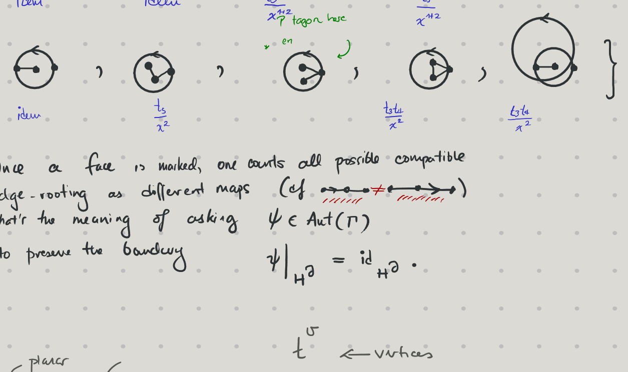
$\mathbb{M}^{g}_{k}(v)$ = set of gen.-g maps with k marked faces & v vertices

$\mathbb{M}^{(g=0)}_{k=1}(2) = \{\ \ ,\ \ \}$

$\mathbb{M}^{(g=0)}_{k=1}(v=3) = \{\ \ \}$

$n_j = \#\, j$-gons

(Why to use TikZ? Because this happens if I don't)



Rem Once a face is marked, one counts all possible compatible edge-rooting as different maps (cf ⟶• ≠ •⟶) That's the meaning of asking $\psi \in \mathrm{Aut}(\Gamma)$ to preserve the boundary $\psi|_{H^\partial} = \mathrm{id}_{H^\partial}$.

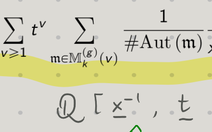
$t^v \leftarrow$ vertices

planar

$$\underset{\text{rooted}}{W^{(g=0)}_{k=1}}(x) = \frac{t}{x} + t^2\left(\frac{1}{x^3} + \frac{t_3}{x^2}\right)$$
$$+ t^3\left(\frac{2}{x^5} + \frac{4t_3}{x^4} + \frac{t_3^2}{x^3} + \frac{2t_4}{x^3} + \frac{2t_5}{x^2} + \frac{2t_3 t_4}{x^2}\right) + \mathcal{O}(t^4)$$

free energy

$$F_0 = W^{(g=0)}_{k=0} = t^3\left[\frac{1}{2}t_4 + \left(\frac{1}{6} + \frac{1}{2}\right)t_3^2\right] + \mathcal{O}(t^3)$$



fix k bdries gen g gen func. formal sum.

$$\boxed{W^{(g)}_k(x_1, \cdots, x_k) = \sum_{v \geqslant 1} t^v \sum_{\mathfrak{m} \in \mathbb{M}^{(g)}_k(v)} \frac{1}{\#\mathrm{Aut}\,(\mathfrak{m})} \frac{t_3^{n_3(\mathfrak{m})} \cdots t_d^{n_d(\mathfrak{m})}}{x_1^{1+\ell_1(\mathfrak{m})} \cdots x_k^{1+\ell_k(\mathfrak{m})}}}$$

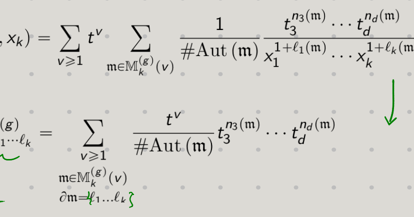
$$\mathbb{Q}[\underline{x}^{-1}, \underline{t}\,][[t]]$$

$\Uparrow$

Def. and Prop. *The following set is finite:*
$\mathbb{M}^{(g)}_k(v) = \{\mathfrak{m} \mid \mathfrak{m}$ *has genus g and k boundaries with v vertices*$\}$

Proof:

$$v - \chi_{g,k} \geqslant \frac{1}{2}\left[\sum_{i=3}^{d} n_i(\mathfrak{m}) + \sum_{\alpha=1}^{k} \ell_\alpha\right]$$



☒

$$W^{(g)}_k(x_1, \cdots, x_k) = \sum_{v \geqslant 1} t^v \sum_{\mathfrak{m} \in \mathbb{M}^{(g)}_k(v)} \frac{1}{\#\mathrm{Aut}\,(\mathfrak{m})} \frac{t_3^{n_3(\mathfrak{m})} \cdots t_d^{n_d(\mathfrak{m})}}{x_1^{1+\ell_1(\mathfrak{m})} \cdots x_k^{1+\ell_k(\mathfrak{m})}}$$

