Matrix Model Techniques in JT-Gravity

String Theory Journal Club

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First Seminar Series Talk

- SYK-Motivation
- Mixture of Sections 2 and 3 of Saad-Shenker-Stanford's *JT-gravity as a Matrix Integral* paper
- Terminology and motivation for *2D-quantum gravity* and *Topological Recursion* talks (terminology)

SYK-model [D. Stanford; Strings 2017]

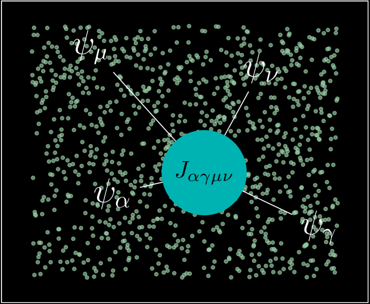
- model (quantum mechanics) for N Majorana fermions $\{\psi_\alpha\}_{\alpha=1}^{N}$

$$\{\psi_\alpha, \psi_\beta\} = \delta_{\alpha\beta}$$

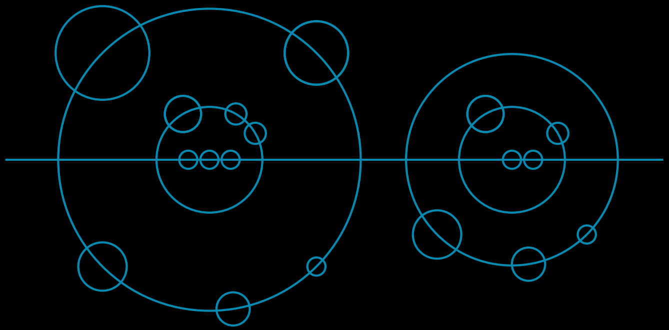
- action

$$S[\psi_\rho] = \int \mathrm{d}\tau \left(\frac{1}{2} \psi_\alpha \frac{\mathrm{d}\psi^\alpha}{\mathrm{d}\tau} + \frac{\mathrm{i}}{4!} J_{\alpha\gamma\mu\nu} \psi_\alpha \psi_\gamma \psi_\mu \psi_\nu \right)$$

coupling whose coupling constants satisfy $\langle J^2_{\alpha\gamma\mu\nu} \rangle = J^2/N^3$



- simple diagramatics (**“melons”**). For the 2-point function

$G \supset$ 

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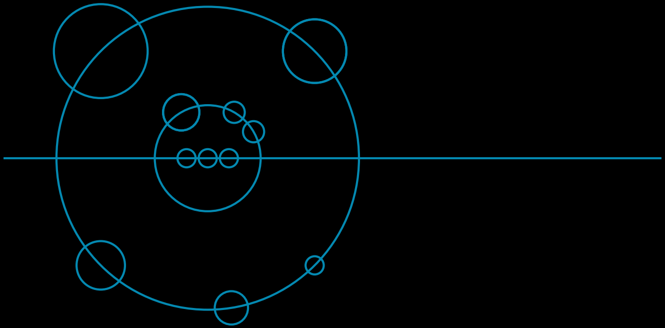
$\Sigma \supset$ 

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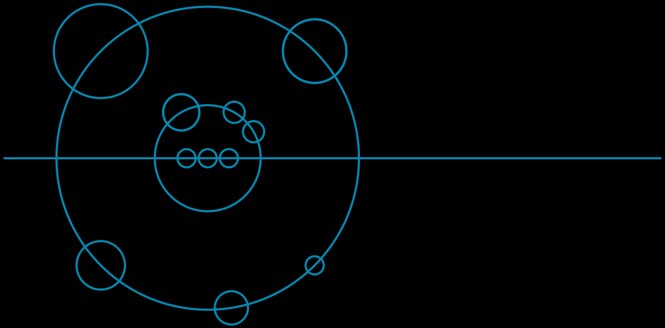
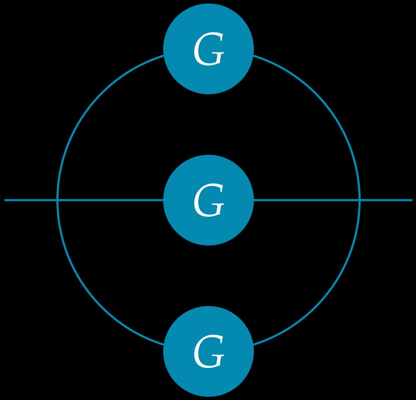
$\Sigma \supset$ 

