Counting Large Maps

Before, we ...

$W_k^g(x_1,\dots,x_n)$ $\longleftrightarrow$ $T^{(g)}_{l_1\dots l_k} = \sum\limits_{\substack{\text{maps } m \\ \text{fixed } g \\ l_1\dots l_k}} \frac{t^{v(m)}}{\# \mathrm{Aut}(m)} t_3^{n_3(m)} \cdots t_d^{n_d(m)}$

Tutte eq. for $T^g_{\dots}$

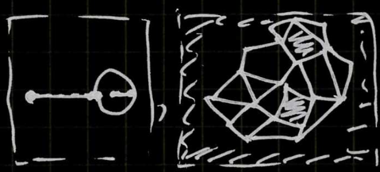
$n_i = \#\{i\text{-agons}\}$ $(d \geq i \geq 3)$

$v = \#$ vertices

ln Z_N generates $W^{(g)}_k(x_1,\dots,x_n)$ [resolvents]

Maps and Gen functions

Ch1

Map 

Formal Matrix Models

Ch2

Solutions of Loop equations & TR

Ch3

Tutte $\Leftrightarrow$ Dyson-Schwinger Eqs. (loop)

Solutions for

$W^{(0)}_1$ (disk)

$W^{(0)}_2$ (cylinder)

generate, via TR, all $W^{(g)}_k(x_1,\dots x_n)$

From JT-gravity Journal Club 2020,

we know a spectral curve $\left(x : \Sigma \to \Sigma_0,\ \omega_{0,1} \text{ on } \Sigma,\ \omega_{0,2} \text{ on } \Sigma^2 \right)$ meromorphic diffs $+ \dots$

- **Ex 1.** $\left(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = z^2,\ y(z)\,dz = \frac{2}{\pi} \sin(\pi z),\ \text{Bergman} \right)$

  known (Mirzakhani; Eynard-Orantin) to generate the recursion for the WP-volumes

- **Ex 2** $\left(\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}},\ x(z) = \alpha + \gamma (z + z^{-1}),\ y(z) = \sum u_j (z^j - z^{-j}) \right.$ and $B(z_1, z_2)$ as before $\left. \frac{dz_1 dz_2}{(z_1 - z_2)^2} \right)$

where

- α, γ and u_j are given in terms of the $V'(x(z)) = \sum_k u_k (z^k + z^{-k})$

  – α and γ solutions to algebraic equations

$$u_k = \alpha\, \delta_{k,0} + \gamma\, \delta_{k,1} + \text{Polynomial}(\gamma, \alpha, t_3, \dots, t_d)$$

- Solutions:

$$W_1^{(0)}(x(z)) = \frac{1}{2} V'(x(z)) + y(z)$$

$$W_1^{(0)}(x) = \frac{1}{2}\Big(V'(x) - \underset{\text{Polyn}}{M(x)} \sqrt{(x-a)(x-b)} \Big) \quad \text{[Brown's Lemma]}$$

Solution for cylinder Amplitude

$$W_2^{(0)}(x(z_1), x(z_2)) = \frac{1}{x'(z_1)\, x'(z_2)} \frac{1}{(z_1 - z_2)^2} - \frac{1}{(x(z_1) - x(z_2))^2}$$

leads to (universal)

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

↳ Fundamental bidiff. of the 2nd kind

↳ Merom. (1,1) form

↳ symmetric ($\Rightarrow$ no residues)

↳ double pole at diagonal

↳ normalized / monic

Large Maps

- string theory: path integrals counted modulo conformal reparam. $\Rightarrow$ Riemann surfaces

- Physicists tackled counting of Riemann surfaces by polygonal approximations:

- detect singularities $\leftrightarrow$ large size asymptotics

$$\langle k \rangle = \frac{\sum_k t \cdot k A_k t^{k-1}}{\sum_k A_k t^k} = \frac{t A'(t)}{A(t)}$$

