UNIVERSITÄT
HEIDELBERG
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SEIT 1386

Una invitación a (los ensembles de Dirac en) la geometría no conmutativa

STRUCTURES
CLUSTER OF
EXCELLENCE

Seminario del CA de Relatividad y Física Matemática FCFM, BUAP

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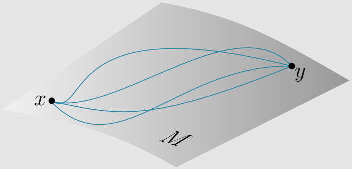
I. MOTIVATING NONCOMMUTATIVE GEOMETRY

Heuristics:

- Path integrals <u>on</u> M



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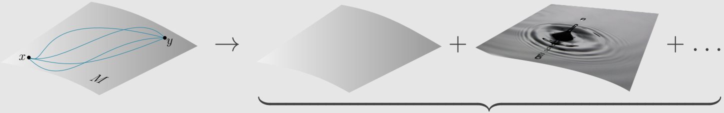

Quantum superposition of geometries

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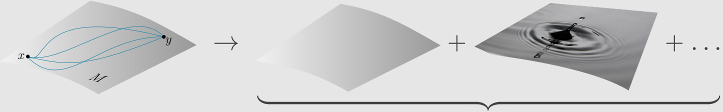

Quantum superposition of geometries

- Quantum Gravity → Random Geometry

$$\mathcal{Z}_{\mathsf{QG}} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\mathsf{EH}}[g]} \mathcal{D}[g] \to \mathcal{Z} = \sum\!\!\!\!\!\!\int_{\substack{\text{topologies}\\ \text{geometries}}} \mathrm{e}^{-\frac{1}{\hbar} S_{\mathsf{EH}}[g]} \mathcal{D}[g]$$

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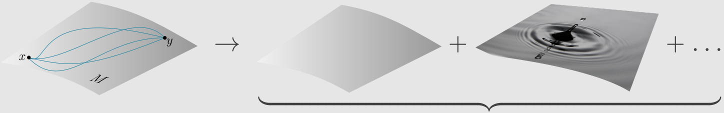

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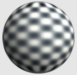
- To access $\mathcal{Z}$, models for ‘quantum space’ are proposed, e.g.

Discrete

single geometry

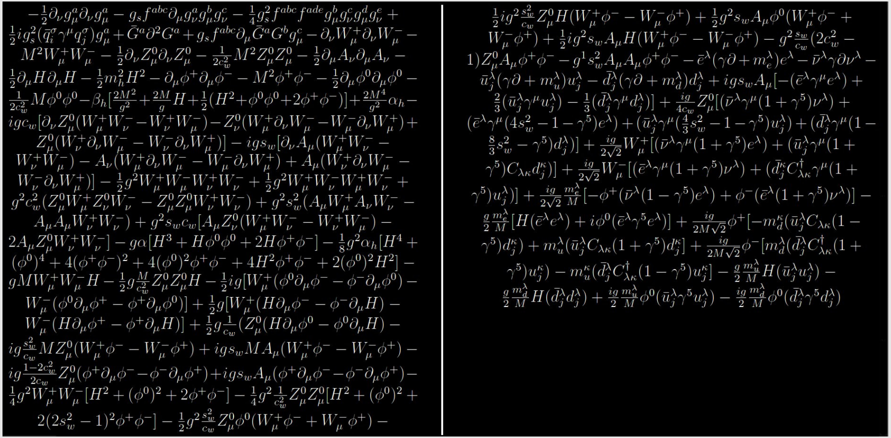
← paradigms →

Algebraic



(↖ from Wikipedia)

The Spectral Standard Model



$$\begin{gathered}
-\tfrac{1}{2}\partial_\nu g^a_\mu\partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
\tfrac{1}{2}ig_s^2(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu\partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu\partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu - \\
\tfrac{1}{2}\partial_\mu H\partial_\mu H - \tfrac{1}{2}m_h^2 H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\tfrac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - \\
igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + \\
Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - \\
A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - \\
2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \tfrac{1}{8}g^2\alpha_h[H^4 + \\
(\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2] - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - \\
W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - \\
ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - \\
ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \tfrac{1}{4}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + \\
2(2s_w^2 - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) -
\end{gathered}$$

$$\begin{gathered}
\tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - \\
1)Z^0_\mu A_\mu\phi^+\phi^- - g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \\
\bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \\
\tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] + \tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + \\
(\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \\
\tfrac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1 + \\
\gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 + \\
\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m_e^\lambda}{M}[-\phi^+(\bar{\nu}^\lambda(1 - \gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1 + \gamma^5)\nu^\lambda)] - \\
\tfrac{g}{2}\tfrac{m_e^\lambda}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1 - \\
\gamma^5)d^\kappa_j) + m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1 + \gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 + \\
\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 - \gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)
\end{gathered}$$

Num. of generations and $\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ ⟶ NCG ⟶ Classical Lagrangian of the Standard Model

[Chamseddine-Connes-Marcolli *ATMP* 2007 (Euclidean); J. Barrett *J. Math. Phys.* 2007 (Lorenzian)]

go to sketch of proof ▻

Commutative spectral triples

A spin manifold M yields (A_M, H_M, D_M)

- $A_M = C^\infty(M)$ is a comm. $*$-algebra
- $H_M := L^2(M, \mathbb{S})$ a representation of A_M
- $D_M = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$ is self-adjoint
- for each $a \in A_M$, $[D_M, a]$ is bounded, and in fact $[D_M, x^\mu] = -\mathrm{i}c(\mathrm{d}x^\mu) = -\mathrm{i}\gamma^\mu$, being c Cliff. mult.
- D_M has compact resolvent

go to more on spin geometry ▻

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go to more on spin geometry ⊳

A *spectral triple* (A, H, D) consists of

- a $*$-algebra A
- a representation H of A
- a self-adjoint operator D on H with compact resolvent and such that $[D, a]$ is bounded for each $a \in A$

  The 'commutative case' motivates

$$\Omega^1_D(A) := \Big\{ \sum_{\text{(finite)}} b[D, a] \mid a, b \in A \Big\}$$

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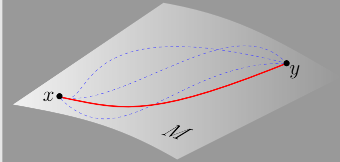
RECONSTRUCTION THEOREM: [A, Connes, *JNCG* '13] *(quite roughly formulated)*
Commutative spectral triples$^{+\textit{some more axioms}}$ are Riemannian manifolds.

The idea is to replace the metric in (M, g) by D_M

Connes' geodesic distance



$\gamma : \mathbb{R} \to M$

$\inf_{\gamma \text{ as above}} \{\int_\gamma \mathrm{d}s\} = d(x, y)$

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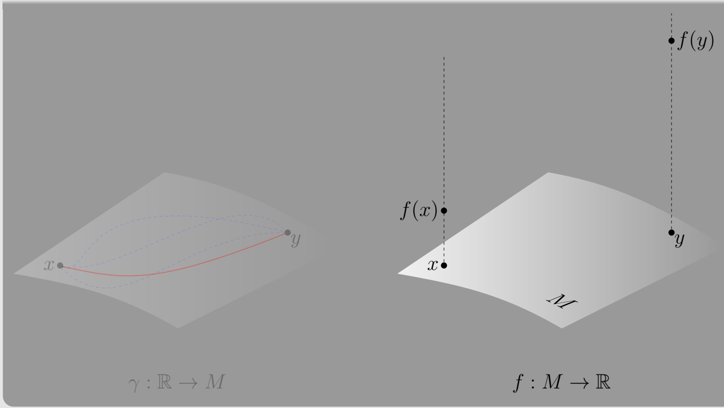

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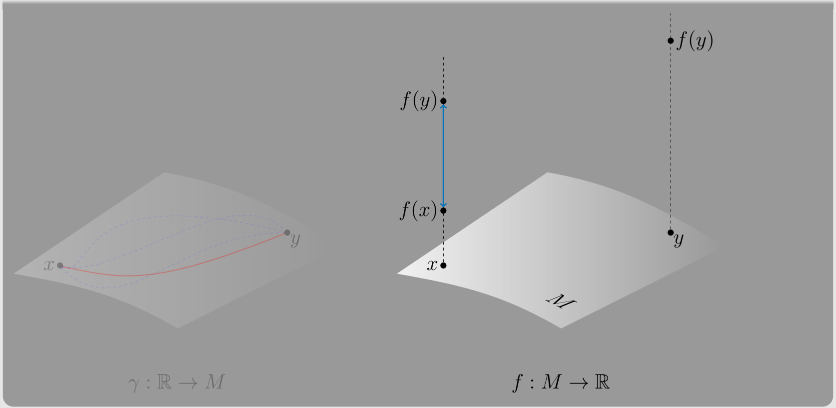

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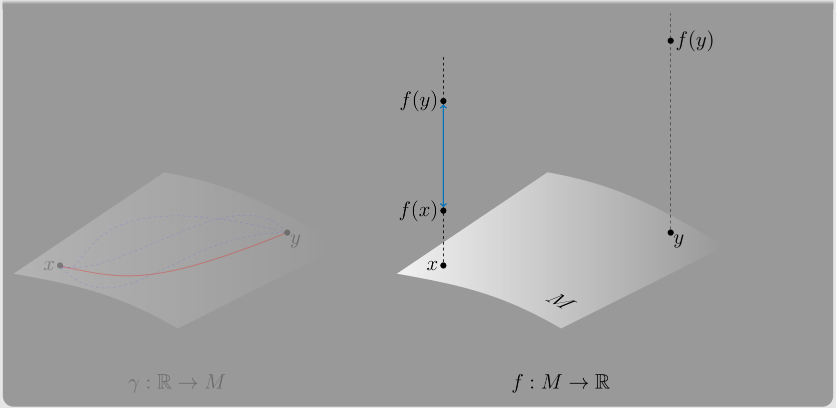

$$\sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \ : \ ||D_M f - f D_M|| \leqslant 1 \}$$

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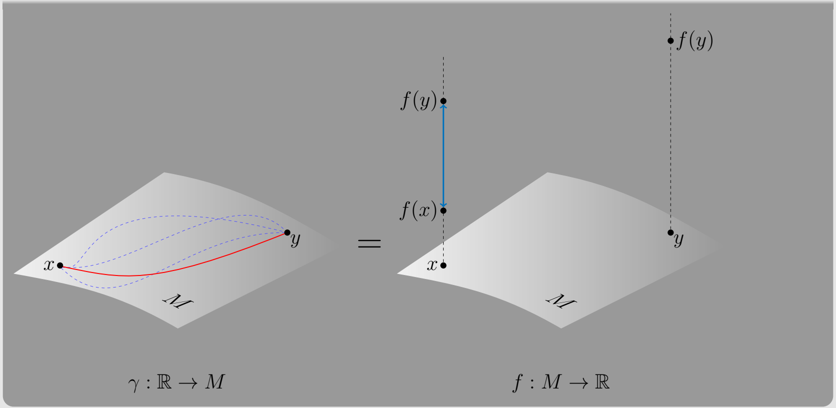

$$\inf_{\gamma \text{ as above}} \{ \textstyle\int_\gamma \mathrm{d}s \} = d(x, y) \;=\; \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| \;:\; ||D_M f - f D_M|| \leqslant 1 \}$$

`go to examples ∇`

NCG toolkit in high energy physics

- On a spectral triple (A, H, D) the (bosonic) classical action is given by

  $$S(D) = \mathrm{Tr}_H\, f(D/\Lambda)$$ [Chamseddine-Connes *CMP* '97]

  for a bump function f around the origin and Λ a cut off scale. It's computed with heat kernel expansion [P. Gilkey, *J. Diff. Geom.* '75]

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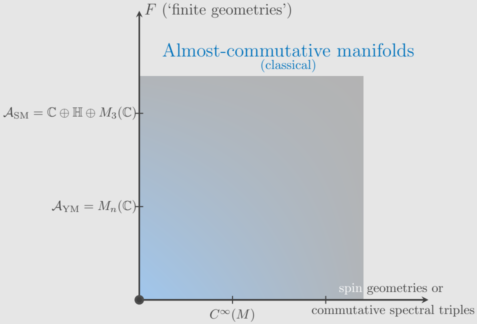