- Σ generating 1PI (2-connected graphs)
- $G(\tau) = \langle \psi_\alpha(\tau)\psi_\alpha(0)\rangle$ in Fourier space:

$$G(\omega) = \frac{1}{-\mathrm{i}\omega - \Sigma(\omega)}$$

$\Sigma =$ 

$$G(\omega) = \frac{1}{-\mathrm{i}\omega - \Sigma(\omega)} \qquad \Sigma(\tau) = J^2[G(\tau)]^3$$

- strong coupling $\beta J \gg 1$ (or infrared regime)

$$G(\tau)\Sigma(\tau) = -1,$$

- the solution in the termic circle $\tau \in (0, \beta)$

$$G(x) \sim \frac{\mathrm{sgn}(\tau)}{\mathrm{sen}^{1/2}(x)} \qquad\qquad (x \equiv \tan(\pi\tau/\beta))$$

  presents $\mathrm{SL}(2,\mathbb{R})$-symmetry $x \mapsto \frac{ax+b}{cx+d}$
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- low energy dynamics of SYK $\leftrightarrow$ ∂-description of 2D JT-gravity

$$I_{JT} = -\overbrace{\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right]}^{\text{topological term} = S_0\,\chi(\mathcal{M})} - \left[\underbrace{\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\varphi(R+2)}_{\text{sets } R=-2} + \underbrace{\int_{\partial\mathcal{M}}\sqrt{h}\varphi(K-1)}_{\text{gives action for boundary}}\right].$$

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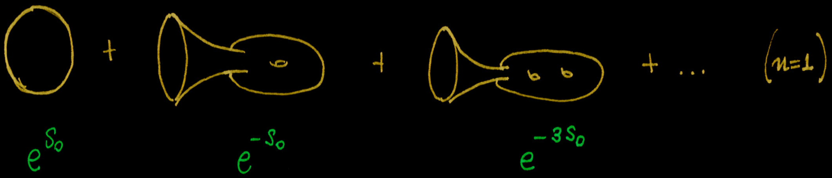
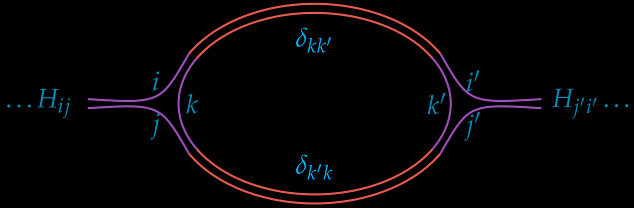
- H = Hamiltonian at the boundary, $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$
- JT-gravity is a matrix model, $(e^{S_0})^{\chi} = e^{S_0(2-2g-n)}$ expansion exists; for $n = 1$ boundaries, a trumpet-expansion:



$$L := \mathrm{e}^N, V = \sum_n v_n H^n, \quad \mathcal{Z} = \int_{\text{formal}} \mathrm{d}H\, \mathrm{e}^{-L \mathrm{Tr} V} \overset{e.g.}{=} \int_{\text{formal}} \mathrm{d}H\, \mathrm{e}^{-\frac{L}{2}\mathrm{Tr}(H^2) + L v_3 \mathrm{Tr}(H^3)}$$



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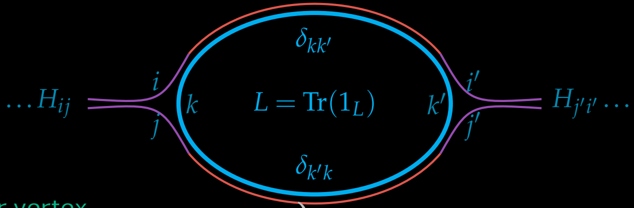