$\downarrow \ \mathrm{Res}_{x_i \to \infty} \underline{x}^{\ell} W^{(g)}_k$

$$\hookrightarrow \quad \underset{\text{fixed perimeter}}{T^{(g)}_{\ell_1 \cdots \ell_k}} = \sum_{\substack{v \geqslant 1 \\ \mathfrak{m} \in \mathbb{M}^{(g)}_k(v) \\ \partial\mathfrak{m} = \{\ell_1 \dots \ell_k\}}} \frac{t^v}{\#\mathrm{Aut}\,(\mathfrak{m})} t_3^{n_3(\mathfrak{m})} \cdots t_d^{n_d(\mathfrak{m})}$$

Tutte's equations

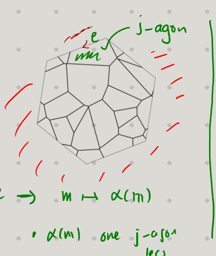
Thm. (Tutte, early 60's) *The generating series of maps satisfy.*

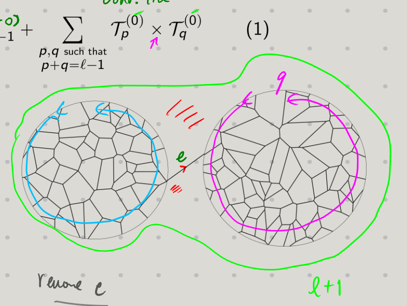
- Planar rooted maps :

$$\mathcal{T}^{(g=0)}_{\ell+1} = \sum_{j\geqslant 3}^{d} t_j \mathcal{T}^{(g=0)}_{\ell+j-1} + \sum_{\substack{p,q \text{ such that} \\ p+q=\ell-1}} \mathcal{T}^{(0)}_p \times \mathcal{T}^{(0)}_q \qquad (1)$$

Conv. like

"Proof": (Planar case)





remove $e \to m \mapsto \alpha(m)$

- $\alpha(m)$ one j-agon less
- $\ell+1 \to \ell+j-1$

remove e

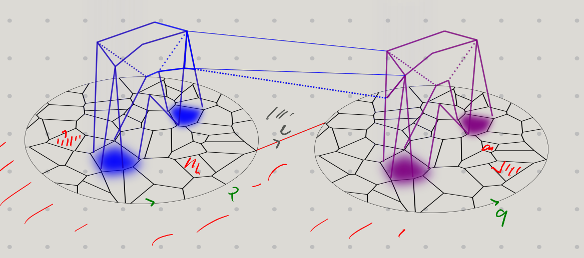
- total perm $\ell-1 = p+q$

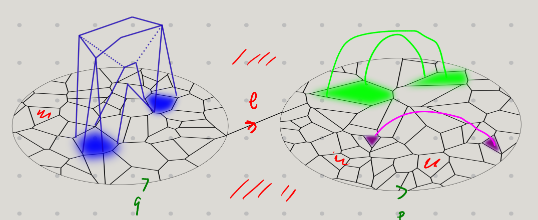
$$\alpha: \mathbb{M}^{(0)}_{\ell+1} \to \bigcup_{j=3}^{d} \mathbb{M}^{(0)}_{\ell+j-1} \cup \left(\mathbb{M}^{(0)}_p \sqcup \mathbb{M}^{(0)}_q\right)$$

- Genus g with arbitrary number (k) of boundaries:

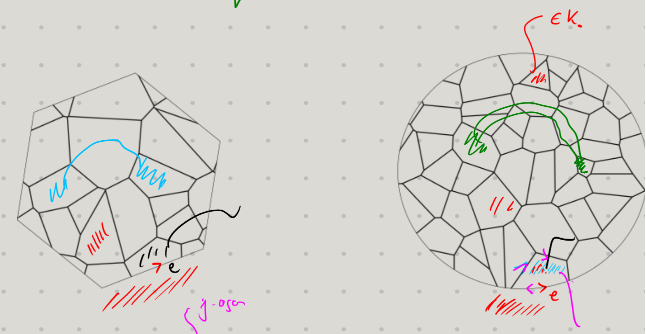
$$\mathcal{T}^{(g)}_{\ell_1+1,\underbrace{\ell_2,\ldots,\ell_k}_{K}} = \sum_{j\geqslant 3} t_j \mathcal{T}^{(g)}_{\ell_1+j-1,K} + \sum_{c=2}^{k} \ell_c \cdot \mathcal{T}^{(g)}_{\ell_1+\ell_c-1,\ell_2,\ldots,\widehat{\ell_c},\ldots,\ell_k}$$
$$+ \sum_{\substack{p,q \text{ such that} \\ p+q=\ell_1-1}} \left\{ \sum_{\substack{h_1+h_2=g \\ I\,\dot\cup\, J=K}} \mathcal{T}^{(h_1)}_{p,I} \times \mathcal{T}^{(h_2)}_{q,J} + \mathcal{T}^{(g-1)}_{p,q,K} \right\}$$