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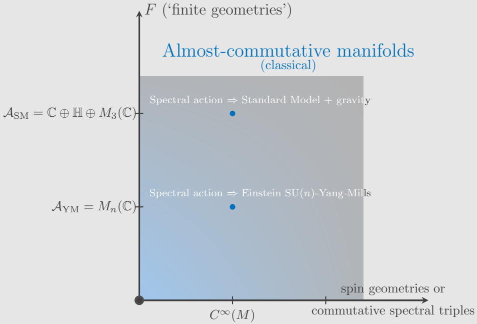
  $$\psi a = J a^* J^{-1} \psi \qquad \psi \in H, a \in A$$



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- let's sketch *connections*: if S^G is a G-invariant functional on M

  $$S^G \rightsquigarrow S^{\mathrm{Maps}(M,G)}$$

  $$\mathrm{d} \rightsquigarrow \mathrm{d} + \mathbb{A} \qquad \mathbb{A} \in \Omega^1(M) \otimes \mathfrak{g}$$

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- given (A, H, D) and a Morita equivalence $A \simeq_{\mathrm{M}} B$ (i.e. $\mathrm{End}_A(E) \cong B$) yields new $(B, E \otimes_A H, D')$. For $A = B$, in fact a tower

$$\{(A, H, D_\omega)\}_{\omega \in \Omega^1_D(A)}$$

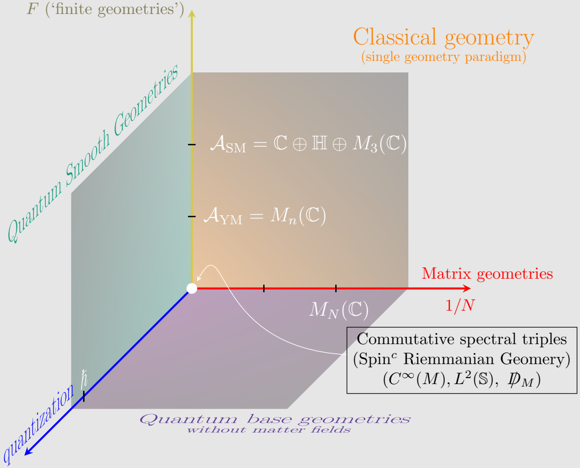
  with $D_\omega = D + \omega$. In presence of J

$$D_\omega = D + \omega \pm J\omega J^{-1}$$

$$D_\omega \mapsto \mathrm{Ad}(u) D_\omega \mathrm{Ad}(u)^* = D_{\omega_u}$$

$$\omega \mapsto \omega_u = u\omega u^* + u[D, u^*] \qquad u \in \mathcal{U}(A)$$

Organization



AIM: Make sense of

$$\mathcal{Z} = \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\operatorname{Tr}_H f(D)} \mathrm{d}D$$

- *Plane* $(\hbar, 1/N, 0)$ *of ‘base geometries’*
- *Plane* $(\hbar, 0, F) = \lim_{N\to\infty}(\hbar, 1/N, F)$
- *Plane* $(0, 1/N, F) = \lim_{\hbar\to 0}(\hbar, 1/N, F)$ *of classical geometries*

[CP 2105.01025]

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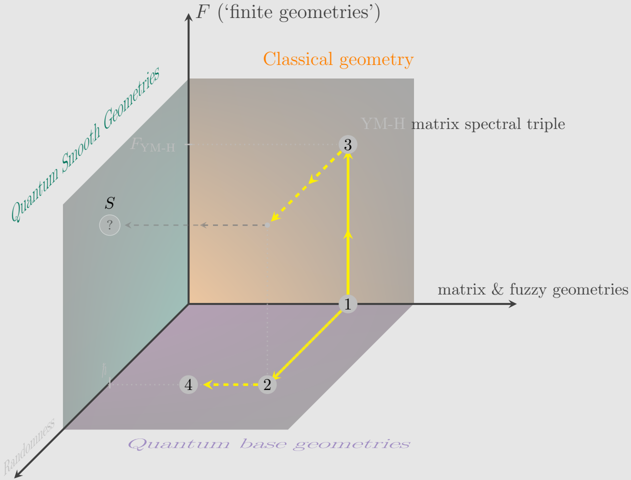

1. Matrix Geometries [J. Barrett, *J. Math. Phys.* 2015]
2. Dirac ensembles [J. Barrett, L. Glaser, *J. Phys. A* 2016] and how to compute the spectral action [CP 1912.13288]
3. Gauge matrix spectral triples [CP 2105.01025]
4. Functional Renormalization [CP 2007.10914] and [CP 2111.02858]

II. FUZZY GEOMETRIES AND MULTIMATRIX MODELS

A *fuzzy geometry* (of signature (p,q) (thus of dim. $p+q$ and KO-dim $q-p$) consists of

- $A = M_N(\mathbb{C})$
- '$H = \mathbb{S} \otimes M_N(\mathbb{C})$, being $\mathbb{S}$ a $\mathbb{C}\ell(p,q)$-module

. . . (axioms for D omitted, `go to axioms` ∇) . . .

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- '$H = \mathbb{S} \otimes M_N(\mathbb{C})$, being $\mathbb{S}$ a $\mathbb{C}\ell(p, q)$-module

  . . . (axioms for D omitted, `go to axioms` ∇) . . .
- Gamma-matrices conventions:
  - $(\gamma^\mu)^2 = +1$,
    $\mu = 1, \ldots, p, \gamma^\mu$ Hermitian
  - $(\gamma^\mu)^2 = -1$,
    $\mu = 1+p, \ldots, q+p, \gamma^\mu$ anti-Hermitian
  - $\Gamma^I := \gamma^{\mu_1} \cdots \gamma^{\mu_r}$ for $\mu_i = 1, \ldots, p+q$,
    $I = (\mu_1, \ldots, \mu_r)$

- Characterization of D in even dimensions:

$$D = \sum_J \Gamma^J_{\text{s.a.}} \otimes \{H_J, \,\cdot\,\} + \sum_J \Gamma^J_{\text{anti.}} \otimes [L_J, \,\cdot\,]$$

multi-index J monot. increasing, $|J|$ odd

[J. Barrett, *J. Math. Phys.* '15], $H_J^* = H_J$, $L_J^* = -L_J$

II. Fuzzy Geometries and Multimatrix Models

A *fuzzy geometry* (of signature (p,q) (thus of dim. $p+q$ and KO-dim $q-p$) consists of

- $A = M_N(\mathbb{C})$
- ‘$H = \mathbb{S} \otimes M_N(\mathbb{C})$, being $\mathbb{S}$ a $\mathbb{C}\ell(p,q)$-module

  . . . (axioms for D omitted, `go to axioms ▽`) . . .
- Gamma-matrices conventions:
  - $(\gamma^\mu)^2 = +1$,
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  multi-index J monot. increasing, $|J|$ odd

  [J. Barrett, *J. Math. Phys.* '15], $H_J^* = H_J$, $L_J^* = -L_J$

- Examples:
  - $D_{(1,1)} = \gamma^1 \otimes [L, \,\cdot\,] + \gamma^2 \otimes \{H, \,\cdot\,\}$
  - $D_{(0,4)} = \sum_\mu \gamma^\mu \otimes [L_\mu, \,\cdot\,] + \gamma^{\hat\mu} \otimes \{H_{\hat\mu}, \,\cdot\,\}$
- [J. Barrett, L. Glaser, *J. Phys. A* 2016]

$$\{H, \,\cdot\,\} \mapsto H \otimes 1_N + 1_N \otimes H^T$$

$$[L, \,\cdot\,] \mapsto L \otimes 1_N - 1_N \otimes L^T$$

so we will get double traces from $\mathrm{Tr}_H = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_{M_N(\mathbb{C})} = \mathrm{Tr}_{\mathbb{S}} \otimes \mathrm{Tr}_N^{\otimes 2}$

Notation: $\mathrm{Tr}_V\, X$ is the trace on operators $X : V \to V$, $\mathrm{Tr}_V\, 1 = \dim V$. So $\mathrm{Tr}_N\, 1 = N$ but $\mathrm{Tr}_{M_N(\mathbb{C})}(1) = N^2$.

Spectral Action: $\mathrm{Tr}_H\, D^{2m}$ [C.P. 1912.13288] Chord diags. $d = 2$ geometries, $\eta = \mathrm{diag}(e_1, e_2)$ `go to more CD's▻`

$$\mathrm{Tr}_H(D^6) = 2N \sum_{\boldsymbol{\mu}} \overline{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \cdots \gamma^{\mu_6})} \times$$

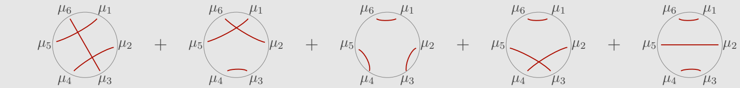
$$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6} \quad +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5} \quad +(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$$

$$+(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_6}\eta^{\mu_4\mu_5} \quad +(-1)^0\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\eta^{\mu_5\mu_6} \quad +(-1)^3\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_6} \quad +(-1)^2\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_5} \quad +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_6}$$

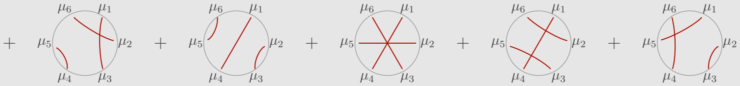
$$+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6} \quad +(-1)^1\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\mu_6} \quad +(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_6}\eta^{\mu_4\mu_5} \quad +(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\eta^{\mu_5\mu_6} \quad +(-1)^2\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_5}\eta^{\mu_4\mu_6}$$

Spectral Action: $\mathrm{Tr}_H\, D^{2m}$ [C.P. 1912.13288] Chord diags. $d = 2$ geometries, $\eta = \mathrm{diag}(e_1, e_2)$ go to more CD's⊳

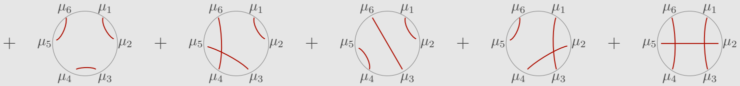
$$\mathrm{Tr}_H(D^6) = 2N \sum_{\boldsymbol{\mu}} \overbrace{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \cdots \gamma^{\mu_6})}^{\text{solid circ.}} \times$$



$$+(-1)^2\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_6} \quad +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_4} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_5} \quad +(-1)^1\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_4}\eta^{\mu_3\mu_5} \quad +(-1)^0\eta^{\mu_6\mu_1}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_4}$$



$$+(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_6}\eta^{\mu_4\mu_5} \quad +(-1)^0\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\eta^{\mu_5\mu_6} \quad +(-1)^3\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_5}\eta^{\mu_3\mu_6} \quad +(-1)^2\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_6}\eta^{\mu_3\mu_5} \quad +(-1)^1\eta^{\mu_1\mu_5}\eta^{\mu_2\mu_3}\eta^{\mu_4\mu_6}$$

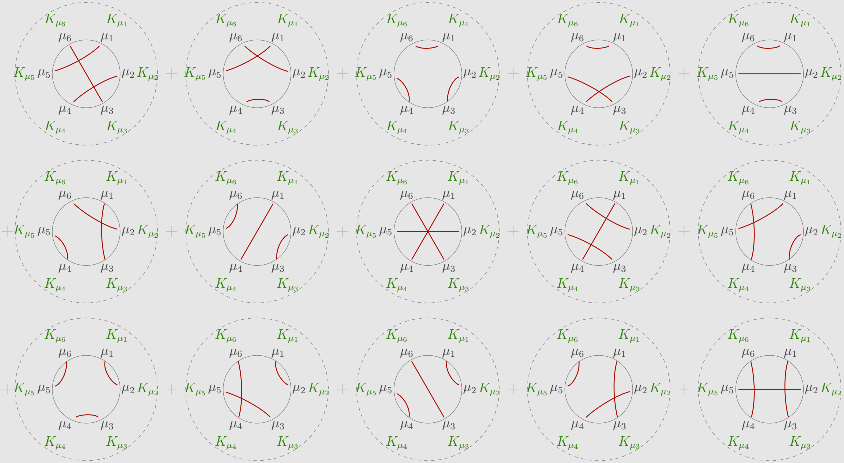


$$+(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\eta^{\mu_5\mu_6} \quad +(-1)^1\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_5}\eta^{\mu_4\mu_6} \quad +(-1)^0\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_6}\eta^{\mu_4\mu_5} \quad +(-1)^1\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\eta^{\mu_5\mu_6} \quad +(-1)^2\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_5}\eta^{\mu_4\mu_6}$$