- L per vertex
- L^{-1} factor per propagator
- L per $\sum_{a,b} \delta_{ab}\delta_{ba}$

$$\mathcal{A}(\mathcal{G}) \sim L^{V(\mathcal{G})-E(\mathcal{G})+F(\mathcal{G})} = L^{\chi(\mathcal{G})}$$

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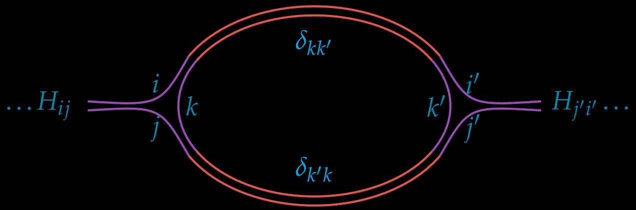
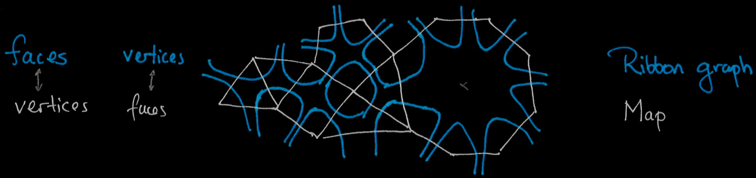


The (maps dual to) ribbon graphs might have boundary:

$$\mathcal{A}(\mathcal{G}) \sim L^{\chi(\mathcal{G})} = L^{2-2g-n} \qquad \leftrightarrow$$

$n = \#\,$(boundaries)

$g =$ genus



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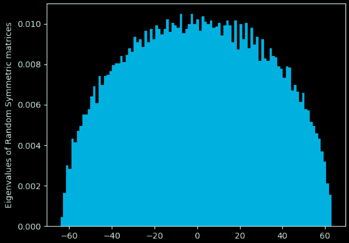

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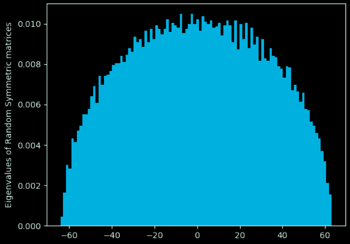

Assumption: One-cut or connected support as in

$$\rho_0(E) = \frac{1}{2\pi}\sqrt{4 - E^2}$$

With $\mathcal{Z} = \int_{H \in M_L(\mathbb{C}), H^* = H} \mathrm{d}H \, \mathrm{e}^{-L \mathrm{Tr} \, V(H)}$ one determines:

$$\langle Z(\beta_1) ... Z(\beta_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{d}H \, \mathrm{e}^{-L \mathrm{Tr} \, V(H)} Z(\beta_1) ... Z(\beta_n).$$

But it will turn out

$$R(E) = -\int_0^\infty \mathrm{d}\beta \mathrm{e}^{\beta E} Z(\beta),$$

$$R(E + i\epsilon) - R(E - i\epsilon) = -2\pi i \rho(E)$$

Convenient:

$$\langle R(E_1) ... R(E_n) \rangle_{\text{conn.}} = \sum_{g=0}^{\infty} \frac{R_{g,n}(E_1, ..., E_n)}{L^{2g+n-2}} \quad (L \to \infty)$$

Relation to Eynard's notation,

$$W_{g,n}(z_1, ..., z_n) \sim R_{g,n}(-z_1^2, ..., -z_n^2) \qquad z_i = \sqrt{-E_i}$$

Disc amplitude $R_{0,1}$ (with 1-cut assumption)

Diagonalization and integration

$$\mathcal{Z} = C_L \int \underbrace{\overbrace{\prod_{i<j}(\lambda_i - \lambda_j)^2}^{\text{repulsion}} \times \overbrace{\exp\left\{ -L \sum_{j=1}^{L} V(\lambda_j) \right\}}^{\text{confinement}}}_{\exp(-\sum_j V_{\text{eff}}(\lambda_j))} \mathrm{d}^L \lambda$$