↑ blows up if $\ln A(t)$ does

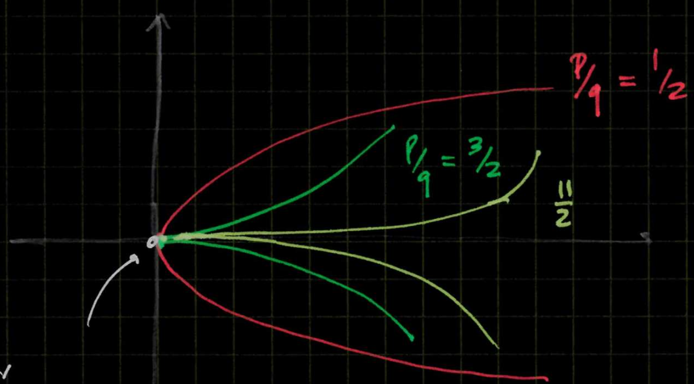
- Let's accept that

$$\frac{d\gamma}{dt} = \frac{1}{4}\left(\frac{1}{y'(1)} + \frac{1}{y'(1)}\right) \qquad \text{[Lemma 3.4.1]}$$

$\Rightarrow$ γ singular whenever (say) $y'(1) = 0$.

Taylor expanding $x(z)$ one finds $\quad z-1 \sim \sqrt{\frac{x-x(1)}{\gamma}} = \sqrt{\frac{x-a}{\gamma}}$

—— „ —— $y(z)$ —— „ —— $\quad y(z) \sim y'(1)\sqrt{\frac{x-a}{\gamma}} + \mathcal{O}((x-a)^{3/2})$



for $p>1$
no tangent space

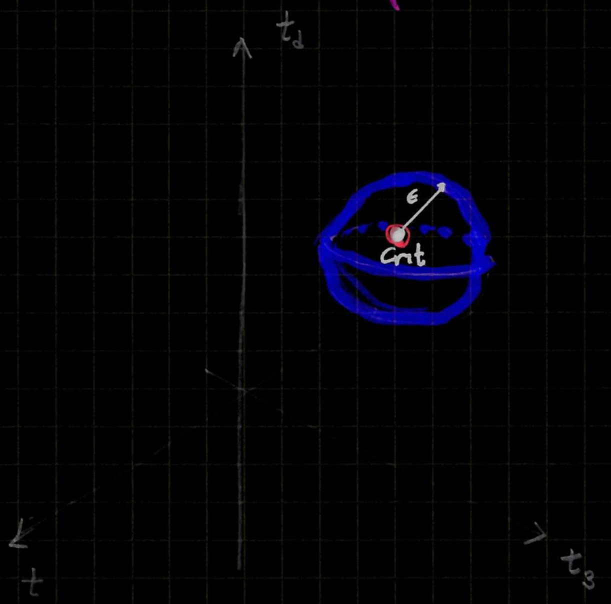
- if more derivatives $y^{(m)}(1)$ vanish then one gets a „cusp"

$$y \sim (x-a)^{\frac{2m+1}{2}} = (x-a)^{p/q}$$

$$\left.\begin{aligned} p &= 2m+1 \\ q &= 2 \end{aligned}\right\} \text{Minimal models pair}$$

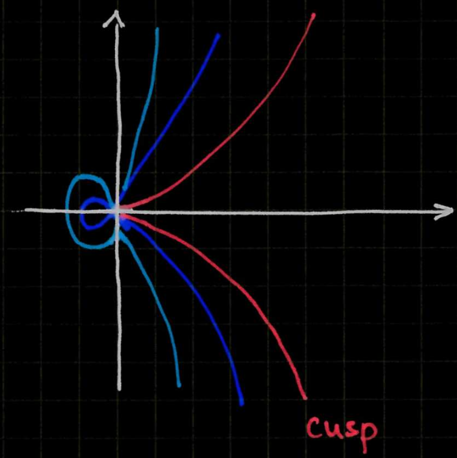
<u>Rem</u>: $O(n)$ - modells allow irrational - exponents