Spectral Action: $\mathrm{Tr}_H\, D^{2m}$ [C.P. 1912.13288] Chord diags. $d = 2$ geometries, $\eta = \mathrm{diag}(e_1, e_2)$

go to more CD's ▻

$$\mathrm{Tr}_H(D^6) = 2N \sum_{\boldsymbol{\mu}} \overbrace{\mathrm{Tr}_{\mathbb{S}}(\gamma^{\mu_1} \cdots \gamma^{\mu_6})}^{\text{solid circ.}} \times \overbrace{\mathrm{Tr}_N\,(K_{\mu_1} \cdots K_{\mu_6})}^{\text{dashed circ.}} + \overbrace{\mathrm{Tr}_N\, P \times \mathrm{Tr}_N\, Q \text{ terms}}^{(1,5),(2,4),(3,3)\text{-partitions}}$$



Spectral Action: $\mathrm{Tr}_H\, D^{2m}$ [C.P. 1912.13288] Chord diags. $d = 2$ geometries, $\eta = \mathrm{diag}(e_1, e_2)$

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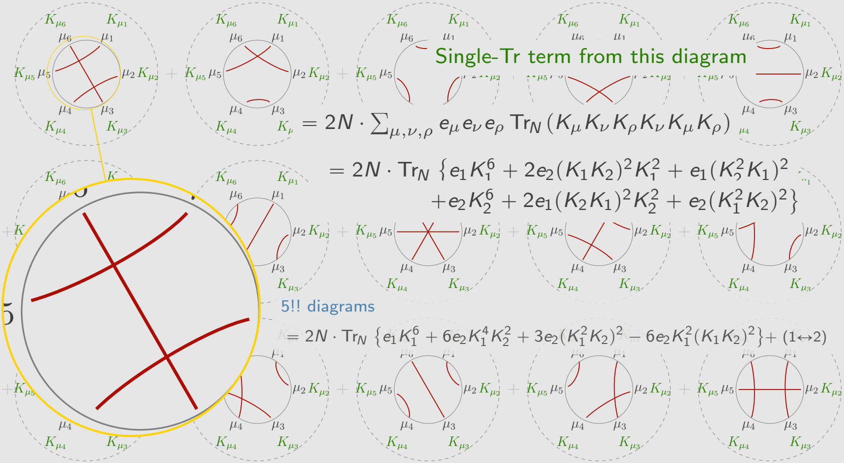


Single-Tr term from this diagram

$$= 2N \cdot \textstyle\sum_{\mu,\nu,\rho} e_\mu e_\nu e_\rho \,\mathrm{Tr}_N\,(K_\mu K_\nu K_\rho K_\nu K_\mu K_\rho)$$

$$= 2N \cdot \mathrm{Tr}_N\,\{e_1 K_1^6 + 2e_2(K_1K_2)^2K_1^2 + e_1(K_2^2K_1)^2$$
$$+e_2K_2^6 + 2e_1(K_2K_1)^2K_2^2 + e_2(K_1^2K_2)^2\}$$

5!! diagrams

$$= 2N \cdot \mathrm{Tr}_N\,\{e_1K_1^6 + 6e_2K_1^4K_2^2 + 3e_2(K_1^2K_2)^2 - 6e_2K_1^2(K_1K_2)^2\} + (1\leftrightarrow 2)$$

Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\mathrm{D\small{IRAC}}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$

$$= \int_{M_{p,q}} \mathrm{e}^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\mathrm{L\small{EB}}}$$

  - $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
  - $\mathrm{d}\mathbb{X}_{\mathrm{L\small{EB}}}$ is the Lebesgue measure on $M_{p,q}$
  - P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle X_1, \ldots, X_k \rangle = \mathbb{C}\langle \mathbb{X} \rangle$ are certain noncommutative polynomials
  - $\mathcal{Z}_{\mathrm{F\small{ORMAL}}}$ leads to colored ribbon graphs

$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$

$\bar{g}_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$

Multimatrix models with multitraces & ribbon graphs

- The chord-diagram description holds in general dimension and signature [CP '19]

$$\mathcal{Z} = \int_{\mathrm{Dirac}} \mathrm{e}^{-\operatorname{Tr}_H f(D)} \mathrm{d}D \quad (\hbar = 1)$$

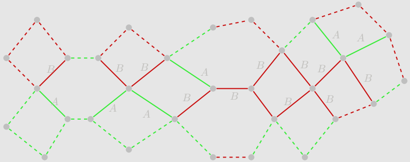
$$= \int_{M_{p,q}} \mathrm{e}^{-N \operatorname{Tr}_N P - \operatorname{Tr}_N^{\otimes 2}(Q_{(1)} \otimes Q_{(2)})} \mathrm{d}\mathbb{X}_{\mathrm{Leb}}$$

  - $\mathbb{X} \in M_{p,q}$ = products of $\mathfrak{su}(N)$ and $\mathcal{H}_N$
  - $\mathrm{d}\mathbb{X}_{\mathrm{Leb}}$ is the Lebesgue measure on $M_{p,q}$
  - P, $Q_{(i)}$ in $\mathbb{C}_{\langle k \rangle} = \mathbb{C}\langle X_1, \ldots, X_k \rangle = \mathbb{C}\langle \mathbb{X} \rangle$ are certain noncommutative polynomials
  - $\mathcal{Z}_{\mathrm{formal}}$ leads to colored ribbon graphs

$\bar{g}_1 \operatorname{Tr}_N(ABBBAB) \quad \leftrightarrow$

$\bar{g}_2 \operatorname{Tr}_N^{\otimes 2}(AABABA \otimes AA) \quad \leftrightarrow$

- **Multitrace:** 'touching interactions' [Klebanov, *PRD* '95], wormholes [Ambjørn-Jurkiewicz-Loll-Vernizzi, *JHEP* '01], 'stuffed maps' [G. Borot *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* '14], AdS/CFT [Witten, `hep-th/0112258`]
- **Ribbon graphs:** Enumeration of maps [Brezin, Itzykson, Parisi, Zuber, *CMP* '78]. The ones we get are 'face-worded'

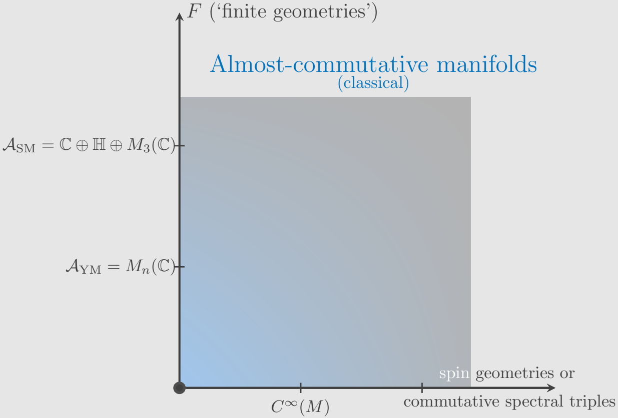


& intersection numbers of ψ-classes [Kontsevich, *CMP*, '92]

$$\sum_{a_1+\ldots+a_n=\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdot \psi_2^{a_2} \cdots \psi_n^{a_n} \prod_{j=1}^{n} \frac{(2a_j-1)!!}{s_j^{2a_j+1}}$$

$$= \sum_{G \text{ trivalent of type } (g,n)} \frac{2^{2g-2+n}}{\#\mathrm{Aut}(G)} \prod_{e \in G} \frac{1}{s_{L(e)} + s_{R(e)}}$$

III. YANG-MILLS-HIGGS MATRIX THEORY



F ('finite geometries')

Almost-commutative manifolds

(classical)

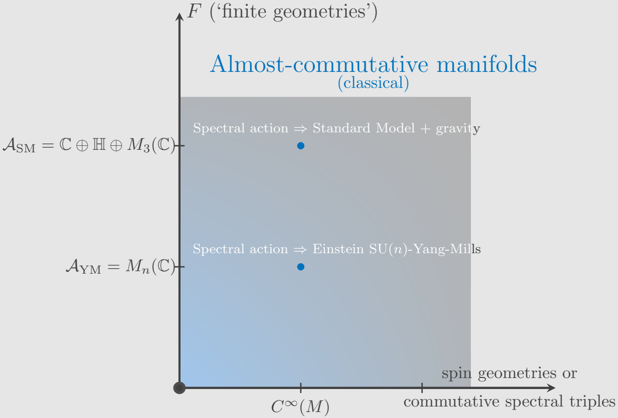
$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

spin geometries or commutative spectral triples

$C^\infty(M)$

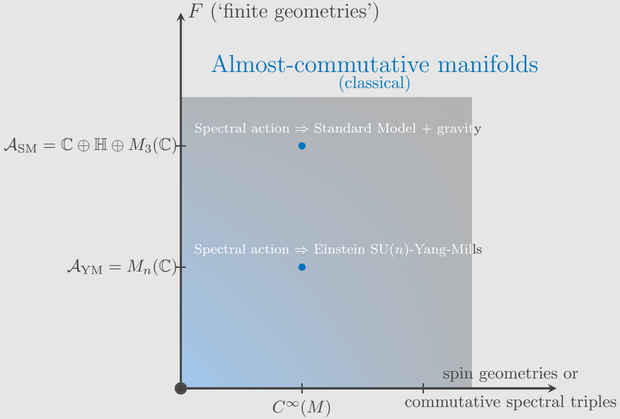
III. YANG-MILLS-HIGGS MATRIX THEORY



$$\mathcal{Z}_{\mathsf{AC}} \overset{?}{=} \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\,\mathsf{Tr}\, f(D)} \mathrm{d}D$$

(hard for almost-commutative manifolds)

III. YANG-MILLS-HIGGS MATRIX THEORY



F (‘finite geometries’)

Almost-commutative manifolds
(classical)

$\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

Spectral action ⇒ Standard Model + gravity

$\mathcal{A}_{\mathrm{YM}} = M_n(\mathbb{C})$

Spectral action ⇒ Einstein SU(n)-Yang-Mills

$C^\infty(M)$

spin geometries or
commutative spectral triples

$$\mathcal{Z}_{\mathrm{AC}} \overset{?}{=} \int_{\mathrm{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\operatorname{Tr} f(D)} \mathrm{d}D$$

(hard for almost-commutative manifolds)

DEFINITION [CP 2105.01025] We define a *gauge matrix spectral triple* $G_\ell \times F$ as the spectral triple product of a fuzzy geometry G_ℓ with a finite geometry $F = (A_F, H_F, D_F)$, $\dim A_F < \infty$.