- V_{eff} is an interesting interaction, e.g. known cases but also:
  - birds on a wire [Šeba]
  - buses without schedule [Krbálek-Šeba 2000,..., Warchoł 2018]

- $V'_{\text{eff}}(E) = 0 \Rightarrow V'(E) = 2 \int d\lambda \frac{\rho_0(\lambda)}{E-\lambda}$ (ρ_0 is L.O. in L of ρ)

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- Migdal's "dispersion relations" trick: $\sigma(x) = \sqrt{(x-a_+)(x-a_-)}$
- residue theorem for $R_{0,1}/\sqrt{\sigma}$ yields

$$R_{0,1}(E) = \oint_E \frac{d\lambda}{2\pi i}\frac{R_{0,1}(\lambda)}{\lambda - E}\sqrt{\frac{\sigma(E)}{\sigma(\lambda)}}$$

$$= -\int_{\mathcal{C}} \frac{d\lambda}{2\pi i}\frac{R_{0,1}(\lambda)}{\lambda - E}\sqrt{\frac{\sigma(E)}{\sigma(\lambda)}} \quad \text{(contour deformed)}$$

$$= -\frac{1}{2}\int_{\mathcal{C}} \frac{d\lambda}{2\pi i}\frac{V'(\lambda)}{\lambda - E}\sqrt{\frac{\sigma(E)}{\sigma(\lambda)}} \quad \text{by } R_{0,1}(E+i\epsilon) + R_{0,1}(E-i\epsilon) = V'(E)$$

so expanding

$$\frac{1}{E} \overset{E\to\infty}{\sim} \int_{\mathcal{C}} \frac{d\lambda}{2\pi i}\frac{V'(\lambda)}{\sqrt{\sigma(\lambda)}} + \frac{1}{2E}\int_{\mathcal{C}} \frac{d\lambda}{2\pi i}\frac{\lambda V'(\lambda)}{\sqrt{\sigma(\lambda)}}$$

can be solved for $a_\pm$ and determine the disk amplitude

Loop equations $\longrightarrow R_{1,1}$ ('lid')

Departing from the loop equations

$$0 = \int d^L\lambda \frac{\partial}{\partial \lambda_a}\left[\mathcal{O}_a(\boldsymbol{\lambda})\prod_{i<j}(\lambda_i - \lambda_j)^2 \mathrm{e}^{-L\sum_j V(\lambda_j)}\right]$$

for $\mathcal{O}_a(\boldsymbol{\lambda}) = \frac{1}{E-\lambda_a}$

$$0 = \sum_a \left\langle \frac{1}{(E-\lambda_a)^2} + \sum_{j\neq a} \underbrace{\frac{1}{E-\lambda_a}\frac{2}{\lambda_a-\lambda_j}}_{\sum_{a\neq j}\frac{1}{\lambda_a-\lambda_j}\left[\frac{1}{E-\lambda_a}-\frac{1}{E-\lambda_j}\right]} - \frac{LV'(\lambda_a)}{E-\lambda_a}\right\rangle$$

Underbracket gives crossed terms, $\sum_{j\neq a}\frac{1}{(E-\lambda_a)(E-\lambda_j)}$ so

$$\left\langle \left(\mathrm{Tr}\frac{1}{E-H}\right)^2 - L\,\mathrm{Tr}\frac{V'(H)}{E-H}\right\rangle = 0$$

$$\left\langle \left(\frac{1}{L} \mathrm{Tr} \frac{1}{E-H} \right)^2 - \frac{1}{L} \mathrm{Tr} \frac{V'(H)}{E-H} \right\rangle = 0$$

and use

- $\frac{1}{L} \mathrm{Tr} \frac{1}{E-H} = R_{0,1} + \frac{1}{L^2} R_{1,1} + \ldots$

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completing the square:

$$\left(R_{0,1}(E) - \frac{V'(E)}{2} \right)^2 = \frac{V'(E)^2}{4} - \left\langle \frac{1}{L} \mathrm{Tr} \frac{V'(E) - V'(H)}{E-H} \right\rangle_0$$