Blown up curve

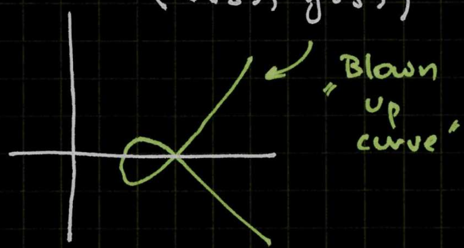


Phase space

$(x, y)_{t, t_3, \dots, t_d}$



Taylor expand
$(x(z), y(z))$
and keep ϵ^1 order
$(\tilde{x}(\zeta), \tilde{y}(\zeta))$



Blown up (quadrangulations)

- Choose ϵ ('mesh size') as $12\, t t_4 = 1 - \epsilon^2$

- The spectral curve (Ch. 3) is

$$y = -\frac{t_4}{2}\left(x^2 - 4\gamma^2 + 3\gamma^2 \frac{\gamma^2 - 2t}{\gamma^2 - t}\right)\sqrt{x^2 - 4\gamma^2}$$

where $\gamma^2 = \dfrac{1 - \sqrt{1 - 12 t t_4}}{6 t_4}$

- Reparametrize Żukowski as $z = 1 + \sqrt{\frac{\epsilon}{2}}\, \zeta + \mathcal{O}(\epsilon)$

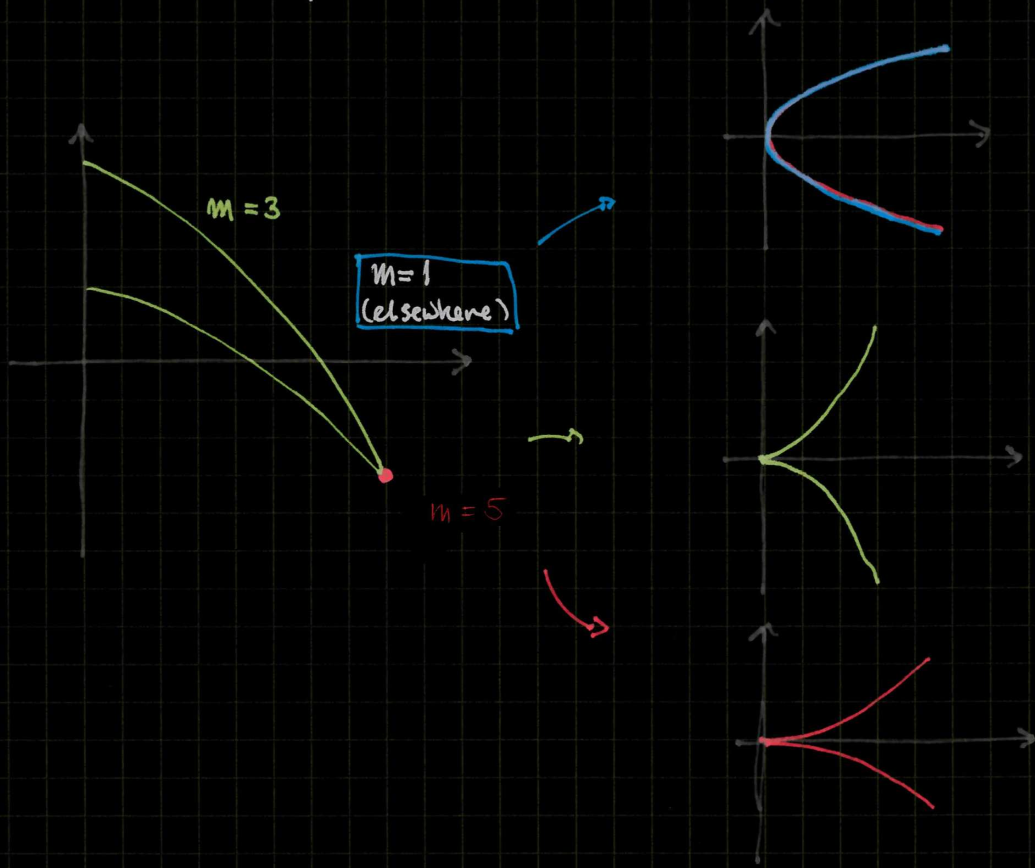
(Joukowski, Zhoukovsky, Жуковский) ...