$m \xrightarrow{\beta}$ remove the marked edge





- Genus $g_1+g_2=g$
- $\ell_1-1=p+q$



- Genus g with arbitrary number (k) of boundaries:

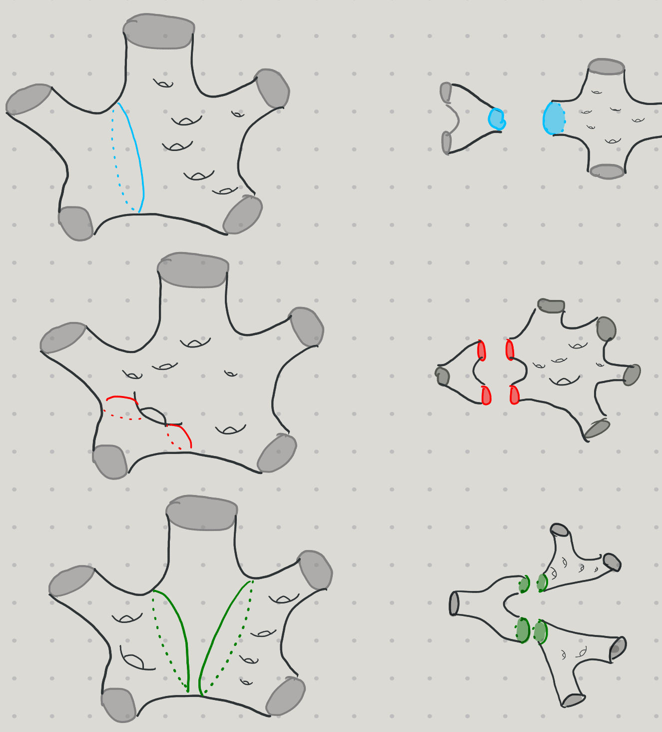
$$\mathcal{T}^{(g)}_{\ell_1+1,\ell_2,\ldots,\ell_k} = \sum_{j\geqslant 3} \boxed{t_j} \mathcal{T}^{(g)}_{\ell_1+j-1,K} + \sum_{c=2}^{k} \ell_c \cdot \mathcal{T}^{(g)}_{\ell_1+\ell_c-1,\ell_2,\ldots,\widehat{\ell_c},\ldots,\ell_k}$$
$$+ \sum_{\substack{p,q \text{ such that} \\ p+q=\ell_1-1}} \left\{ \sum_{\substack{h_1+h_2=g \\ I\,\dot\cup\, J=K}} \mathcal{T}^{(h_1)}_{p,I} \times \mathcal{T}^{(h_2)}_{q,J} + \mathcal{T}^{(g-1)}_{p,q,K} \right\}$$

j-agon

ℓ_c

$\Sigma_{g=6,n=5}$

Excisions of
Pair of pants leaving
stable ($\chi < 0$) surfaces
has interesting combinatorics



⋆ Genus g with arbitrary number (k) of boundaries:

$$\mathcal{T}^{(g)}_{\ell_1+1,\ell_2,\dots,\ell_k} = \sum_{j\geqslant 3} \overset{\text{washed away}}{t_j}\,\mathcal{T}^{(g)}_{\ell_1+j-1,K} + \sum_{c=2}^{k} \ell_c \cdot \mathcal{T}^{(g)}_{\ell_1+\ell_c-1,\ell_2,\dots,\widehat{\ell_c},\dots,\ell_k}$$

$$+ \sum_{\substack{p,q \text{ such that} \\ p+q=\ell_1-1}} \left\{ \sum_{\substack{h_1+h_2=g \\ I\cup J=K}} \mathcal{T}^{(h_1)}_{p,I} \times \mathcal{T}^{(h_2)}_{q,J} + \boxed{\mathcal{T}^{(g-1)}_{p,q,K}} \right\}$$