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yields for $y = R_{0,1}(E) - \frac{V'(E)}{2}$ the *spectral curve*

$$y^2 = f(E)$$

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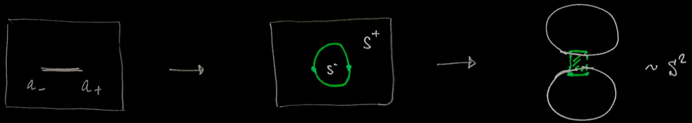


Next to LO in L of $\left\langle \left(\frac{1}{L}\mathrm{Tr}\frac{1}{E-H}\right)^2 - \frac{1}{L}\mathrm{Tr}\frac{V'(H)}{E-H} \right\rangle = 0$ yields similarly[Exercise]

$$2y(E)R_{1,1}(E) = -R_{0,2}(E,E) - \left\langle \frac{1}{L}\mathrm{Tr}\frac{V'(E)-V'(H)}{E-H} \right\rangle_{L^{-2}\text{ terms}}$$

- Combinatorial derivative $\mathrm{Tr}\frac{V'(E)-V'(H)}{E-H}$ is not singular at endpoints: so dispersion relations (again) yield

$$R_{1,1}(E)\sqrt{\sigma(E)} = \sum_{\pm}\oint_{a_\pm}\frac{d\lambda}{2\pi i}\frac{R_{0,2}(\lambda,\lambda)}{\lambda-E}\frac{\sqrt{\sigma(\lambda)}}{2y(\lambda)}.$$

- to find $R_{0,2}$ one uses the loop insertion operators $\frac{\mathrm{d}}{\mathrm{d}V(E)} = -\sum_{n=0}^{\infty}\frac{1}{E^{n+1}}\frac{\partial}{\partial v_n}$ acted on the free energy F_L:

$$R_{*,s}(E_1,\ldots,E_s) = \frac{\mathrm{d}}{\mathrm{d}V(E_s)}\cdots\frac{\mathrm{d}}{\mathrm{d}V(E_1)}F_L$$

so $R_{0,2}(E_1,E_2) = \frac{\mathrm{d}}{\mathrm{d}V(E_2)}\frac{\mathrm{d}}{\mathrm{d}V(E_1)}F_L = -\sum_{n=0}^{\infty}\frac{1}{E_2^{n+1}}\partial_{v_n}R_{0,1}(E_1)$

This method works for all higher $R_{g,n}$'s!

Higher topologies and Topological Recursion

- new coordinate $z^2 = x = -E$, the locus $(x(z), y(z)) \subset \mathbb{C}^2$ is the spectral curve, which is a hyperelliptic curve ($y = R_{0,1}(E) - \frac{V'(E)}{2}$ is polynomial)
- Saad-Shenker-Stanford $\leftrightarrow$ Eynard's notation:

$$W_{g,n}(z_1, ..., z_n) = (-1)^n 2^n z_1 ... z_n \, R_{g,n}(-z_1^2, ..., -z_n^2) \qquad z_i = \sqrt{-E_i}$$

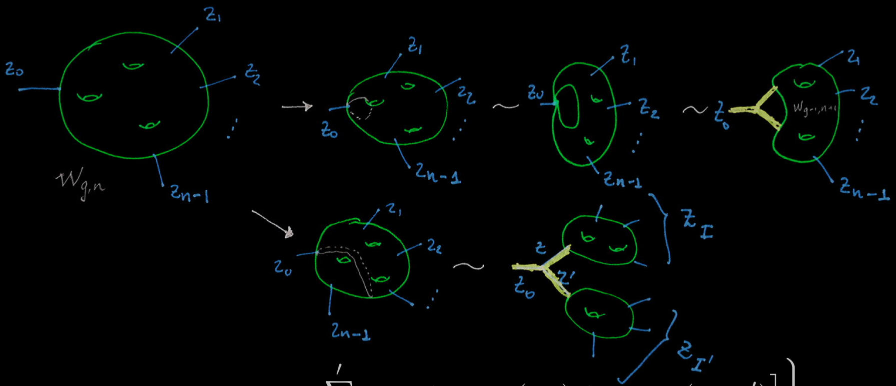
$$W_{0,1}(z) = 2z\, y(z), \qquad W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$$