- inserting $z(\zeta)$ in the spectral curve and keeping $\mathcal{O}(\epsilon)$ coefficients,

$$\left.\begin{aligned} \tilde{x}(\zeta) &= \sqrt{\tfrac{t}{2}}\,(\zeta^2 - 2) \\ \tilde{y}(\zeta) &= -\frac{\sqrt{t}}{3}\,(\zeta^3 - 3\zeta) \end{aligned}\right\}$$

„blown-up" of spectral curve for quadrangulations

- Enlarging the phase space might lead to multi-critical points (critical submanifolds)



- Critical subm.: reparametrization $t_i = t_i(\epsilon, \underline{\tilde{t}_1, \dots, \tilde{t}_m})$

$$\begin{cases} \tilde{x}(\zeta) = \zeta^2 - 2u \\ \tilde{y}(\zeta) = \sum_{j=0}^{\boxed{m}} \tilde{t}_j\, Q_j(\zeta) \end{cases}$$

The blown up spectral curve will be related to $(2m+1, 2)$ minimal model (Painlevé $\mathrm{I}_{(m+1)}$)

Asymptotic $W_n^{(g)}$

- We now study $W_n^{(g)}(x_1, \dots, x_n)$ in a δ-neighbourhood of a branchpoint, say $z = 1$.

$$x_i = x(1) + \gamma\, \delta^2 \zeta_i^2 + O(\delta^3)$$

where $\quad z_i = 1 + \delta \zeta_i$

- Want: asymptotics $W_n^{(g)}$ as $\delta \to 0$
  first without assuming dependence of δ on parameters of phase space.

- Taylor expansion around branchpoint $z=1$, $O(\delta')$

$$\begin{cases} x(z) \sim a + \gamma\, \delta^q \, (\overset{\zeta^2 - 2w}{\tilde{x}(\zeta)}) + O(\delta^2) \\ y(z) \sim \frac{t}{\gamma}\, \delta^p\, \tilde{y}(\zeta) \end{cases}$$

and this yields

$$B(z_0, z) = \begin{cases} \dfrac{\delta^{-2}}{(\zeta - \zeta_0)^2} \equiv \delta^{-2} \tilde{B}(\zeta_0, \zeta) & z, z_0 \text{ near } 1 \\ O(1) & \text{otherwise} \end{cases}$$

Similarly

$$K(z_0, z) \sim \begin{cases} \dfrac{1}{t}\, \delta^{-(p+q)}\, \tilde{K}(\zeta_0, \zeta) & z, z_0 \text{ near } +1 \\ \text{suppressed } [O(1) \text{ or } O(\delta^{1-p-q})] & \text{otherwise} \end{cases}$$

- with $\tilde{K}(\zeta, \zeta_0) = \frac{1}{2}\left(\frac{1}{\zeta_0 - \zeta} - \frac{1}{\zeta_0 + \zeta}\right) \frac{1}{2\hat{y}(\zeta)\,\tilde{x}'(-\zeta)}$

this was not in book

THEOREM

$$\omega_n^{(g)}(1+\delta\zeta_1, \dots, 1+\delta\zeta_n) \sim t^{2-2g-n}\,\delta^{(2-2g-n)(p+q)}\,\delta^{-n} \cdot \tilde{\omega}_n^{(g)}(\zeta_1, \dots, \zeta_n)$$

where $\tilde{\omega}_n^{(g)}$ can be computed recursively

$$\hat{\omega}_{n+1}^{(g)}(\zeta_0, J) = \operatorname*{Res}_{\zeta \to 0} \tilde{K}(\zeta_0, \zeta) \left\{ \hat{\omega}_{n+2}^{(g)}(\zeta, -\zeta, J) + \sum_{\substack{g=h+h' \\ J = I \uplus I'}}^{\text{no } (0,\emptyset)} \tilde{\omega}^{h}_{1+|I|}(\zeta, I)\, \tilde{\omega}^{h'}_{1+|I'|}(-\zeta, I') \right\}$$