LEMMA-DEFINITION [CP 2105.01025] Consider a gauge matrix spectral triple $G_\ell \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_ℓ Riemannian ($d = 4$) fuzzy geometry on the algebra $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + a_\mu) + \gamma^{\hat{\mu}} \otimes (x_\mu + s_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}},$$

The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\underbrace{\ell_\mu + a_\mu}_{d_\mu}, \ell_\nu + a_\nu] =: [\mathrm{F}_{\mu\nu}, \ \cdot\ ]$$

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The *field strength* is given by

$$\mathscr{F}_{\mu\nu} := [\underbrace{\mathscr{l}_\mu + \mathscr{a}_\mu}_{\mathscr{d}_\mu}, \mathscr{l}_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu}, \ \cdot\ ]$$

LEMMA The gauge group $\mathsf{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \text{ for all } u \in \mathsf{G}(A)$$

Lemma-Definition [CP 2105.01025] Consider a gauge matrix spectral triple $G_{\ell} \times F$ with

$$F = (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$$

and G_{ℓ} Riemannian ($d = 4$) fuzzy geometry on the algebra $M_N(\mathbb{C})$, whose fluctuated Dirac op. is

$$D_\omega = \sum_{\mu=0}^{3} \overbrace{\gamma^\mu \otimes (\ell_\mu + \mathscr{a}_\mu) + \gamma^{\hat\mu} \otimes (\mathscr{x}_\mu + \mathscr{s}_\mu)}^{D_{\text{gauge}}} + \overbrace{\gamma \otimes \Phi}^{D_{\text{Higgs}}} ,$$

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$$\mathscr{F}_{\mu\nu} := [\underbrace{\ell_\mu + \mathscr{a}_\mu}_{\mathscr{d}_\mu}, \ell_\nu + \mathscr{a}_\nu] =: [\mathsf{F}_{\mu\nu},\ \cdot\ ]$$

Lemma The gauge group $\mathsf{G}(A) \cong \mathrm{PU}(N) \times \mathrm{PU}(n)$ acts as follows

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}^u_{\mu\nu} = u\mathsf{F}_{\mu\nu}u^* \text{ for all } u \in \mathsf{G}(A)$$

Theorem For a Yang-Mills–Higgs matrix spectral triple on a 4-dimensional flat ($\mathscr{x} = 0 = \mathscr{s}$) Riemannian ($p = 0$) fuzzy base, the Spectral Action for a real polynomial $f(x) = \frac{1}{2}\sum_{i=1}^{4} a_i x^i$ reads

$$\frac{1}{4}\operatorname{Tr}_H f(D) = S^{\ell}_{\mathrm{YM}} + S^{\ell}_{\mathrm{H}} + S^{\ell}_{\mathrm{g\text{-}H}} + S^{\ell}_{\vartheta} ,$$

where each sector is defined as follows:

$$S^{\ell}_{\mathrm{YM}}(\ell, \mathscr{a}) := -\frac{a_4}{4}\operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}) ,$$

$$S^{\ell}_{\mathrm{g\text{-}H}}(\ell, \mathscr{a}, \Phi) := -a_4 \operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}}(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi) ,$$

$$S^{\ell}_{\mathrm{H}}(\Phi) := \operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}} f_{\mathrm{e}}(\Phi) ,$$

$$S^{\ell}_{\vartheta}(\ell, \mathscr{a}) := \operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}} f_{\mathrm{e}}\left(\vartheta^{1/2}\right) . \quad \vartheta = \mathscr{d}_\mu \mathscr{d}^\mu$$

Moreover, one obtains positivity for each of the following functionals, independently:

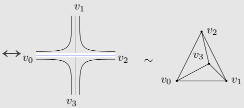
$$S^{\ell}_{\vartheta},\ S^{\ell}_{\mathrm{YM}},\ S^{\ell}_{\mathrm{H}} \geqslant 0 , \qquad \text{if } a_4 \geqslant 0 .$$

Meaning	Random matrix case, flat $d = 4$ Riem.	Smooth operator
	$\mathrm{Tr} =$ trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot \ ]$	∂_i
Gauge potential	$\mathscr{a}_\mu = [A_\mu, \ \cdot \ ]$	$\mathbb{A}_i$
Higgs field	Φ	h
Covariant Derivative	$\mathscr{d}_\mu = \ell_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$

Meaning	Random matrix case, flat $d = 4$ Riem. Tr = trace of ops. $M_N \otimes M_n \to M_N \otimes M_n$	Smooth operator
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot\ ]$	∂_i
Gauge potential	$\mathscr{a}_\mu = [A_\mu, \ \cdot\ ]$	$\mathbb{A}_i$
Higgs field	Φ	h
Covariant Derivative	$\mathscr{d}_\mu = \ell_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[\mathscr{d}_\mu, \mathscr{d}_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} +$ $[\ell_\mu, \mathscr{a}_\nu] - [\ell_\nu, \mathscr{a}_\mu] + [\mathscr{a}_\mu, \mathscr{a}_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} +$ $\partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Higgs potential	$\mathrm{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$	$\int_M \left(f_2 \lvert h \rvert^2 + f_4 \lvert h \rvert^4\right) \mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi)$	$-\int_M \lvert \mathbb{D}_i h \rvert^2 \mathrm{vol}$
Yang-Mills action	$-\frac{1}{4} \mathrm{Tr}(\mathscr{F}_{\mu\nu} \mathscr{F}^{\mu\nu})$	$-\frac{1}{4} \int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij} \mathbb{F}^{ij}) \mathrm{vol}$

MEANING	RANDOM MATRIX CASE, FLAT $d=4$ RIEM.	SMOOTH OPERATOR
	Tr = TRACE OF OPS. $M_N \otimes M_n \to M_N \otimes M_n$	
Derivation	$\ell_\mu = [L_\mu \otimes 1_n, \ \cdot \ ]$	∂_i
Gauge potential	$\mathscr{a}_\mu = [A_\mu, \ \cdot \ ]$	$\mathbb{A}_i$
Higgs field	Φ	h
Covariant Derivative	$\mathscr{d}_\mu = \ell_\mu + \mathscr{a}_\mu$	$\mathbb{D}_i = \partial_i + \mathbb{A}_i$
Field strength	$[\mathscr{d}_\mu, \mathscr{d}_\nu] = \overbrace{[\ell_\mu, \ell_\nu]}^{\not\equiv 0} +$ $[\ell_\mu, \mathscr{a}_\nu] - [\ell_\nu, \mathscr{a}_\mu] + [\mathscr{a}_\mu, \mathscr{a}_\nu]$	$[\mathbb{D}_i, \mathbb{D}_j] = \overbrace{[\partial_i, \partial_j]}^{\equiv 0} +$ $\partial_i \mathbb{A}_j - \partial_j \mathbb{A}_i + [\mathbb{A}_i, \mathbb{A}_j]$
Higgs potential	$\mathrm{Tr}(f_2 \Phi^2 + f_4 \Phi^4)$	$\int_M \left(f_2 \lvert h \rvert^2 + f_4 \lvert h \rvert^4\right) \mathrm{vol}$
Gauge-Higgs coupling	$-\mathrm{Tr}(\mathscr{d}_\mu \Phi \mathscr{d}^\mu \Phi)$	$-\int_M \lvert \mathbb{D}_i h \rvert^2 \mathrm{vol}$
Yang-Mills action	$-\frac{1}{4}\mathrm{Tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu})$	$-\frac{1}{4}\int_M \mathrm{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{ij}\mathbb{F}^{ij})\mathrm{vol}$

+ Propagators and $(\ell_\mu)_{ij}(\ell_\nu)_{jm}(\ell^\mu)_{ml}(\ell^\nu)_{li} \leftrightarrow$



IV. FRG FOR MULTIMATRIX MODELS WITH MULTITRACES

Motivation from '2D-Quantum Gravity'

discrete surfaces	↔	matrix integrals $\mathcal{Z}(\lambda)$ [B. Eynard, *Counting Surfaces* '16]
smooth surface	↔	$\langle$area$\rangle$ finite & infinitesimal mesh $\mathfrak{a}$ $\langle \text{area} \rangle_g \sim \frac{\mathfrak{a}^2(2-2g)}{\lambda/\lambda_c-1}$
all topologies	↔	$\mathcal{Z}(\lambda) = \sum_g N^{2-2g} \underset{(\lambda_c-\lambda)^{(2-2g)/\theta}}{\underset{\wr}{\mathcal{Z}_g(\lambda)}}$
⇑		
double-scaling limit		$N(\lambda_c-\lambda)^{1/\theta} = C$
lin. RG-flow near a fixed point	↔	$\lambda(N) = \lambda_c + (N/C)^{-\theta}$ $\theta = -(\partial\beta/\partial\lambda)\vert_{\lambda_c}$ [Eichhorn-Koslowski, PRD, '13]

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$\Uparrow$

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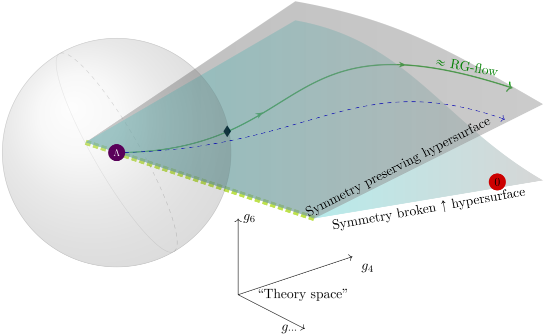

Λ Chosen bare action $S = \Gamma_{N=\Lambda}$

0 Full effective action $\Gamma = \Gamma_{N=0}$

◆ Interpolating action $\Gamma_{N=\Lambda-\rho}$ (projected & truncated)

⟶ RG-flow with truncation and projection

⋯⋯ Moduli of Dirac operators ↪ theory space

- - → RG-flow without truncation nor projection

$g_{\dots}$ Rest of coupling constants

Two approaches

1. Mathematical construction:

[CP 2007.10914, *Ann. Henri Poincaré* 2021]

- prove Wetterich Equation, or FRGE, "$\dot{\Gamma} = \frac{1}{2}\,\mathrm{STr}\left\{\dot{R}_N/[\Gamma^{(2)} + R_N]\right\}$"
- its proof determines the algebra that governs geometric series in the Hessian of Γ (this expansion is originally from [D. Benedetti, K. Groh, P. F. Machado and F. Saueressig, *JHEP* 2011])
- to determine the scalings of the couplings with Z and N we use [A. Eichhorn, T. Koslowski, *Phys. Rev. D* '13], but the proof of the FRGE dictates an algebra not reported there
- fixed-point solution to β-equations for a sextic truncation (48 running operators)
- for the unique real solution g^* leading to a single relevant direction (positive e.v. of $-(\partial\beta_i/\partial g_j)_{i,j}|_{g*})$ yields an R_N-**dependent**

  $g^*_{A^4} = 1.002 \times \big(g^*_{A^4}|$[Kazakov-Zinn-Justin, *Nucl. Phys. B* '99]$\big)$

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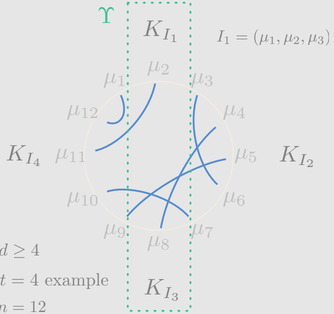
$$g^*_{A^4} = 1.002 \times \left(g^*_{A^4}|_{\text{[Kazakov-Zinn-Justin, \textit{Nucl. Phys. B} '99]}}\right)$$

2. Pragmatic approach: [CP 2111.02858]

- write down Wetterich Equation (*Generatio Sponanea*)
- assume an expansion of its rhs in unitary-invariant operators ($\neq$ exact RG)
- impose the one-loop structure
- determine from it the 'algebra of functional renormalization'; it is unique and the one reported in [CP 2007.10914]

Double-traces makes our *invariants for 'free' two-matrix models* grow quite fast [CP '19]: in dimension d

$$\frac{1}{\dim \mathbb{S}} \operatorname{Tr}(D^{2t}) = \overbrace{\sum_{I_1,\ldots,I_{2t}\in\Lambda_d^-}}^{\text{if } J\in\Lambda_d,\, \mathrm{d}x^J \neq 0 \text{ on } \mathbb{R}^d} \left\{ \overbrace{\sum_{\substack{\chi\in \mathrm{CD}_{2n}\\ 2n=\sum_i |I_i|}} \chi^{I_1\ldots I_{2t}}}^{\text{decorated chord diags}} \times \left(\sum_{\Upsilon\in\mathscr{P}_{2t}} \operatorname{sgn}(I_\Upsilon)\times \operatorname{Tr}_N\left(K_{I_{\Upsilon^c}}\right)\times \operatorname{Tr}_N\left[(K^T)_{I_\Upsilon}\right]\right)\right\}$$

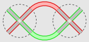


`go to examples of complexity of double-traces ▻`

Handwaving Functional Renormalization for k-matrix models (w/multitrace-measures)

Quantum theories 'flow' with energy, here in RG-time $t = \log N$, $1 \ll N < \mathscr{N}$. E.g. for $k = 2$ and with bare action

$$S[A, B] = \mathscr{N}\, \mathrm{Tr}_{\mathscr{N}} \left\{ \frac{1}{2}A^2 + \frac{1}{2}B^2 + g_{A^4}\frac{1}{4}A^4 + g_{B^4}\frac{1}{4}B^4 + \frac{1}{2}g_{ABAB}ABAB \right\}$$

radiative corrections 'generate' *effective vertices*. For instance  generates $\mathscr{N}\, \mathrm{Tr}_{\mathscr{N}}(ABBA)$.

$$\Gamma_N[A, B] = \mathrm{Tr}_N \left\{ \overbrace{\frac{Z_A}{2}A^2 + \frac{Z_B}{2}B^2 + \bar{g}_{A^4}\frac{1}{4}A^4 + \bar{g}_{B^4}\frac{1}{4}B^4 + \frac{1}{2}\bar{g}_{ABAB}ABAB}^{\text{operators from the bare action (but with 'running couplings')}} + \overbrace{\frac{1}{2}\bar{g}_{ABBA}ABBA + \frac{1}{2}\bar{g}_{A|A}\,\mathrm{Tr}_N(A) \times A + \dots}^{\text{radiative corrections}} \right\}$$

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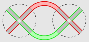
radiative corrections 'generate' *effective vertices*. For instance  generates $\mathcal{N}\,\mathrm{Tr}_{\mathcal{N}}(ABBA)$.