- excluding $(I = J, h = g)$ and $(I' = J, h' = g)$ in $\sum'$,

$$W_{g,n}(z_1, \overbrace{z_2, \ldots, z_n}^{J}) = \operatorname*{Res}_{z \to 0} \left\{ \frac{1}{(z_1^2 - z^2)} \frac{1}{4y(z)} \left[W_{g-1,n+1}(z, -z, J) + \sum_{I \cup I' = J; h + h' = g}' W_{h,1+|I|}(z, I) \, W_{h',1+|I'|}(-z, I') \right] \right\}$$

... but rather, it's two different ways to choose a cycle and pinch it:

$$W_{g,n}(z_0,\overbrace{z_1,\ldots,z_{n-1}}^{J}) = \operatorname*{Res}_{z\to 0}\left\{ \frac{1}{(z_0^2 - z^2)}\,\frac{1}{4y(z)}\,\Big[W_{g-1,n+1}(z,-z,J)\right.$$



$$\left. + \sum_{I\cup I'=J;\, g'+g''=g}^{\prime} W_{g',1+|I|}(z,I)\,W_{g'',1+|I'|}(-z,I')\Big]\right\}$$

The TR machinery

- input, the density of ev./spectral curve: e.g. $\rho_0^{\text{total}}(E) = \frac{e^{S_0}}{\pi}\sqrt{E}, E > 0$ or $y = z$

- output, the $W_{g,n}$'s:

$$W_{0,1} = 2z_1^2,$$
$$W_{0,2} = \frac{1}{(z_1 - z_2)^2},$$
$$W_{0,3} = \frac{1}{2z_1^2 z_2^2 z_3^2}$$
$$W_{1,1} = \frac{1}{16z_1^4},$$
$$W_{1,2} = \frac{5z_1^4 + 3z_1^2 z_2^2 + 5z_2^4}{32z_1^6 z_2^6}$$
$$W_{2,1} = \frac{105}{1024z_1^{10}}.$$

- input, the density of ev./spectral curve: e.g. $y = \frac{\sin(2\pi z)}{4\pi}$
- output, the $W_{g,n}$'s:

$$W_{0,1} = 2z_1 \frac{\sin(2\pi z_1)}{4\pi},$$
$$W_{0,2} = \frac{1}{(z_1 - z_2)^2},$$
$$W_{0,3} = \frac{1}{z_1^2 z_2^2 z_3^2}$$
$$W_{1,1} = \frac{3 + 2\pi^2 z_1^2}{24 z_1^4},$$
$$W_{1,2} = \frac{5(z_1^4 + z_2^4) + 3z_1^2 z_2^2 + 4\pi^2 (z_1^4 z_2^2 + z_2^4 z_1^2) + 2\pi^4 z_1^4 z_2^4}{8 z_1^6 z_2^6}$$
$$W_{2,1} = \left(\frac{105}{128 z_1^{10}} + \frac{203\pi^2}{192 z_1^8} + \frac{139\pi^4}{192 z_1^6} + \frac{169\pi^6}{480 z_1^4} + \frac{29\pi^8}{192 z_1^2} \right).$$

Relation to volumes $V_{g,n}$ of the moduli of genus g bordered Riemann surfaces with geodesic n boundaries of lengths b_i,

$$W_{g,n}(z_1, \ldots, z_n) = \int_0^\infty b_1 \mathrm{d}b_1 \mathrm{e}^{-b_1 z_1} \cdots \int_0^\infty b_n \mathrm{d}b_n \mathrm{e}^{-b_n z_n} \underbrace{V_{g,n}(b_1, \ldots, b_n)}_{\text{Mirzakhani}}.$$