Notice the involution $z \to z^{-1}$ was replaced by $\zeta \mapsto -\zeta$.

Double scaling limit of $\mathcal{F}_g$

- We now make $\delta \sim (1 - t/t_c)^{\nu}$; δ now depends on

  and $t_k = t_k^{crit} + \sum_j C_{k,j}\, \delta^{\nu_j}\, \tilde{t}_j$

  [unclear to me how to practically obtain C, $\tilde{t}$, ...]

- Assume proven Lemma 3.4.1,

$$\partial_z x\, \partial_t y - \partial_t x\, \partial_z y = 1/z$$

- Rewriting in terms of Q_k (← appear in blown up curve) yields

$$\nu = \frac{1}{p+q-1} \quad \& \quad \nu_j = 2(m-j)$$

THEOREM

$$F_g \sim \left(1-\frac{t}{t_c}\right)^{(2-2g)\frac{p+q}{p+q-1}} t_c^{2-2g}\, \tilde{F}_g$$

obtained by TR from $\hat{\omega}_1^{(g)}$ and $(\tilde{x}(\zeta), \tilde{y}(\zeta))$

In quantum gravity

$$F_g \sim (1-t/t_c)^{2-\gamma_g}\, t_c^{2-2g}\, \tilde{F}_g \qquad [\,\gamma_0 = \gamma_{string}\,]$$

so

$$\boxed{2-\gamma_g = (2-2g)\frac{p+q}{p+q-1}}$$

Critical exponents

- Another „dressing exponents"

$$t_k = t_k^{crit} + \sum_j c_{j,k}\,(1-t/t_c)^{\frac{\Delta_{j,1}-\Delta_{m,1}}{1-\Delta_{m,1}}}\,\tilde{t}_j$$

$\gamma\nu_j$

$$\therefore \quad \frac{\Delta_{j,1}-\Delta_{m,1}}{1-\Delta_{m,1}} = \frac{2(m-j)}{p+q-1} = \frac{p-(2j+1)}{p+q-1}$$

Solving
$$\Delta_{j,1} = \frac{|p-jq|-|p-q|}{p+q-|p-q|}$$

in agreement with KPZ formula

$$\frac{\kappa}{4}\Delta_{r,s}^2 + \left(1-\frac{\kappa}{4}\right)\Delta_{r,s} = h_{r,s}$$

SLE_κ

κ usually called γ in proba

Relation to integrability

- Korteweg-de Vries (1895) eq.

  [KdV] $\partial_t \phi + \partial_x^3 \phi - 6 \phi \partial_x \phi = 0$

  describes, e.g cnoidal waves

- The (2+1) generalization

  $$\lambda \partial_y^2 \phi + \partial_x (\partial_t \phi + \phi \partial_x \phi + \epsilon^2 \partial_x^3 \phi) = 0$$

  is called KP-equation

  Kadomtsev Petviashvili
  in 1970

- KP-eq. is just the first non-trivial case of an ∞-tower of non-linear PDE's (the „KP-hierarchy")

- This is also found from a τ-function ($\tau(\vec{t})$ satisfies Hirota eq.) for specialization of

  $$t_1 = x, \quad t_2 = y, \quad t_3 = t,$$

  then $u = 2 \partial_x^2 \ln \tau(\vec{t})$

  satisfies KP. eq.