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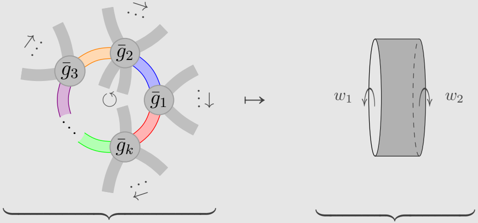
We are interested in *one-loop graphs*. These are graphs G whose 1-dim skeleton $G^\circ = G/\{$interaction vertices collapsed to points$\}$ has $b_1(G^\circ) = 1$. The *effective vertex* O_G^{eff} of such Feynman graph is formed by reading off each word w_i traveling around all ribbon edges (propagators) by both sides:

$$O_G^{\text{eff}} = \overbrace{\mathrm{Tr}_N(w_1)\times\mathrm{Tr}_N(w_2)\times\cdots\times\mathrm{Tr}_N(w_s)}^{\text{from vertices contracted with propagators}}$$
$$\times\underbrace{\mathrm{Tr}_N(U_1)\times\mathrm{Tr}_N(U_2)\ldots\times\mathrm{Tr}_N(U_r)}_{\text{from vertices uncontracted with propagators}}$$



gral. situation (yet, without multitraces)

inner word $w_1 \otimes$ outer word w_2

- *nc-derivative* $\partial_A : \mathbb{C}_{\langle k\rangle} \to \mathbb{C}_{\langle k\rangle}^{\otimes 2}$ sums over 'replacements of A by $\otimes$'[Rota-Sagan-Stein+Voiculescu]:

$$\partial_A(PAAR) = P \otimes AR + PA \otimes R\,, \text{ but}$$

$$\partial_A(ALGEBRA) = 1 \otimes LGEBRA + ALGEBR \otimes 1$$

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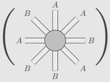
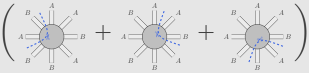
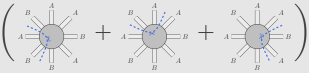
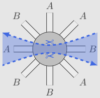
- for $W \in \mathbb{C}_{\langle k\rangle}$, $\operatorname{Hess}\operatorname{Tr}_N W \in M_k(\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle})$ is the *nc-Hessian* [CP 2007.10914], whose entries are $\operatorname{Hess}_{b,a}\operatorname{Tr} W = (\partial_{X_b} \circ \partial_{X_a})\operatorname{Tr}_N W$. These are computed by 'cuts': e.g. $W = ABAABABB$

$\partial_B\partial_A$ 

go to examples of nc-Hessians ▽

$= 1_N \otimes$ 

$+$  $\otimes 1_N + \dots$

in ellipsis $\sum_{cuts}$  $\to BAA \otimes ABB$

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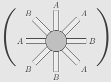

$\Big)$

go to examples of nc-Hessians ▽

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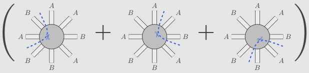
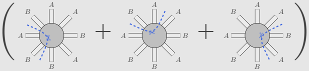


$\Big)$

$+ \Big($


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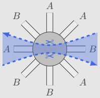


$\to BAA \otimes ABB$

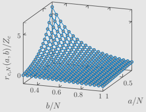
- products of traces $\Rightarrow$ extend by $\boxtimes$, $\mathcal{A}_k = (\mathbb{C}_{\langle k\rangle} \otimes \mathbb{C}_{\langle k\rangle}) \oplus (\mathbb{C}_{\langle k\rangle} \boxtimes \mathbb{C}_{\langle k\rangle})$

$$\mathrm{Hess}_{a,b}(\mathrm{Tr}\, P \cdot \mathrm{Tr}\, Q)$$
$$= \mathrm{Tr}\, P \cdot \mathrm{Hess}_{a,b}[\mathrm{Tr}\, Q] + (\partial_{X_a}\mathrm{Tr}\, P) \boxtimes (\partial_{X_b}\mathrm{Tr}\, Q)$$
$$+ (P \leftrightarrow Q)$$

- Wetterich Eq. governs the functional RG

$t = \log N$

$$\partial_t \Gamma_N[\mathbb{X}] = \frac{1}{2}\,\mathrm{STr}\left\{\frac{\partial_t R_N}{\mathrm{Hess}\,\Gamma_N[\mathbb{X}] + R_N}\right\}$$

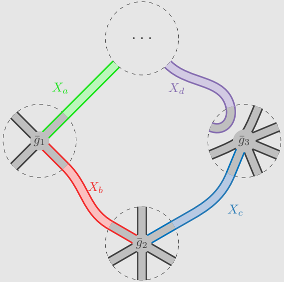


$$\overset{assume}{=} \sum_{k=0}^{\infty} \bar{h}_k(N, \eta_1, \dots, \eta_n)$$
$$\times \underbrace{\frac{1}{2}(-1)^k\,\mathrm{STr}\left\{(\mathrm{Hess}\,\Gamma_N^{\mathrm{INT}}[\mathbb{X}])^{\star k}\right\}}_{\text{regulator-independent part}}$$

- $\mathrm{STr} = \mathrm{Tr}_k \otimes \mathrm{Tr}_{\mathcal{A}_n}$. Tadpoles

imply $\mathrm{Tr}_{\mathcal{A}_n}(P \otimes Q) = \mathrm{Tr}_N\, P \cdot \mathrm{Tr}_N\, Q$, $\mathrm{Tr}_{\mathcal{A}_n}(P \boxtimes Q) = \mathrm{Tr}_N(PQ)$

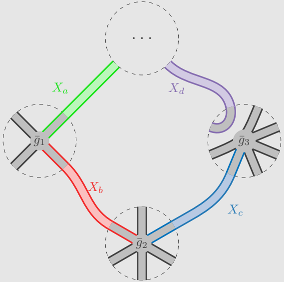
Finding $\star$

Want: 

$\subset \mathrm{Hess}_{a,b}\, O_1 \star \mathrm{Hess}_{b,c}\, O_2 \star \mathrm{Hess}_{c,d}\, O_3 \star \dots \star \mathrm{Hess}_{*,a}\, O_\ell$

Finding $\star$

Want:



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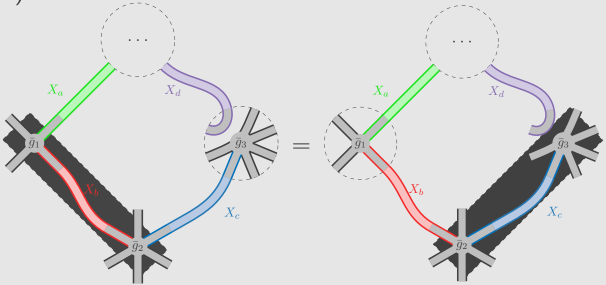
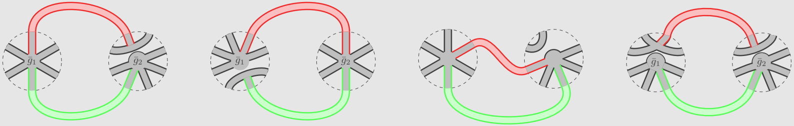
Associativity (trivial check):





$$\begin{tikzcd}
\text{Case I} \arrow[d] & \text{Case II} \arrow[d] & \text{Case III} \arrow[d] & \text{Case IV} \arrow[d] \\
\mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \boxtimes Q)] \arrow[ddd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \otimes Q)] \arrow[ddd] & \mathrm{Tr}_{\mathcal{A}_n}[(U \otimes W) \star (P \otimes Q)] \arrow[d] & \mathrm{Tr}_{\mathcal{A}_n}[(U \boxtimes W) \star (P \boxtimes Q)] \arrow[dl] \arrow[dd] \\
 & & (U \otimes W) \star (P \otimes Q)/\sim \arrow[dr] \arrow[ddd] \arrow[dddr] & \\
 & & & (U \boxtimes W) \star (P \boxtimes Q)/\sim \arrow[dlll, bend right] \arrow[dll] \arrow[ddl] \arrow[dd] \\
(U \otimes W) \star (P \boxtimes Q) \arrow[drr] \arrow[drrr, bend right] & (U \boxtimes W) \star (P \otimes Q) & & \\
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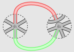

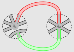

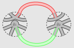

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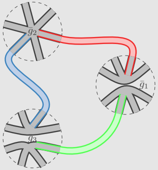

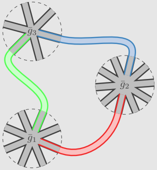

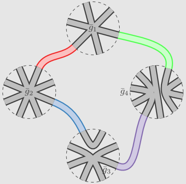

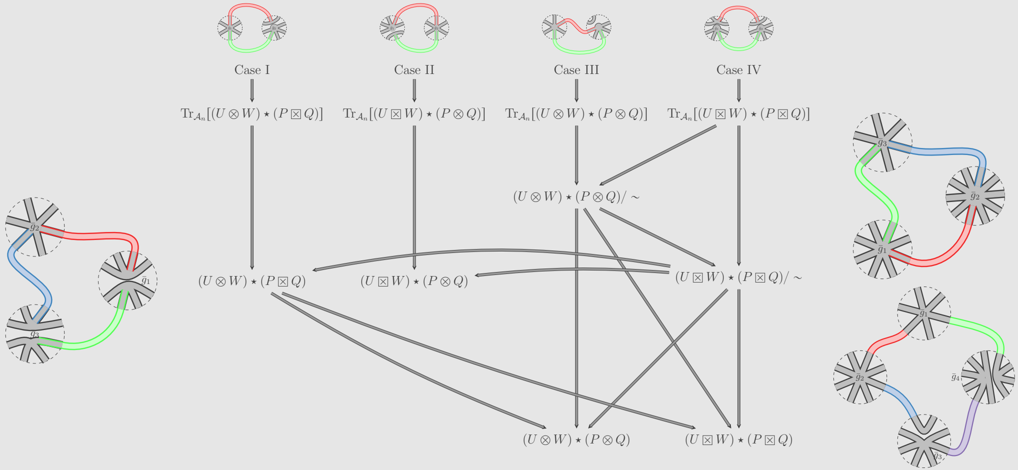

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\end{tikzcd}$$

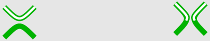
THM. [CP 2111.02858] If the RG-flow is computable in terms of $\mathrm{U}(N)$-invariants, the algebra of Functional Renormalization is $M_k(\mathcal{A}_{N,k},\star)$ where $\mathcal{A}_{N,k} = (\mathbb{C}^{(N)}_{\langle k\rangle}\otimes\mathbb{C}^{(N)}_{\langle k\rangle})\oplus(\mathbb{C}^{(N)}_{\langle k\rangle}\boxtimes\mathbb{C}^{(N)}_{\langle k\rangle})$ whose product in homogeneous elements reads:

$$\begin{aligned}
(U\otimes W)\star(P\otimes Q) &= PU\otimes WQ\,,\\
(U\boxtimes W)\star(P\otimes Q) &= U\boxtimes PWQ\,,\\
(U\otimes W)\star(P\boxtimes Q) &= WPU\boxtimes Q\,,\\
(U\boxtimes W)\star(P\boxtimes Q) &= \mathrm{Tr}(WP)U\boxtimes Q\,.
\end{aligned}$$

Example: Hermitian ‘free’ 3-matrix model

Consider two operators $O_1 = \frac{\bar{g}_1}{2}[\mathrm{Tr}_N(\frac{A^2}{2})]^2$ and $O_2 = \bar{g}_2\,\mathrm{Tr}_N(ABC)$. We compute $g_1 g_2^2$-coefficients:

$$\mathrm{Hess}_{I,J}\,O_1 = \delta_I^J\delta_I^A\bar{g}_1\{\underbrace{\mathrm{Tr}_N(A^2/2)\cdot[1_N\otimes 1_N]} + \underbrace{A\boxtimes A}\}\,,$$



where a ‘filled ribbon’ means contracted in the one-loop graph, and ‘black ribbon’ uncontracted.

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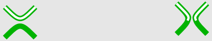

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$$\mathrm{Hess}\, O_2 = \bar{g}_2 \begin{bmatrix} 0 & C\otimes 1_N & B\otimes 1_N \\ 1_N\otimes C & 0 & A\otimes 1_N \\ 1_N\otimes B & 1_N\otimes A & 0 \end{bmatrix} \Rightarrow [\mathrm{Hess}\, O_2]^{\star 2} = \bar{g}_2^2 \begin{bmatrix} \overbrace{C\otimes C} + \overbrace{B\otimes B} & B\otimes A & C\otimes A \\ A\otimes B & A\otimes A + C\otimes C & C\otimes B \\ A\otimes C & B\otimes C & B\otimes B + A\otimes A \end{bmatrix}.$$

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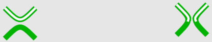

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Extracting coefficients

$$[\bar{g}_1\bar{g}_2]\,\mathrm{STr}\{\mathrm{Hess}\, O_1 \star [\mathrm{Hess}\, O_2]^{\star 2}\} = \mathrm{Tr}_N(A^2/2) \times \left[(\mathrm{Tr}_N\, C)^2 + (\mathrm{Tr}_N\, B)^2\right] + \mathrm{Tr}_N(ACAC + ABAB),$$

which are effective vertices of the four one-loop graphs that can be formed with the contractions of (the filled ribbon half-edges of) any of {, } with any of {, }. This is a toy example in [CP '21]; in [CP '20] 48 such operators run ⇒ (48 less friendly Hessians)^3. go to nc-Hessians examples ▷

CONCLUSION

- spectral triple $\equiv$ spin without commutativity of the ‘algebra of functions’
- spin $M \times \{$finite spectral triple$\} \equiv$ almost-commutative

  (reproduces classical Standard Model, but hard to quantize)
- *fuzzy* or *matrix* geometry $\approx$ finite spectral triple $+$ $\mathbb{C}\ell$-action

  ($\mathcal{Z}_{\text{FUZZY}} = \int_{\text{DIRAC}} \mathrm{e}^{-\frac{1}{\hbar}\operatorname{Tr}_H f(D)} \mathrm{d}D$ is a multimatrix model with multitraces)
- $G_\ell \times F =$ fuzzy $\times$ finite $=$ gauge matrix spectra triple

  (is $\mathrm{PU}(n)$-Yang-Mills-Higgs-like if F is over $M_n(\mathbb{C})$;

  partition function is a k-matrix model, k large)
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thank you!

References: [CP 1912.13288] [CP 2007.10914] [CP 2102.06999] [CP 2102.06999] [CP 2111.02858]

THE CLASSICAL DIRAC OPERATOR IN RIEMANNIAN GEOMETRY

For us M will be a Riemannian closed manifold, $\dim M = d$.
In physics, M is a spacetime (for this first part, still deterministic).
Q: When do Dirac operators exist on M?
A: Only if the *obstruction* to a spin-structure $w_2(M) \in \check{H}^2(M, \mathbb{Z}_2)$ is trivial.

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$\mathbb{Z}_2$-Čech cohomology in short,

- $U = \{U_i\}_i$ good open cover of M
- j-simplices are $\sigma = (k_0, \ldots, k_j)$ such that $U_{k_0 k_1 \ldots k_j} = U_{k_0} \cap U_{k_1} \cap \cdots \cap U_{k_j} \neq \varnothing$
- j-cochains, maps $f : \{j\text{-simplices}\} \to \mathbb{Z}_2$ satisfying invariance $\tau^* f = f$ under $\tau \in \mathfrak{S}(j+1)$, form an abelian group $\check{C}^j(U, \mathbb{Z}_2)$
- coboundary maps $\delta^j : \check{C}^j(U, \mathbb{Z}_2) \to \check{C}^{j+1}(U, \mathbb{Z}_2)$ given by

$$(\delta^j f)(k_0, \ldots, k_{j+1}) := f(k_1, \ldots, k_{j+1}) f(k_0, k_1 \ldots, k_{j+1}) \cdots f(k_0, \ldots, k_j)$$

- $\check{H}^j(U, \mathbb{Z}_2) = \ker \delta^j / \operatorname{im} \delta^{j-1}$

A more familiar $\mathbb{Z}_2$-Čech cohomology class is the orientability obstruction

- $\{U_j\}_j$ good open cover of M
- pick $s_j : U_j \to F(M)$ sections on the frame bundle $\mathrm{O}(d) \hookrightarrow F(M) \to M$
- on a 1-simplex (k_0, k_1), $s_{k_0} = s_{k_1} G_{k_0 k_1}$

$$\begin{aligned} f(k_0, k_1) &= \det(G_{k_0,k_1}) \\ &= \det(G_{k_1,k_0}) = f(k_1, k_0) \end{aligned}$$

- since $\{G_{j,l}\}_{j.l}$ are transition functions,

$$(\delta^1 f)(j, l, m) = G_{j,l} G_{l,m} G_{m,j} = 1$$

- other choice of sections s'_k yields $G'_{kl} = g_k G_{kl} g_l^{-1}$ and $f' = (\delta^0 h) f$ where $h = \det g_k$
- other choice $\{V\}_j$ of a good open cover yields a cochain complex map $\check{C}^*(V, \mathbb{Z}_2) \to \check{C}^*(U, \mathbb{Z}_2)$,

Low Stiefel-Whitney classes (first two floors of Whitehead tower)

THEOREM M is orientable iff the 1st Stiefel-Whitney class $w_1(M) := [f] = 1$.

Proof of 'if'. If $w_1(M) = 1$, $f(k_0, k_1) = \det(G_{k_0,k_1})$ is a 0-coboundary, $f = \delta^0 h$. We can pick sections $\{s_k : U_k \to F(M)\}_k$ and $g_k \in \mathrm{O}(d)$ with $\det g_k = h(k)$, so that the transition functions for $s'_k := s_k \cdot g_k$ satisfy

$$\begin{aligned} \det(G'_{k_0,k_1}) &= \det(g_{k_0}^{-1} G_{k_0,k_1} g_{k_1}) \\ &= [(\delta^0 h) \cdot f](k_0, k_1) = 1. \quad \square \end{aligned}$$

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In similar way, a *spin structure* λ

$$\begin{tikzcd}
\mathrm{Spin}(d) \times P(M) \arrow[r] \arrow[dd] & P(M) \arrow[dd, "\lambda"] \arrow[dr] & \\
 & & M \\
\mathrm{SO}(d) \times F_{\mathrm{SO}}(M) \arrow[r] & F_{\mathrm{SO}}(M) \arrow[ur] &
\end{tikzcd}$$

exists when the SO-frame bundle can be lifted in compatible way with the double cover $\mathbb{Z}_2 \to \mathrm{Spin}(d) \xrightarrow{\rho} \mathrm{SO}(d)$. Transition functions $g_{ij} : U_{ij} \to \mathrm{SO}(d)$ can be lifted to $\mathrm{Spin}(d)$-valued $\tilde{g}_{ij}$. For $U_{ijk} \neq \varnothing$, let

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} =: z(i,j,k)\mathrm{id}_{\mathrm{Spin}(d)}$$

Theorem [A. Haefliger, '56] Orientable M is spin iff its second Stiefel-Whitney class $w_2(M) := [z] = 1$.

Classical Dirac operators (assume d even)

- M (space~~time~~) will be a closed, Riemannian manifold
- if M is spin, there is a vector bundle $\mathbb{S}$ with fibers satisfying $\mathrm{End}(\mathbb{S}_x) \cong \mathbb{C}\ell(d)$ $(x \in M)$. The sections $\Gamma(\mathbb{S})$ are spinors
- the Levi-Civita connection ∇^{LC} can be also lifted to the *spin connection* $\nabla^s : \Gamma(\mathbb{S}) \to \Omega^1(M) \otimes \Gamma(\mathbb{S})$

$$\nabla^s c(\omega)\psi = c(\nabla^{\mathrm{LC}}\omega)\psi + c(\omega)\nabla^s\psi$$

$$\psi \in \Gamma(\mathbb{S}), \omega \in \Omega^1(M)$$

being c Clifford multiplication, basically $c(\mathrm{d}x^\mu) = \gamma^\mu$

- on the space of square integrable spinors $L^2(M, \mathbb{S})$ there is an (ess.) self-adjoint operator, the *Dirac operator*,

$$D_M = -\mathrm{i}c \circ \nabla^s \overset{\mathrm{loc.}}{=} -\mathrm{i} \sum_{\mu=1}^{d} \gamma^\mu(\partial_\mu + \omega_\mu)$$

and by Leibniz rule

$$[D_M, a] = -\mathrm{i}c(\mathrm{d}a) \quad a \in C^\infty(M)$$

which is bounded

`back to ‘spectral triples’` ⋘

Matrix or Fuzzy Geometries

Definition(“condensed” from [J. Barrett, *J. Math. Phys.* 2015]).
A *fuzzy geometry* of *signature* $(p, q) \in \mathbb{Z}^2_{\geqslant 0}$ is given by

- a simple matrix algebra $\mathcal{A}$ – we take always $\mathcal{A} = M_N(\mathbb{C})$
- a Hermitian $\mathbb{C}\ell(p, q)$-module $\mathbb{S}$ with a *chirality* γ. That is a linear map $\gamma : \mathbb{S} \to \mathbb{S}$ satisfying $\gamma^* = \gamma$ and $\gamma^2 = 1$
- a Hilbert space $\mathcal{H} = \mathbb{S} \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w)\, \mathrm{Tr}_N\,(R^* S)$ for each $R, S \in M_N(\mathbb{C})$, being $(\cdot\,, \cdot)$ the inner product of $\mathbb{S}$
- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}$, $a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

$\ldots$

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- a left-$\mathcal{A}$ representation $\rho(a)(v\otimes R)=v\otimes(aR)$ on $\mathcal{H}$, $a\in\mathcal{A}$ and $v\otimes R\in\mathcal{H}$
- three signs $\epsilon,\epsilon',\epsilon''\in\{-1,+1\}$ determined through $s:=q-p$ by the following table:

$s\equiv q-p \pmod 8$	0	1	2	3	4	5	6	7
ϵ	+	+	−	−	−	−	+	+
ϵ'	+	−	+	+	+	−	+	+
ϵ''	+	+	−	+	+	+	−	+

- a real structure $J=C\otimes *$, where $*$ is complex conjugation and C is an anti-unitarity on $\mathbb{S}$ satisfying $C^2=\epsilon$ and $C\gamma^\mu=\epsilon'\gamma^\mu C$ for all the gamma matrices $\mu=1,\ldots,p+q$.
- a self-adjoint operator D on $\mathcal{H}$ satisfying the *order-one condition*

$$[\,[D,\rho(a)]\,,J\rho(b)J^{-1}]=0 \qquad \text{for all } a,b\in\mathcal{A}$$

- a chirality $\Gamma=\gamma\otimes 1_{\mathcal{A}}$ for $\mathcal{H}$, where γ is the chirality of $\mathbb{S}$. The signs above impose:

$$J^2=\epsilon\,,\qquad JD=\epsilon' DJ\,,\qquad J\Gamma=\epsilon''\Gamma J\,.$$

GO TO characterization ⋘

Two-matrix model from $\mathrm{Tr}_H D^6$, $\eta = \mathrm{diag}(e_1, e_2)$, $K_i^* = e_i K_i$ [CP '19]

$$\begin{aligned}\mathcal{S}_6[K_1, K_2] = 2 \cdot \mathrm{Tr}_N \{ & e_1 K_1^6 + 6e_2 K_1^4 K_2^2 - 6e_2 K_1^2 (K_1 K_2)^2 + 3e_2 (K_1^2 K_2)^2 \\ & + e_2 K_2^6 + 6e_1 K_2^4 K_1^2 - 6e_1 K_2^2 (K_2 K_1)^2 + 3e_1 (K_2^2 K_1)^2 \}\end{aligned}$$