(Brutal) sketch towards τ

String equation

[Douglas – Shenker '90]

$[P, Q] = \frac{1}{N}\,\mathrm{Id}$ $\qquad d = \frac{1}{N}\frac{\partial}{\partial s}$

$P = d^p - p\, u(s)\, d^{p-2} + \dots$

$Q = d^q + \dots = d^2 - 2u(s)$

Solution of the string eq.

- $P = \sum \tilde{t}_j \left(Q^{j+1/2}\right)_+ + \sum_{j=0}^{m-1} c_j\, Q^j$
- and $\sum_{j=0}^{m} \tilde{t}_j R_{j+1}(u) = s$ (Painlevé prop.)

where $\{R_j\}$ are the Gelfand-Dikij polynomials. (homogeneous of deg j ; ∂_s counts as $\sqrt{\ }$)

$R_0 = 2$

$R_1 = -2u$

$R_2 = 3u^2 - \frac{1}{2N^2}\ddot{u}$

e.g. $R_2(u) - 2\tilde{t}_0\, u = s$ (Painleve I)

and $\left[\left[Q^{j+1/2}_+\right]^2, Q^{2j+1}\right] \leq 2j$ (defines DO's $Q^{j+1/2}_+$)

Lax equation $\left[\frac{1}{N}\partial_x + D(x,s),\ R(x,s) - \frac{1}{N}\partial_s\right] = 0$

for $R(x,s) = \begin{pmatrix} 0 & 0 \\ x+2u(s) & 0 \end{pmatrix}$

and $D(x,s) = \begin{pmatrix} A_k & B_k \\ C_k & -A_k \end{pmatrix}$ in terms of $R_0, \dots, R_k(u)$ and derivatives.

- Linear Ψ-system

$$\begin{cases} \frac{1}{N}\partial_x \Psi = -\mathcal{D}\,\Psi \\ \frac{1}{N}\partial_s \Psi = \mathcal{R}\,\Psi \end{cases}$$

$\exists$ common sol^n for $\Psi(x,s) = \begin{pmatrix} \psi & \phi \\ \tilde{\psi} & \tilde{\phi} \end{pmatrix}$ where ψ satisfies Schrödinger for the potential $x + 2u(s)$.

- Christoffel Darboux kernel

$$K(x_1,x_2) = \frac{[\Psi(x_1,s)^{-1}\,\Psi(x_2,s)]_{2,2}}{x_1 - x_2}$$

- Correlators...

$$\hat{W}_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} + (-1)^{n-1}\sum_{\substack{\sigma \\ \sigma \text{ cycle} \\ \text{of } \mathfrak{S}_n}} \prod_{i=1}^{n} K(x_i, x_{\sigma(i)})$$

- ... turn out to be $\tilde{W}_n$

$$\tilde{W}_n = \hat{W}_n \qquad (\text{equality in } \mathbb{C}[[1/N]])$$

- τ-function

$$\frac{1}{N^2}\partial_s^2 \log \tau = u(s)$$

where $\sum_{j=0}^{m} \tilde{t}_j\, R_j(u) = s$

- topological expansions

$$N^2 u(s) = \sum_g N^{2-2g} u_g(s) \qquad u_g \in \mathbb{C}(u_0)$$

rat. fct.

$$\Downarrow$$

$$\ln \tau = \sum_g N^{2-2g} F_g(u_0) \qquad \ddot{F}_g = u_g$$

τ - function and maps

Near an m-th order critical point

$$F_g \sim (t-t_c)^{2-2g \, \frac{2m+3}{2m+2}} \tilde{F}_g$$

are such that

$$\tau = e^{\sum_g N^{2-2g} F_g}$$ is the τ-function of the $(2m+1, 2)$ minimal model

$u(s) = d^2 \ln \tau / ds^2$ satisfying

GD_{m+1}: $\quad R_{m+1}(u(s)) = s$

The asymptotic gen. function counting large maps near an m-th order critical point is the $(2m+1, 2)$ reduction of τ-function of the KdV hierarchy.