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and the double-trace part is

$$\begin{aligned}\mathcal{B}_6[K_1, K_2] = {} & 6\, \mathrm{Tr}_N\,(K_1) \Big\{ 2\, \mathrm{Tr}_N\,(K_1^5) + 2\, \mathrm{Tr}_N\,(K_1 K_2^4) + 6 e_1 e_2\, \mathrm{Tr}_N\,(K_1^3 K_2^2) - 2 e_1 e_2\, \mathrm{Tr}_N\,(K_1^2 K_2 K_1 K_2)] \Big\} \\ & + 6\, \mathrm{Tr}_N\,(K_2) \Big\{ 2\, \mathrm{Tr}_N\,(K_2^5) + 2\, \mathrm{Tr}_N\,(K_2 K_1^4) + 6 e_1 e_2\, \mathrm{Tr}_N\,(K_2^3 K_1^2) - 2 e_1 e_2\, \mathrm{Tr}_N\,(K_2^2 K_1 K_2 K_1)] \Big\} \\ & + 48\, \mathrm{Tr}_N\,(K_1 K_2) \cdot \big[e_1\, \mathrm{Tr}_N\,(K_1^3 K_2) + e_2\, \mathrm{Tr}_N\,(K_2^3 K_1) \big] \\ & + 6\, \mathrm{Tr}_N\,(K_1^2) \cdot \Big\{ e_2 \big[8\, \mathrm{Tr}_N\,(K_1^2 K_2^2) - 2\, \mathrm{Tr}_N\,(K_2 K_1 K_2 K_1) \big] + e_1 \big[5\, \mathrm{Tr}_N\,(K_1^4) + \mathrm{Tr}_N\,(K_2^4) \big] \Big\} \\ & + 6\, \mathrm{Tr}_N\,(K_2^2) \cdot \Big\{ e_1 \big[8\, \mathrm{Tr}_N\,(K_1^2 K_2^2) - 2\, \mathrm{Tr}_N\,(K_1 K_2 K_1 K_2) \big] + e_2 \big[5\, \mathrm{Tr}_N\,(K_2^4) + \mathrm{Tr}_N\,(K_1^4) \big] \Big\} \\ & + 4 \big(5 [\mathrm{Tr}_N\,(K_1^3)]^2 + 6 e_1 e_2\, \mathrm{Tr}_N\,(K_1 K_2^2)\, \mathrm{Tr}_N\,(K_1^3) + 9 [\mathrm{Tr}_N\ K_1^2 K_2]^2 \\ & \quad + 5 [\mathrm{Tr}_N\,(K_2^3)]^2 + 6 e_1 e_2\, \mathrm{Tr}_N\,(K_1^2 K_2)\, \mathrm{Tr}_N\,(K_2^3) + 9 [\mathrm{Tr}_N\ K_1 K_2^2]^2 \big) \,.\end{aligned}$$

back to main presentation ◅◅

Sketch of the Standard Model derivation from NCG

One starts with the $M \times_{\text{s.t.}} F$ and $\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$

- $F = (\mathcal{A}_{LR}, \mathcal{M}_F^{\#\text{generations}}, D_F)$, $\mathcal{M}_F$ an $\mathcal{A}_{LR}$-module
- $\mathcal{M}_F$ has to be of the form $\mathcal{M}_F = \mathcal{E} \otimes \mathcal{E}^o$, with

$$\mathcal{E} = (2_L \otimes 1^o) \oplus (2_R \otimes 1^o) \oplus (2_L \otimes 3^o) \oplus (2_L \otimes 3^o)\,, \quad \dim_{\mathbb{C}} \mathcal{E} = 16$$

- Thus the $\mathcal{H}_F \cong \mathbb{C}^{32\times 3}$. The 96×96 matrix D_F can have off-diagonal elements only for the maximal subalgebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

- Lie group part of $\mathrm{SU}(\mathcal{A}_F) = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$

Sketch of the Standard Model derivation from NCG

With $Q : \mathbb{C} \hookrightarrow \mathbb{H}$, $Q_\lambda = \mathrm{diag}(\lambda, \bar{\lambda})$ and $Q_\lambda \, |\pm\rangle = \pm\lambda \, |\pm\rangle$,

- Weak hypercharge:

	ν	e	u	d
Y	$\lvert+\rangle \otimes 1^o$	$\lvert-\rangle \otimes 1^o$	$\lvert+\rangle \otimes 3^o$	$\lvert-\rangle \otimes 3^o$
L	-1	-1	$+1/3$	$+1/3$
R	0	-2	$+4/3$	$-2/3$

- $\mathrm{SU}(2)$-adjoint action is 2 on $\mathcal{H}_L$ or trivial in the $\mathcal{H}_R$ sector
- $\mathrm{SU}(3)$-adjoint action is the color action on $\mathcal{H}_q$ and trivial on $\mathcal{H}_\ell$

$$\mathrm{Lie}(\mathrm{SU}(\mathcal{A}_F)) = \mathrm{U}(1)_Y \times \mathrm{SU}(2)_L \times \mathrm{SU}(3)_{\mathrm{color}}$$

- All D_F such that $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ is a spectral triple are

$$D_F(\Upsilon_R, \Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d)$$

the moduli of such Dirac operators has dimension $31 =$ num. Yukawa couplings in νMSM.

Fermionic Spectral Action

- The fermionic part is not treated here but is essentially given by [not needed here]

$$S_f(D) = \frac{1}{2}\langle J\psi \mid D\psi\rangle$$

where ψ are classical fermions, J implements charge-conjugation (J fixes the spin structure)

Dirac F_{SM} operator

$$D_F = \begin{pmatrix}
0 & 0 & \Upsilon_\nu^* & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_R^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Upsilon_e^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^* & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^* & \otimes 1_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Upsilon_u & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Upsilon_d & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Upsilon_R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_\nu^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_e^T & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_u^T & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_d^T & \otimes 1_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_u & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\Upsilon_d & 0 & 0 &
\end{pmatrix} \in M_{96}(\mathbb{C})_{\mathrm{s.a.}}$$

$\sim 10^4$ zeroes from geometry.

back to spectral standard model ⋘

- Example 1: (Finite spectral triples) [Exercise from W. van Suijlekom's book]
  - $\mathcal{A} = \mathbb{C}^3$
  - $\mathcal{H} = \mathbb{C}^3 \leftharpoondown \mathcal{A}$, in defining representation
  - $D = \begin{pmatrix} 0 & 1/d & 0 \\ 1/d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (0 \neq d \in \mathbb{R})$

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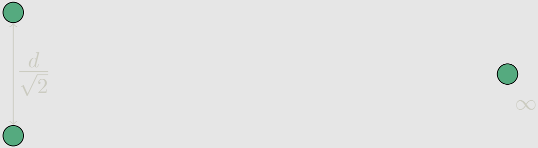
  - $D = \begin{pmatrix} 0 & 1/d & 0 \\ 1/d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (0 \neq d \in \mathbb{R})$

looks like



- **Example 1:** (Finite spectral triples) [Exercise from W. van Suijlekom's book]
  - $\mathcal{A} = \mathbb{C}^3 =$ functions $a : \{1,2,3\} \to \mathbb{C}$
  - $\mathcal{H} = \mathbb{C}^3 \leftharpoondown \mathcal{A}$, in defining representation $\mathbb{C}^3$ diag $\subset M_3(\mathbb{C})$
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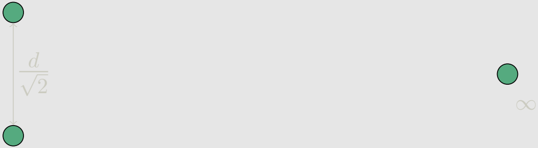

$$\text{for } d_{ij} = \sup_{a\in\mathcal{A}} \{ |a(i) - a(j)| \ : \ \| [D,a] \| \leqslant 1 \}$$

$$= \sup_{a\in\mathcal{A}} \{ |a(i) - a(j)| \ : \ |a(1) - a(2)|^2 \leqslant d^2 \} \qquad \forall i,j = 1,2,3$$

- **Example 2:** (Finite spectral triples)
  - $\mathcal{A} = M_3(\mathbb{C})$
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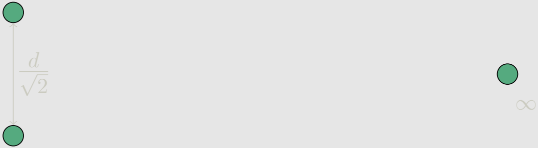

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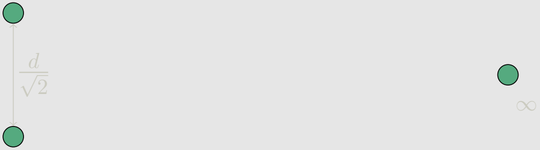

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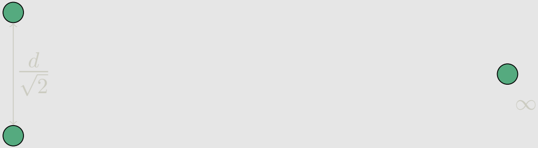

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  - $D = 0$

$= \{1,2,3\}/\sim$

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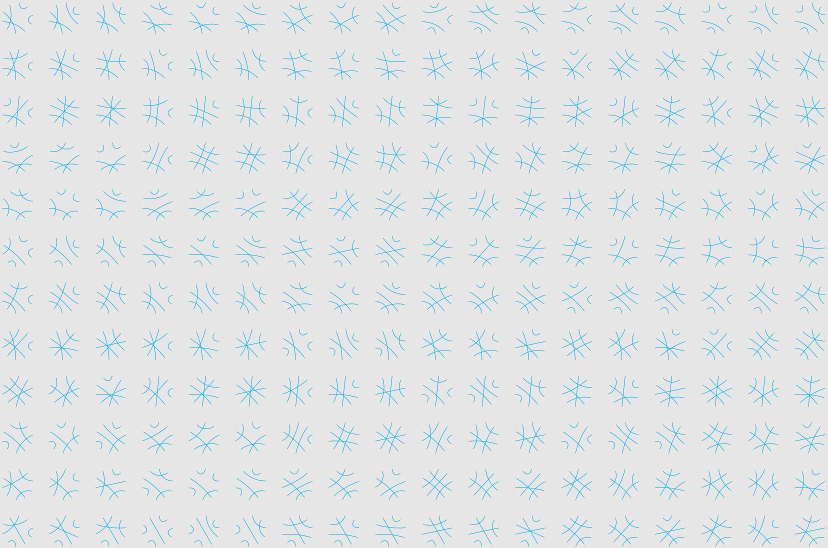
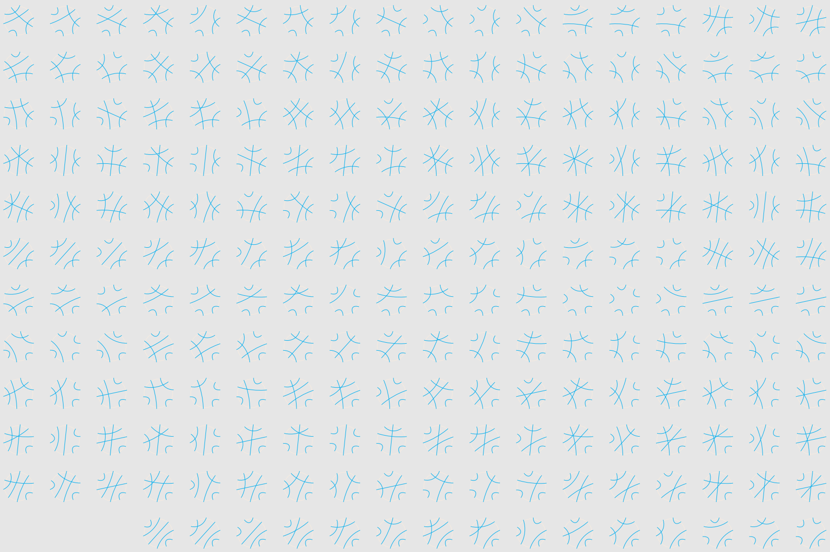
Chord Diagrams of 10 points (1/4)

(appear in $D^{10}|_{d=2}$, $D^4|_{d=4}$, $D^2|_{d=6}$)

Chord Diagrams of 10 points (2/4)

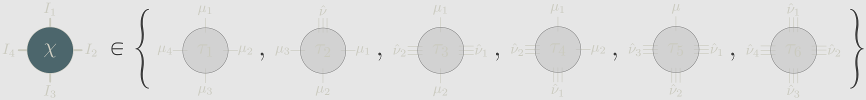


Chord Diagrams of 10 points (3/4)



Chord Diagrams of 10 points (4/4)

Non-vanishing ones… $\frac{1}{4}\operatorname{Tr} D^4 = N\mathcal{S}_4^{\mathrm{Riemann}} + \sum_i \operatorname{Tr}_N{}^{\otimes 2}(A_i \otimes B_i)$



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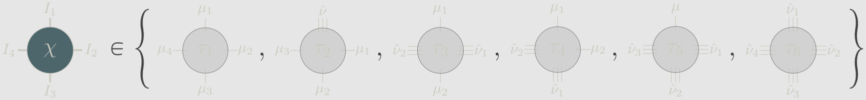

$$
\begin{aligned}
\mathcal{S}_4^{\mathrm{Riemann}} = \mathrm{Tr}_N \Big\{ & 2\sum_{\mu}(L_\mu^4 + H_\mu^4) + 4\sum_{\mu<\nu}(2L_\mu^2 L_\nu^2 + 2H_\mu^2 H_\nu^2 - L_\mu L_\nu L_\mu L_\nu - H_\mu H_\nu H_\mu H_\nu) \\
& - \sum_{\mu\neq\nu}\left[2(L_\mu H_\nu)^2 + 4L_\mu^2 H_\nu^2\right] + \sum_{\mu}\left[2(L_\mu H_\mu)^2 - 4L_\mu^2 H_\mu^2\right] \\
& + 8\Big[H_1\big(L_2[L_3,L_4] + L_3[L_4,L_2] + L_4[L_2,L_3]\big) \\
& \quad + H_2\big(L_1[L_3,L_4] + L_3[L_4,L_1] + L_4[L_1,L_3]\big) \\
& \quad + H_3\big(L_1[L_2,L_4] + L_2[L_4,L_1] + L_4[L_1,L_2]\big) \\
& \quad + H_4\big(L_1[L_2,L_3] + L_2[L_3,L_1] + L_3[L_1,L_2]\big)\Big] + 8[H \leftrightarrow L]\Big\}
\end{aligned}
$$

[C.P. '19]

L_μ, H_μ are random matrices!

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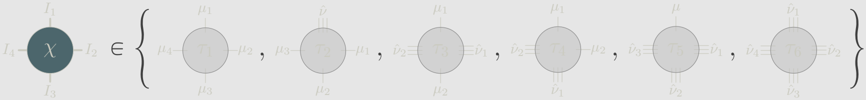

$$\begin{aligned}
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& + 8\Big[H_1\big(L_2[L_3,L_4] + L_3[L_4,L_2] + L_4[L_2,L_3]\big) \\
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\end{aligned}$$

[C.P. '19]

L_μ, H_μ are random matrices!

- Analogy $[L_\mu, \cdot\,] \to \partial_\mu \quad \{H_\mu, \cdot\,\} \to \omega_\mu$ [J. Barrett, L. Glaser, *J. Phys. A* 2016]
- Obtained for any signature, also the A_i, B_i noncommutative-polynomials [C.P. '19]
- Lorentzian signature [L. Glaser, *J. Phys. A* '17]

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Operator	Its noncommutative Hessian
$\operatorname{Tr}(A)\operatorname{Tr}(A^3)$	$3\cdot\begin{pmatrix} \operatorname{Tr} A\cdot(A\otimes 1+1\otimes A) + 1\boxtimes A^2 + A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\operatorname{Tr}(ABAB)$	$2\cdot\begin{pmatrix} B\otimes B & (1\otimes BA+AB\otimes 1) \\ (1\otimes AB+BA\otimes 1) & A\otimes A \end{pmatrix}$
$\operatorname{Tr} A\operatorname{Tr}(AAABB)$	$\begin{pmatrix} (\operatorname{Tr}(A)1\otimes(ABB)+\operatorname{Tr}(A)1\otimes(BBA)+\operatorname{Tr}(A)(ABB)\otimes 1+\operatorname{Tr}(A)(BBA)\otimes 1+1\boxtimes(AABB)+1\boxtimes(ABBA)+1\boxtimes(BBAA)+(AABB)\boxtimes 1+(ABBA)\boxtimes 1+(BBAA)\boxtimes 1+\operatorname{Tr}(A)A\otimes B^2+\operatorname{Tr}(A)B^2\otimes A) & \operatorname{Tr}(A)(1\otimes(AAB)+(BAA)\otimes 1+A\otimes AB+B\otimes A^2+A^2\otimes B+BA\otimes A)+1\boxtimes(A^3B)+1\boxtimes(BA^3) \\ \operatorname{Tr}(A)(1\otimes(BAA)+(AAB)\otimes 1+A\otimes BA+B\otimes A^2+A^2\otimes B+AB\otimes A)+(A^3B)\boxtimes 1+(BA^3)\boxtimes 1 & \operatorname{Tr}(A)(1\otimes(A^3)+(A^3)\otimes 1) \end{pmatrix}$

Table: Some Hessians order operators. Here $\operatorname{Tr} = \operatorname{Tr}_N$.

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OPERATOR	ITS NONCOMMUTATIVE HESSIAN
$\mathrm{Tr}(A)\,\mathrm{Tr}(A^3)$	$3\cdot\begin{pmatrix} \mathrm{Tr}\,A\cdot(A\otimes 1+1\otimes A) +1\boxtimes A^2+A^2\boxtimes 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathrm{Tr}(ABAB)$	$2\cdot\begin{pmatrix} B\otimes B & (1\otimes BA+AB\otimes 1) \\ (1\otimes AB+BA\otimes 1) & A\otimes A \end{pmatrix}$
$\mathrm{Tr}\,A\,\mathrm{Tr}(AAABB)$	$\begin{pmatrix} (\mathrm{Tr}(A)1\otimes(ABB)+\mathrm{Tr}(A)1\otimes(BBA)+\mathrm{Tr}(A)(ABB)\otimes 1+\mathrm{Tr}(A)(BBA)\otimes 1+1\boxtimes(AABB)+1\boxtimes(ABBA)+1\boxtimes(BBAA)+(AABB)\boxtimes 1+(ABBA)\boxtimes 1+(BBAA)\boxtimes 1+\mathrm{Tr}(A)A\otimes B^2+\mathrm{Tr}(A)B^2\otimes A) & \mathrm{Tr}(A)(1\otimes(AAB)+(BAA)\otimes 1+A\otimes AB+B\otimes A^2+A^2\otimes B+BA\otimes A)+1\boxtimes(A^3B)+1\boxtimes(BA^3) \\ \mathrm{Tr}(A)(1\otimes(BAA)+(AAB)\otimes 1+A\otimes BA+B\otimes A^2+A^2\otimes B+AB\otimes A)+(A^3B)\boxtimes 1+(BA^3)\boxtimes 1 & \mathrm{Tr}(A)(1\otimes(A^3)+(A^3)\otimes 1) \end{pmatrix}$

Table: Some Hessians order operators. Here $\mathrm{Tr}=\mathrm{Tr}_N$.

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go to conclusions ⋘

β-functions of ‘free’ two-matrix models, signature $\eta = \mathrm{diag}(e_1, e_2)$

$$2h_1(a_4 + c_{22} + 2d_{2|02} + 6d_{2|2}) = \eta_a$$

$$2h_1(b_4 + c_{22} + 6d_{02|02} + 2d_{2|02}) = \eta_b$$

$$-h_1[e_a(a_4 - c_{1111}) + 2d_{1|12} + 6d_{1|3}] + d_{1|1}(\eta + 1) = \beta(d_{1|1})$$

$$-h_1[e_b(b_4 - c_{1111}) + 6d_{01|03} + 2d_{01|21}] + d_{01|01}(\eta + 1) = \beta(d_{01|01})$$

The next block encompasses the connected quartic couplings:

$$h_2\left(4a_4^2 + 4c_{22}^2\right) + a_4(2\eta + 1)$$
$$-h_1(24a_6 e_a + 4c_{42} e_b + 4d_{02|4} e_b + 4d_{2|4} e_a) = \beta(a_4)$$

$$h_2\left(4b_4^2 + 4c_{22}^2\right) + b_4(2\eta + 1)$$
$$-h_1(24b_6 e_b + 4c_{24} e_a + 4d_{02|04} e_b + 4d_{2|04} e_a) = \beta(b_4)$$

$$-h_1\left(2e_a c_{1212} + e_b 2c_{2121} + 3e_a c_{24} + 3e_b c_{42} + e_a d_{02|22} + e_b d_{2|22}\right)$$
$$+h_2\left(2a_4 c_{22} + 2b_4 c_{22} + 2e_a e_b c_{1111}^2 + 2e_a e_b c_{22}^2\right) + c_{22}(2\eta + 1) = \beta(c_{22})$$

$$8e_a e_b c_{1111} c_{22} h_2 + c_{1111}(2\eta + 1)$$
$$+h_1\left(4e_a c_{1311} + 4e_b c_{3111} + 2e_a d_{02|1111} + 2e_b d_{2|1111}\right) = \beta(c_{1111})$$

$$2h_2(6a_4a_6 + e_ae_bc_{22}c_{42}) + a_6(3\eta + 2) = \beta(a_6)$$

$$2h_2(6b_4b_6 + e_ae_bc_{22}c_{24}) + b_6(3\eta + 2) = \beta(b_6)$$

$$\begin{aligned} 4h_2\{a_4c_{3111} + e_ae_b[c_{22}(c_{1311} + 2c_{3111}) \\ -c_{1111}(2c_{2121} + c_{42})]\} + c_{3111}(3\eta + 2) = \beta(c_{3111}) \end{aligned}$$

$$\begin{aligned} 2h_2[2a_4c_{2121} + e_ae_b(-2c_{1111}c_{3111} \\ +4c_{2121}c_{22} + c_{22}c_{24})] + c_{2121}(3\eta + 2) = \beta(c_{2121}) \end{aligned}$$

$$\begin{aligned} 2h_2[a_4c_{24} + 3b_4c_{24} + 2e_ae_b(c_{22}(3b_6 + c_{2121} + c_{24} + c_{42}) \\ -c_{1111}c_{1311})] + c_{24}(3\eta + 2) = \beta(c_{24}) \end{aligned}$$

$$\begin{aligned} 4h_2\{b_4c_{1311} + e_ae_b[c_{22}(2c_{1311} + c_{3111}) \\ -c_{1111}(2c_{1212} + c_{24})]\} + c_{1311}(3\eta + 2) = \beta(c_{1311}) \end{aligned}$$

$$\begin{aligned} 2h_2[2b_4c_{1212} + e_ae_b(c_{22}(4c_{1212} + c_{42}) \\ -2c_{1111}c_{1311})] + c_{1212}(3\eta + 2) = \beta(c_{1212}) \end{aligned}$$

$$\begin{aligned} 2h_2[3a_4c_{42} + 2e_ae_b(3a_6c_{22} - c_{1111}c_{3111} + c_{1212}c_{22} \\ +c_{22}c_{24} + c_{22}c_{42}) + b_4c_{42}] + c_{42}(3\eta + 2) = \beta(c_{42}) \end{aligned}$$

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